Controlled teleportation of qubit states: Relation between teleportation faithfulness, controller's authority, and tripartite entanglement

Artur Barasiński[1,2,*] and Jiří Svozilík[1,3,4]

[1]RCPTM, Joint Laboratory of Optics of Palacký University and Institute of Physics of CAS, Faculty of Science, Palacký University, 17. Listopadu 12, 771 46 Olomouc, Czech Republic

[2]Institute of Physics, University of Zielona Góra, Z. Szafrana 4a, 65-516 Zielona Góra, Poland

[3]Yachay Tech, School of Physical Sciences & Nanotechnology, 100119, Urcuquí, Ecuador

[4]Centro de Física Aplicada y Tecnología Avanzada, Universidad Nacional Autónoma de México, Boulevard Juriquilla 3001, Juriquilla Querétaro 76230, Mexico

(Received 4 October 2018; published 4 January 2019)

We consider the controlled quantum teleportation performed on a single copy of entangled three-qubit pure states and show how the fidelity of teleportation and the control power are related to the tripartite entanglement measures, namely, three-tangle and genuine concurrence. We characterize the states with extreme properties, and use them to derive tight lower and upper bounds on both the teleportation fidelity and control power for a given amount of entanglement. Furthermore, we discuss the usefulness of these two quantities for experimental detection and/or quantification of tripartite entanglement. In particular, we present the results which imply that the control power cannot be considered as a candidate for a degree of three-qubit entanglement complementary to the three-tangle as conjectured recently [K. Jeong, J. Kim, and S. Lee, Phys. Rev. A **93**, 032328 (2016)]. Furthermore, we prove a mutual relation between teleportation fidelity and control power which provides attainable limits of the controller's authority for existing controlled teleportation schemes. Although those relations are derived for pure states, their extensions for mixed states are also examined.

DOI: 10.1103/PhysRevA.99.012306

I. INTRODUCTION

Quantum teleportation is one of the most important protocols in quantum information science due to its essential role in the development of formal quantum information theory [1,2] and quantum technology, such as quantum computing [3], quantum repeaters [4,5], and quantum network teleportation [6–8]. The concept of quantum teleportation involves the transfer of a quantum state from one location to another without having to exchange the physical system, which requires classical communication and the shared resources of quantum entanglement [9,10]. Without such resources, the transfer of a quantum state would not be possible within the laws of quantum mechanics. The cost of constructing an exact replica of the original unknown state in the remote location is the destruction of the original state.

The reliability of teleportation is quantified by the fidelity F which measures the overlap of the states $|\psi_t\rangle$ to be teleported and the output state depicted by the density operator $\rho_{\rm out}$. Since the state $|\psi_t\rangle$ is generally unknown, it is more appropriate to consider the average of the fidelity $F = \overline{\langle\psi_t|\rho_{\rm out}|\psi_t\rangle}$, taken over all possible input states to obtain a quantitative description of the efficiency of the protocol which is independent of $|\psi_t\rangle$ [10,11]. In the ideal quantum teleportation procedure based on the maximally entangled states, $|\psi_t\rangle$ can be reconstructed with fidelity $F = 1$, whereas the resemblance of the teleportation attainable by a purely classical channel cannot exceed $F = \frac{2}{3}$ [9,10] (hereafter, the classical limit).

Although quantum teleportation is a typically bipartite process, it has been also extended to a multipartite case. For instance, a tripartite variant of quantum teleportation called controlled quantum teleportation (CQT) has been proposed by Karlsson and Bourennane [12]. The essential idea of the CQT protocol is to transfer an arbitrary quantum state from the sender to the receiver, but only with the permission of the controller [12–15]. Specifically, the controller determines the success or failure of the teleportation by restricting the access to his measurement information. Therefore, the CQT protocol is always characterized by two quantities: the conditioned fidelity $F_{\rm CQT}$ (i.e., with controller's permission) which should be greater than the classical limit and the nonconditioned fidelity $F_{\rm NC}$ (i.e., without controller's permission) which cannot exceed $\frac{2}{3}$. At this point, a substantial question arises of the achievable teleportation fidelity $F_{\rm CQT}$ which guarantees better than classical teleportation with simultaneous fulfillment of the second requirement of CQT protocol. Although a lot of work has been devoted to studying controlled teleportation on maximally entangled GHZ state, only a few studies are directed to nonmaximally entangled states, despite the importance of such states in practical experiments. On the other hand, as the teleportation faithfulness is directly related with the quantity called fully entangled fraction [16,17], the above question is essential also in the context of entanglement distillation, where the fully entanglement fraction quantifies how close a state is to a maximally entangled one [18–21].

*artur.barasinski@upol.cz

Recently, an additional parameter which provides a quantitative characteristic of the controller's authority has been defined [14,15]. This quantity, called control power, is measured by the deference between conditioned and nonconditioned fidelity and it is meaningful if and only if both requirements of CQT are satisfied. It is also conjectured [15] that a special variant of control power (so-called minimal control power) can be considered as a candidate for a measure of three-party correlation complementary to the tripartite entanglement monotony called three-tangle [22]. However, this hypothesis is based on a study conducted on just a few examples of pure states and no general analysis has been carried out in this field. Only recently it has been shown that the tripartite entanglement is not a necessary recourse for the CQT protocol if one considers mixed-state channels [21]. This means that nonzero control power appears also for biseparable states. For that reason, it is important to reexamine the relationship between control power and tripartite entanglement for an arbitrary pure state. Such relation could (potentially) be further used to witness the tripartite entanglement without measuring the correlations between parties, which is strongly demanding experimentally [23,24].

This paper aims at filling all these gaps and presents an extensive analysis of the CQT protocol. We derive the tight upper and lower bounds on the fidelity and the control power for given three-tangle and genuine multipartite (GME) concurrence [25]. The need to use two entanglement measures is caused by the fact that the tripartite entanglement has no unambiguous definition. Specifically, the three-tangle corresponds to the so-called residual entanglement and takes nonzero value only for the GHZ-class states [26]. Consequently, one cannot distinguish the biseparable state from the tripartite entangled W-class state in the frame of the three-tangle. In contrast, the GME concurrence which is based on the bipartite concurrence of all possible bipartitions vanishes only for biseparable and fully separable states. Our results provide important information on the usefulness of CQT protocol for experimental detection and/or quantification of tripartite entanglement. Moreover, by means of the Karush-Kuhn-Tucker (KKT) method, we show that previous results concerning the relation between control power and three-tangle are incorrect and hence lead to inappropriate conclusions.

II. CONTROLLED TELEPORTATION PROTOCOL

Consider a system which is partitioned into three subsystems distributed between Alice (the sender), Charlie (the controller), and Bob (the receiver). All subsystems are far from each other, and they share an entangled three-qubit quantum state ρ. The entangled state is delivered to all participants in such a way that Alice receives qubit 1, Charlie keeps qubit 2, and Bob gets qubit 3. The only manipulations that all participants are allowed to do are local quantum operations and classical communications.

Now, suppose that Alice wants to transfer an unknown quantum state represented by the state of qubit 4 to Bob with the Charlie participation (permission). For this purpose, the controller, Charlie, makes a one-qubit orthogonal measurement on qubit No. 2 with the measurement outcome t. As a result, the entangled channel ρ is projected onto two-qubit state ρ_{13}^t [27]:

$$\rho_{13}^t = \frac{\mathrm{Tr}_2[\mathbb{1}_2 \otimes |t\rangle\langle t| U \otimes \mathbb{1}_2 \rho \mathbb{1}_2 \otimes U^\dagger |t\rangle\langle t| \otimes \mathbb{1}_2]}{\langle t|U\rho_2 U^\dagger|t\rangle}, \quad (1)$$

where U is a 2×2 unitary matrix, $\mathbb{1}_2$ stands for a 2×2 identity matrix, and $\rho_2 = \mathrm{Tr}_{13}(\rho)$ is a reduced state of qubit 2. In the next step, Alice performs a joint orthogonal measurement on qubits 1 and 4. Then, Bob applies an accordingly chosen unitary transformation on qubit 3 in order to reconstruct the input state in the most optimal way. Naturally, the selection of the appropriate unitary operation requires a classical communication with both Alice and Charlie. For such a scenario, the teleportation fidelity F_{CQT} can be written in a general form [27]

$$F_{\mathrm{CQT}}(\rho) = \frac{2\max_U \left[\sum_{t=0}^{1} \langle t|U\rho_2 U^\dagger|t\rangle f\left(\rho_{13}^t\right)\right] + 1}{3}, \quad (2)$$

where $\langle t|U\rho_2 U^\dagger|t\rangle$ denotes the probability of receiving outcome t and $f(\varrho) = \max_e \langle e|\varrho|e\rangle$ is the fully entangled fraction (FEF) [16,17]. Note that the first term in Eq. (2) can be interpreted as an average FEF localized between qubits 1 and 3 after the one-qubit measurement of the controller. It provides information about the distance between ρ_{13}^t and the maximally entangled state. By analogy to Refs. [28,29] we call that quantity as localizable FEF (hereafter denoted as $f^L(\rho) = \max_U[\sum_{t=0}^{1}\langle t|U\rho_2 U^\dagger|t\rangle f(\rho_{13}^t)]$). Recently, the relation between localizable FEF and localizable concurrence in the context of CQT has been derived in Ref. [21].

On the other hand, if Charlie decides not to send his information to Alice and Bob, i.e., to forbid the teleportation, Alice can transfer a quantum state to Bob through the teleportation channel described by the density matrix $\rho_{13} = \mathrm{Tr}_2\rho$, where we traced out the qubit No. 2. The faithfulness of the process without controller's participation is given by the nonconditioned fidelity

$$F_{\mathrm{NC}}(\rho) = \frac{2f(\rho_{13}) + 1}{3}. \quad (3)$$

Based on these two fidelities, the controller's measurable authority can be characterized by the parameter called control power [14,15]

$$P(\rho) = F_{\mathrm{CQT}}(\rho) - F_{\mathrm{NC}}(\rho). \quad (4)$$

We recall that the CQT protocol and, hence, the control power is meaningful if and only if the nonconditioned fidelity $F_{\mathrm{NC}}(\rho) \leqslant \frac{2}{3}$ and the conditioned fidelity $F_{\mathrm{CQT}} > \frac{2}{3}$.

Suppose now that the teleportation channel prepared by Charlie is a pure tripartite state. For convenience and without loss of generality we write such state in a canonical form proposed by Acín *et al.* [30]:

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \quad (5)$$

with real $\lambda_i \geqslant 0$, $0 \leqslant \phi \leqslant \pi$ and standard normalization. For such state, the conditioned fidelity F_{CQT} in Eq. (2) can be expressed [27] as

$$F_{\mathrm{CQT}}(\psi) = \frac{2 + \sqrt{\tau + \mathcal{C}_{13}^2}}{3}, \quad (6)$$

where the three-tangle $\tau = 4\lambda_0^2\lambda_4^2$ and $\mathcal{C}_{13} = 2\lambda_0\lambda_2$ is the Wootters' concurrence between qubits 1 and 3. Furthermore, the nonconditioned fidelity can be computed by means of Horodecki theorem [31] as

$$F_{\rm NC}(\psi) = \frac{||T_{13}||_1 + 3}{6}, \tag{7}$$

where $||\cdot||_1$ is a trace norm and T_{13} is a correlation tensor for the state ρ_{13}. In particular, for the pure state in Eq. (5) the correlation tensor writes

$$T_{13} = \begin{pmatrix} 2\lambda_0\lambda_2 & 0 & 2\lambda_0\lambda_1\cos\phi \\ 0 & -2\lambda_0\lambda_2 & 2\lambda_0\lambda_1\sin\phi \\ \Lambda & 2\lambda_1\lambda_2\sin\phi & 1-2(\lambda_1^2+\lambda_3^2) \end{pmatrix}, \tag{8}$$

where $\Lambda = -2(\lambda_3\lambda_4 + \lambda_1\lambda_2\cos\phi)$.

III. PURE STATE CHANNELS

First, we explicitly derive the possible range of values of $F_{\rm CQT}$ as a function of its three-tangle τ under a constraint $F_{\rm NC} \leqslant \frac{2}{3}$. It is easy to notice that the lower bound of $F_{\rm CQT}$ in Eq. (6) with given τ is reached when $\mathcal{C}_{13} \equiv 0$. It is because the three-tangle and concurrence are always positive and, hence, any value $\mathcal{C}_{13} \neq 0$ can only increase the fidelity. This assumption also implies that the two-qubit channel ρ_{13} achieved by tracing over qubit No. 2 is a product state. Consequently, the teleportation protocol performed without Charlie's participation cannot exceed the classical limit $F_{\rm NC} \leqslant \frac{2}{3}$, as is required for the CQT protocol. The lower bound of $F_{\rm CQT}$ can be written then as

$$\frac{2+\sqrt{\tau}}{3} \leqslant F_{\rm CQT}. \tag{9}$$

We note that there are few important examples of states which provide the lower bound, namely, the generalized GHZ and the maximal slice (MS) states

$$|\psi_G\rangle = \lambda_0|000\rangle + \lambda_4|111\rangle,$$
$$|\psi_{{\rm MS}_1}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \frac{1}{\sqrt{2}}|111\rangle,$$
$$|\psi_{{\rm MS}_3}\rangle = \frac{1}{\sqrt{2}}|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{10}$$

where the standard normalization condition is assumed. For these states, both fidelities can be easily computed (see Table I). We also note that the MS state of the form $|\psi_{{\rm MS}_2}\rangle = \frac{1}{\sqrt{2}}|000\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle$ with qubit 2 kept by the controller is not suitable for CQT protocol. It is because for any $\lambda_2, \lambda_4 \geqslant 0$, the teleportation fidelity achieved without controller's permission is greater than the classical limit $F_{\rm NC}(\psi_{{\rm MS}_2}) > \frac{2}{3}$ [14].

Determination of the upper limit of attainable $F_{\rm CQT}$ for given value τ is much more complicated. In Ref. [15] a special case of $\tau = 0$ has been discussed. It has been shown that all W-class pure states are suitable for CQT and the maximum of $F_{\rm CQT}$ is reached for standard W state. However, this result is incorrect as we show in the Appendix. In order to find the upper bound, we use the Karush-Kuhn-Tucker (KKT) conditions. For this purpose, we consider the Lagrange function $\mathcal{L}(\psi) = \mathcal{F}(\psi) + \alpha[\tau(\psi) - \tau_0] + \beta[\sum_{m=0}^4 \lambda_m^2 - 1] + \mu[F_{\rm NC}(\psi) - \frac{2}{3}]$, where $\mathcal{F} = \tau + \mathcal{C}_{AB}^2$ denotes a target function, τ_0 is a constant value in the range $\langle 0, 1\rangle$, and we discuss all states $|\psi\rangle$ defined in Eq. (5) which refer to a given value of the three-tangle $\tau(\psi) - \tau_0 = 0$. The third term in the Lagrange function corresponds to normalization condition while the last term implies $F_{\rm NC} \leqslant \frac{2}{3}$. We note that the nonconditioned fidelity $F_{\rm NC}$ in Eq. (7) can be calculated analytically, even for a general pure state $|\psi\rangle$. However, a direct expression for $||T_{13}||_1$ takes a long form. For that reason, to simplify our presentation let us consider a special case of $\phi = \pi$ [cf. Eq. (5)]. Then,

TABLE I. Exact results of fidelities (controlled fidelity $F_{\rm CQT}$ given in the second column and nonconditioned fidelity $F_{\rm NC}$ shown in the third column) and control power P for extremal states (see text for details).

State	$F_{\rm CQT}$	$F_{\rm NC}$	P
$\lvert\psi_G\rangle$	$\frac{2}{3} + \frac{2\lambda_0\sqrt{1-\lambda_0^2}}{3}$	$\frac{2}{3}$	$\frac{2\lambda_0\sqrt{1-\lambda_0^2}}{3}$
$\lvert\psi_{{\rm MS}_1}\rangle$	$\frac{2}{3} + \frac{\sqrt{2}\lambda_0}{3}$	$\frac{1}{2} + \frac{\sqrt{2}\lambda_0}{6}$	$\frac{1}{6} + \frac{\sqrt{2}\lambda_0}{6}$
$\lvert\psi_{{\rm MS}_3}\rangle$	$\frac{2}{3} + \frac{\sqrt{2}\lambda_4}{3}$	$\frac{1}{2} + \frac{\sqrt{2}\lambda_4}{6}$	$\frac{1}{6} + \frac{\sqrt{2}\lambda_4}{6}$
$\lvert\psi_T\rangle$	$\frac{5}{6} + \frac{2\sqrt{2d_0^2-4d_0^4}}{6}$	$\frac{2}{3}$	$\frac{1}{6} + \frac{2\sqrt{2d_0^2-4d_0^4}}{6}$
$\lvert\psi_W\rangle$	$\frac{2}{3} + \frac{2c^2}{3}$	$\frac{2}{3}$	$\frac{2c^2}{3}$

$$||T_{13}||_1 = 2\lambda_0\lambda_2 + \sqrt{\max\{A^+, A^-\}}, \tag{11}$$

where

$$A^\pm = [(\lambda_1+\lambda_4)^2 + (\pm\lambda_0 + \lambda_2 + \lambda_3)^2] \times [(\lambda_1-\lambda_4)^2 + (\lambda_0 \pm \lambda_2 \mp \lambda_3)^2]. \tag{12}$$

With the above assumptions the Lagrange function can be recast as

$$\mathcal{L}(\psi) = 4\lambda_0^2(\lambda_2^2+\lambda_4^2) + \alpha[4\lambda_0^2\lambda_4^2 - \tau_0] + \beta\left[\sum_{m=0}^4 \lambda_m^2 - 1\right] + \mu_1[A^+ - (1-2\lambda_0\lambda_2)^2] + \mu_2[A^- - (1-2\lambda_0\lambda_2)^2]. \tag{13}$$

The complementary slackness conditions are given by

$$\frac{\partial\mathcal{L}}{\partial\lambda_i} \leqslant 0, \quad \lambda_i\frac{\partial\mathcal{L}}{\partial\lambda_i} = 0, \quad \lambda_i \geqslant 0, \quad 4\lambda_0^2\lambda_4^2 = 0,$$
$$\sum_{m=0}^4 \lambda_m^2 = 1, \quad A^\pm \leqslant (1-2\lambda_0\lambda_2)^2,$$
$$\mu_{1,2}[A^\pm - (1-2\lambda_0\lambda_2)^2] = 0, \quad \mu_{1,2} \geqslant 0 \tag{14}$$

for $i = \{0, \dots, 4\}$.

By straightforward calculations one can find that the complementary slackness conditions are satisfied when

$$\lambda_0 = \frac{\sqrt{1+4d_0d_1}}{2}, \quad \lambda_1 = \frac{(1-4d_0^2)\sqrt{d_0d_1}}{2(d_0+d_1)\lambda_0},$$
$$\lambda_2 = \frac{(1-4d_0^2)(d_1-d_0)}{4(d_0+d_1)\lambda_0}, \quad \lambda_3 = \frac{(1-4d_0^2)}{4(d_0+d_1)\lambda_0},$$
$$\lambda_4 = \frac{\sqrt{d_0d_1}}{\lambda_0}, \tag{15}$$

where $d_0 = \frac{1}{2}(1 - \sqrt{1 - \tau_0^2})$ and $d_1 = \sqrt{\frac{1-2d_0^2}{2}}$. We note that although the above-described upper bound is derived with the assumption $\phi = \pi$, it represents a global maximum of the analyzed target function; other values of $\phi \neq \pi$ do not yield a greater outcome of the target function.

Further simplifications of the upper bound state based on local unitary operations allow us to rewrite it as

$$|\psi_T\rangle = d_0|000\rangle + \sqrt{\frac{1 - 2d_0^2}{2}}|101\rangle + \frac{1}{2}(|110\rangle + |011\rangle), \tag{16}$$

where $0 \leqslant d_0 \leqslant \frac{1}{2}$. For that state one can find

$$\tau(\psi_T) = 2\sqrt{2d_0^2 - 4d_0^4}, \quad \mathcal{C}(\psi_T) = \frac{1 - \tau(\psi_T)}{2}. \tag{17}$$

Substituting these results into Eq. (6) and using Eq. (7), one has

$$F_{\rm CQT}(\psi_T) = \frac{5 + 2\sqrt{2d_0^2 - 4d_0^4}}{6}, \quad F_{\rm NC}(\psi_T) = \frac{2}{3}, \tag{18}$$

which confirms the usefulness of the $|\psi_T\rangle$ state for CQT protocol.

At this point, it should be noted that the state $|\psi_T\rangle$ represents a special example of the tetrahedral states [32–34]. When $d_0 = \frac{1}{2}$, Eq. (16) is equivalent to the GHZ state, up to the local unitary transformation. On the other hand, for $d_0 = 0$ one has $|\psi_T\rangle_{d_0=0} = \frac{1}{\sqrt{2}}|101\rangle + \frac{1}{2}(|110\rangle + |011\rangle)$ (hereafter denoted as $|W_1\rangle$) which evidently is not the prototype W state [cf. Eq. (A1)]. Interestingly, $|W_1\rangle$, in contrast to the prototype W state, can also be used for perfect ($F = 1$) two-partite teleportation performance on three-qubit channel [35]. In this protocol, Alice has qubits 1 and 3 while Bob gets qubit 2. When Alice wants to transfer an unknown state of qubit 4 to Bob, she makes a joint measurement on the three qubits "413" and broadcasts her result to Bob through classical communication channel. The optimal set of orthogonal states used by Alice is defined in the W-state category (see Ref. [35] for details). Naturally, such an optimal set cannot be determined in the CQT protocol as a combination of Alice (two-qubit) and Charlie (one-qubit) measurements. For that reason, the maximal fidelity of CQT protocol for W class is $F_{\rm CQT} = \frac{5}{6} < 1$. Finally, we note that the question whether the state $|W_1\rangle$ is suitable for CQT was posed in Ref. [35]. Our calculations provide the affirmative answer for this question, but only if the qubits are distributed among all participants in the sequence given in Sec. II. Any permutation of qubits makes the state $|W_1\rangle$ unsuitable for CQT.

Now, combining together Eqs. (9) and (17) the range of attainable values $F_{\rm CQT}$ is given by

$$\frac{2 + \sqrt{\tau}}{3} \leqslant F_{\rm CQT} \leqslant \frac{5 + \tau}{6}. \tag{19}$$

In Fig. 1(a), we illustrate the results of our calculation. As we see, the difference between the upper and lower bounds in terms of τ becomes very small (approximately $\epsilon^2/24$) for large three-tangle $\tau = 1 - \epsilon$. Therefore, one can reverse this relation in order to quantify a tripartite entanglement experimentally. Specifically, by measuring $F_{\rm CQT}$ one can estimate the three-tangle as $6F_{\rm CQT} - 5 \leqslant \tau \leqslant (3F_{\rm CQT} - 2)^2$, where the difference between the upper and lower bounds in terms of $F_{\rm CQT}$ is approximately equal to $9\varepsilon^2$ for large $F_{\rm CQT} = 1 - \varepsilon$. The higher the $F_{\rm CQT}$ value, the better the τ estimate.

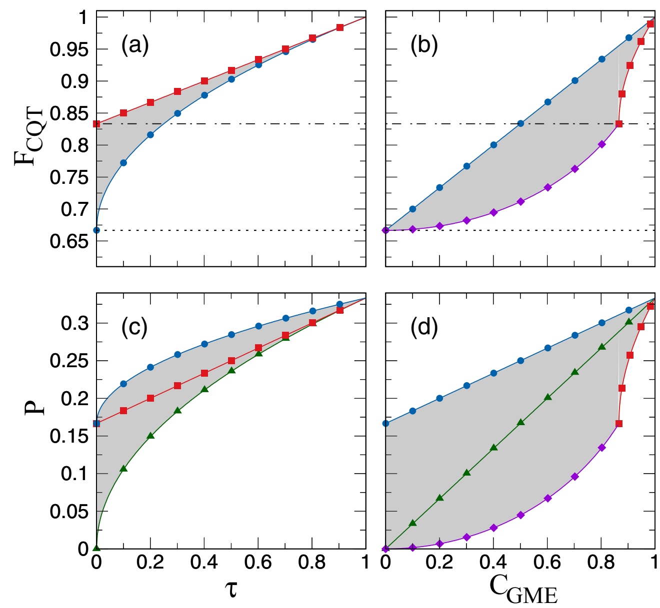

FIG. 1. (a), (b) Range of values of fidelity $F_{\rm CQT}$ for given value of tripartite entanglement measure. (c), (d) Range of values of control power P for given value of tripartite entanglement measure. We consider two kinds of entanglement measure, namely, the three-tangle and genuine concurrence. In all panels, the blue line with circle symbols depicts the MS states $|\psi_{\rm MS}\rangle$, the green line with triangle symbols corresponds to the GHZ state $|\psi_G\rangle$, the red line with square symbols denotes tetrahedral state $|\psi_T\rangle$, and the violet line with diamond symbols represents the W-class states $|\psi_W\rangle$. Gray areas correspond to all admissible values of pure states. We note that in (a) and (b) $F_{\rm CQT}(\psi_{\rm MS}) \equiv F_{\rm CQT}(\psi_G)$ and the dotted line corresponds to classical limit while the dashed-dotted line refers to no-cloning limit [36].

From relation (19) one can also notice that when $\tau > 0$ the fidelity of the upper bound is always greater than $\frac{5}{6}$. Consequently, the controlled quantum teleportation proceeded through $|\psi_T\rangle$ always ensures the nonexistence of any other copy of the output state with better fidelity [36]. On the other hand, the GHZ and MS states beat the no-cloning limit if and only if τ exceeds $\frac{1}{4}$. Since these states correspond to the lower bound of $F_{\rm CQT}$ with given τ, this result implies a more general conclusion, i.e., the three-tangle $\tau \geqslant \frac{1}{4}$ is a sufficient condition to guarantee that CQT performed through an arbitrary pure state is secure.

In the next step, we analyze the controller's authority measured by the *control power* $P(\rho)$ given in Eq. (4). Once again, we use the KKT method with the target function $\mathcal{F} = 2\sqrt{\tau + \mathcal{C}_{AB}^2} - ||T_{13}||_1$. We also keep the assumption $\phi = \pi$ which provides $||T_{13}||_1$ given by Eq. (11). Furthermore, in order to simplify the calculations, we consider two independent situations, namely, $A^+ \geqslant A^-$ and $A^+ < A^-$. The first case is ensured by the inequality $\lambda_2(1 - 2\lambda_3^2) \geqslant 2\lambda_1\lambda_3\lambda_4$ and the analyzed Lagrange function takes a form

$$\mathcal{L}^{+}(\psi) = 4\lambda_0\sqrt{\lambda_2^2+\lambda_4^2} - 2\lambda_0\lambda_2 - \sqrt{A^+} + \alpha\big[4\lambda_0^2\lambda_4^2 - \tau_0\big] + \beta\left[\sum_{m=0}^{4}\lambda_m^2 - 1\right] + \mu_1[A^+ - (1-2\lambda_0\lambda_2)^2] + \mu_2\big[2\lambda_1\lambda_3\lambda_4 - \lambda_2\big(1-2\lambda_3^2\big)\big]. \tag{20}$$

In the same manner, we construct the second Lagrange function $\mathcal{L}^{-}(\psi)$ for the situation when $A^+ < A^-$. By making appropriate calculations, we have found that the control power P is limited by the MS (upper border) and GHZ (lower border) states [see Fig. 1(c)]. Note that for these two extreme states $A^+ = A^-$ and the choice of Lagrange function has no influence on the solutions. All exact results of our calculations are presented in Table I and they imply the following relation linking P and τ:

$$\frac{\sqrt{\tau}}{3} \leqslant P \leqslant \frac{1+\sqrt{\tau}}{6}. \tag{21}$$

We note that Eq. (21) is true even if one replaces control power P by the minimal control power [15]. It is because any permutation of the qubits in the GHZ and MS states either does not change the controller's authority or leads to the meaningless result (i.e., $F_{\rm NC} > \frac{2}{3}$). Moreover, it is worth noting that when τ becomes arbitrary close to 0, the maximum of P approaches $\frac{1}{6}$. This value is provided either by the W-class state $|W_1\rangle$ or biseparable state, for instance, $|\psi_{\rm MS_3}\rangle_{\lambda_4\to 0} = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)$. In other words, despite that both states $|W_1\rangle$ and $|\psi_{\rm MS_3}\rangle_{\lambda_4\to 0}$ exhibit different entanglement properties, for $\tau \approx 0$ the control power P is not sufficient to distinguish these two states. Taking into account the fact that the control power cannot be applied for all W-class pure states (for instance, the prototype W state) and it may take nonzero value for biseparable mixed states, the above discussion provides another argument that control power cannot be considered as a measure of three-party correlation complementary to the three-tangle as conjuncted in Ref. [15].

In the same way, we estimate the extreme states of $F_{\rm CQT}$ and P for given genuine tripartite concurrence $C_{\rm GME}$ [25]. We recall that $C_{\rm GME} = \min\{C_{1(23)}, C_{2(13)}, C_{3(12)}\}$, where $C_{i(jk)}$ stands for a bipartite concurrence of qubit i and joint qubits jk. The exact expressions for $C_{i(jk)}$ are given by $C_{1(23)} = 2\lambda_0\sqrt{\lambda_2^2+\lambda_3^2+\lambda_4^2}$, $C_{2(13)} = 2\sqrt{\lambda_0^2(\lambda_2^2+\lambda_4^2) + |\lambda_1\lambda_4 e^{i\phi} - \lambda_2\lambda_3|^2}$, $C_{3(12)} = 2\sqrt{\lambda_0^2(\lambda_3^2+\lambda_4^2) + |\lambda_1\lambda_4 e^{i\phi} - \lambda_2\lambda_3|^2}$. As a result of our calculations, we have found that the upper bound for $F_{\rm CQT}$ and P for given $C_{\rm GME}$ is realized by the MS states, whereas the lower bound is either the $|\psi_T\rangle$ state (when $C_{\rm GME} \geqslant \frac{\sqrt{3}}{2}$) or the W-class state

$$|\psi_W\rangle = \sqrt{1-2c_0^2}|101\rangle + c_0(|110\rangle + |011\rangle), \tag{22}$$

with $0 \leqslant c \leqslant 1/2$ [see Figs. 1(b) and 1(d)]. The above outcomes lead to the following sharp bounds for $F_{\rm CQT}$ and P as a function of its genuine concurrence $C_{\rm GME}$:

$$\frac{3-\sqrt{1-C_{\rm GME}^2}}{3} \leqslant F_{\rm CQT} \leqslant \frac{2+C_{\rm GME}}{3}, \quad \frac{1-\sqrt{1-C_{\rm GME}^2}}{3} \leqslant P \leqslant \frac{1+C_{\rm GME}}{6} \tag{23}$$

for $C_{\rm GME} < \frac{\sqrt{3}}{2}$ and

$$\frac{5+\sqrt{4C_{\rm GME}^2-3}}{6} \leqslant F_{\rm CQT} \leqslant \frac{2+C_{\rm GME}}{3}, \quad \frac{1+\sqrt{4C_{\rm GME}^2-3}}{6} \leqslant P \leqslant \frac{1+C_{\rm GME}}{6} \tag{24}$$

otherwise. In contrast to previous analysis, here, the difference between the upper bound and the lower bound (for both $F_{\rm CQT}$ and P) is much greater. Moreover, the fidelity $F_{\rm CQT}$ of an arbitrary pure state exceeds the no-cloning limit $\frac{5}{6}$ if the genuine concurrence exceeds $\frac{\sqrt{3}}{2} \approx 0.866$.

Finally, let us consider the CQT protocol from the controller's point of view. Note that for Charlie the optimal solution is not only the one that maximizes the fidelity of teleportation, but also one that simultaneously maximizes his measurable authority. Therefore, combining together previously discussed results we derive the sharp bounds for the fidelity $F_{\rm CQT}$ versus the control power P:

$$\max\left\{\frac{2}{3}, \frac{1}{3} + 2P\right\} \leqslant F_{\rm CQT} \leqslant \frac{2}{3} + P, \tag{25}$$

where the second inequality directly corresponds to the requirement $F_{\rm NC} \leqslant \frac{2}{3}$ and, hence, is always satisfied according to the definition of CQT protocol. The first inequality, however, implies $F_{\rm CQT} \leqslant 2F_{\rm NC} - \frac{1}{3}$ (or, alternatively, $P \leqslant F_{\rm NC} - \frac{1}{3}$). In other words, this relation provides an upper bound of $F_{\rm CQT}$ (or P) based on $F_{\rm NC}$. Moreover, since both fidelities are linear functions of FEF, one can estimate upper and lower bounds of localizable FEF in a function of standard FEF, $\frac{1}{2} < f^L(\rho) \leqslant 2f(\rho_{13})$, where $f(\rho_{13}) \leqslant \frac{1}{2}$ to satisfy the requirement $F_{\rm NC} \leqslant \frac{2}{3}$.

A plot of bounds in Eq. (25) is presented in Fig. 2. As we see, the optimal teleportation channel which maximizes $F_{\rm CQT}$ for given P is the GHZ state (or the $|\psi_T\rangle$ state if $P > \frac{1}{6}$). However, if Charlie wants to maximize his authority, then the MS states are the best choice.

IV. SYMMETRIC MIXED STATES

Let us now verify whether the above-described relations are also satisfied when mixed states are taken as a teleportation resource. Unfortunately, such analysis for a general situation represents an NP-hard problem because there is no exact analytical formula for tripartite entanglement measures. However, one can always simplify this task and limits the investigation to a certain family of symmetric states (see, for instance, [37,38] and references therein).

In this paper, as an example of symmetric mixed family we analyze the three-qubit X-matrix states proposed by Yu and

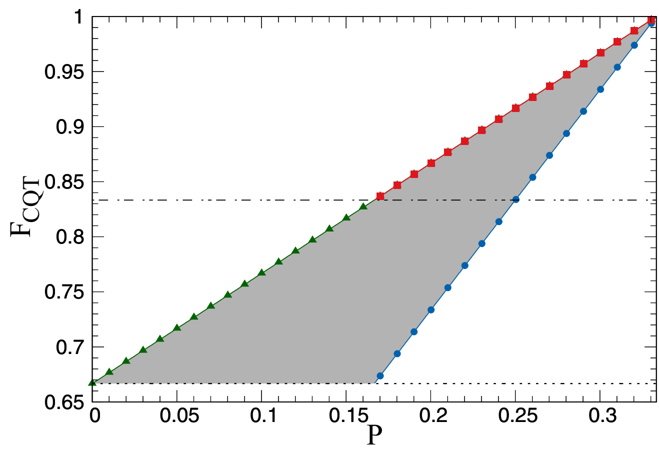

FIG. 2. Range of values of fidelity $F_{\rm CQT}$ for given control power P. Here, we use the same symbols as in Fig. 1, namely, the blue line with circle symbols depicts $|\psi_{\rm MS}\rangle$, the green line with triangle symbols corresponds to $|\psi_G\rangle$, and the red line with square symbols denotes $|\psi_T\rangle$. The dotted line corresponds to classical limit, whereas the dashed-dotted line corresponds to no-cloning limit.

Eberly [39]. These states are represented by a density matrix of three qubits, written in an orthonormal product basis, whose nonzero elements are only diagonal elements (denoted by $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \geqslant 0$) and/or antidiagonal elements (given by $z_1, z_2, z_3, z_4, z_1^*, z_2^*, z_3^*, z_4^*$). The X-matrix states are positive if $|z_j| \leqslant \sqrt{a_j b_j}$ and we also expect $\sum_j (a_j + b_j) = 1$ to ensure the normalization of ρ_X. Moreover, it can be shown that the genuine multipartite concurrence of the X-matrix states is given by $C_{\rm GME}(\rho_X) = 2\max\{0, |z_j| - \omega_j\}$, where $\omega_j = \sum_{k\neq j} \sqrt{a_k b_k}$ and $1 \leqslant k \leqslant 4$ [40].

Recently, both fidelities in Eqs. (2) and (3) have been determined for three-qubit X matrices as [21]

$$F_{\rm NC}(\rho_X) = \frac{3 + |\Delta_1|}{6}, \qquad (26)$$

$$F_{\rm CQT}(\rho_X) = \max\left\{F_{\rm CQT}^{(1)}, F_{\rm CQT}^{(2)}, F_{\rm CQT}^{(3)}, F_{\rm CQT}^{(4)}\right\}, \qquad (27)$$

where $\Delta_1 = a_1 - a_2 + a_3 - a_4 + b_1 - b_2 + b_3 - b_4$ and

$$F_{\rm CQT}^{(1)} = \frac{3 + |\Delta_1| + 4(|z_1| + |z_3|)}{6},$$
$$F_{\rm CQT}^{(2)} = \frac{3 + |\Delta_1| + 4(|z_2| + |z_4|)}{6},$$
$$F_{\rm CQT}^{(3)} = \frac{3 + \sqrt{\Delta_2^2 + 16(|z_1| + |z_3|)^2}}{6},$$
$$F_{\rm CQT}^{(4)} = \frac{3 + \sqrt{\Delta_2^2 + 16(|z_2| + |z_4|)^2}}{6}, \qquad (28)$$

with $\Delta_2 = a_1 - a_2 - a_3 + a_4 - b_1 + b_2 + b_3 - b_4$.

Based on these results and the KKT method we determine the range of attainable values of $F_{\rm CQT}$ and P as a function of genuine tripartite concurrence

$$\max\left\{\frac{3 + 2C_{\rm GME}}{6}, \frac{1 + 2C_{\rm GME}}{3}\right\} \leqslant F_{\rm CQT} \leqslant 1,$$
$$\frac{C_{\rm GME}}{3} \leqslant P \leqslant \frac{1}{3}, \qquad (29)$$

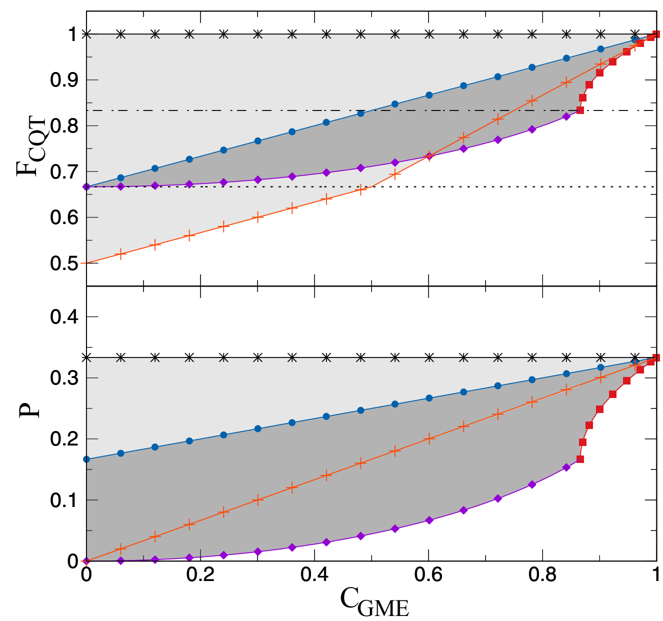

FIG. 3. Range of values of fidelity $F_{\rm CQT}$ and control power P for given value of genuine multipartite concurrence $C_{\rm GME}$. Blue line with circle symbols depicts the MS states $|\psi_{\rm MS}\rangle$, the red line with square symbols denotes tetrahedral state $|\psi_T\rangle$, and the violet line with diamond symbols represents the W-class states $|\psi_W\rangle$. Black line with stars symbol represents the X-matrix state given in Eq. (32) and the X-matrix state given in Eq. (32) is marked by the orange line with plus symbols. Dark gray areas correspond to all admissible values achievable by all pure states while light gray areas denote all attainable values of X-matrix states. The dotted line corresponds to the classical limit, while the dashed-dotted line refers to no-cloning limit.

illustrated also in Fig. 3. As one can see, the area defined by Eq. (29) does not coincide with results presented in the previous section. In particular, pure states do not provide the maximization of the attainable values of both $F_{\rm CQT}$ and P. This outcome contrasts with the standard teleportation protocol, where the upper bound of teleportation faithfulness for given degree of entanglement is achieved for all pure states [19]. In our case, the upper bound of both $F_{\rm CQT}$ and P versus $C_{\rm GME}$ is realized by the X-matrix state being a statistical mixture of two GHZ states

$$\rho_u(p) = p\big|\psi_G^{(1)}\big\rangle\big\langle\psi_G^{(1)}\big| + (1-p)\big|\psi_G^{(4)}\big\rangle\big\langle\psi_G^{(4)}\big|, \qquad (30)$$

where $0 \leqslant p \leqslant 1$, $|\psi_G^{(1)}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and $|\psi_G^{(4)}\rangle = \frac{1}{\sqrt{2}}(|011\rangle + |100\rangle)$ [21]. We note that $\rho_u(p)$ also ensures the upper bound of $F_{\rm CQT}$ in terms of its $\tau \in \langle 0, 1\rangle$ [21]. For that reason, the previous discussion concerning experimental estimation of τ based on $F_{\rm CQT}$ cannot be extended to a general mixed state. It should rather be limited to pure states affected by the presence of a small amount of noise.

The states which minimize the fidelity $F_{\rm CQT}$ and control power P are, up to local unitaries, of the form

$$\rho_d^{(1)}(x) = \frac{C_{\rm GME}}{2}[|001\rangle + |110\rangle][\langle 001| + \langle 110|] + x|000\rangle\langle 000| + (1 - C_{\rm GME} - x)|010\rangle\langle 010| \qquad (31)$$

for $C_{\mathrm{GME}} \geqslant \frac{1}{2}$, and

$$\rho_d^{(2)}(x) = \left(\frac{1}{2} - x\right)[|000\rangle\langle 000| + |001\rangle\langle 001|] + x[|111\rangle\langle 111| + |011\rangle\langle 011|] + \frac{C_{\mathrm{GME}}}{2}[|000\rangle\langle 111| + |111\rangle\langle 000|] \quad (32)$$

otherwise. Here, x is an independent variable in the range $\frac{1}{4}(1 - \sqrt{1 - 4C_{\mathrm{GME}}^2}) \leqslant x \leqslant \frac{1}{4}(1 + \sqrt{1 - 4C_{\mathrm{GME}}^2})$.

Similarly, we derive tight upper and lower bounds on fidelity for a given amount of control power for three-qubit X-matrix states

$$\max\left\{\tfrac{2}{3}, g(P)\right\} \leqslant F_{\mathrm{CQT}} \leqslant \tfrac{2}{3} + P, \quad (33)$$

where

$$g = \begin{cases} \frac{1}{2} + P & \text{for } F_{\mathrm{CQT}} \leqslant \frac{9+2\sqrt{3}}{18}, \\ \frac{1}{6} + 4P - \sqrt{2P(6P-1)} & \text{for } \frac{9+2\sqrt{3}}{18} < F_{\mathrm{CQT}} \leqslant \frac{6+\sqrt{6}}{12}, \\ \frac{1}{3} + 2P & \text{for } F_{\mathrm{CQT}} > \frac{6+\sqrt{6}}{12}. \end{cases} \quad (34)$$

Interestingly, the relation F_{CQT} versus P in Eq. (33) remains unchanged with respect to Eq. (25) when $F_{\mathrm{CQT}} > \frac{6+\sqrt{6}}{12}$. Otherwise, the X-matrix states provide a small enhancement of achievable P for given F_{CQT} (see Fig. 4). This means that for mixed states, inequality $F_{\mathrm{CQT}} \leqslant 2F_{\mathrm{NC}} - \frac{1}{3}$ is no longer satisfied. Consequently, the relation $f_L(\rho) \leqslant 2f(\rho_{13})$ also does not hold if $f_L(\rho) \leqslant \frac{2+\sqrt{6}}{8}$.

Now, it only remains to show the existence of X-matrix states which correspond to the limits given in Eq. (33). By the very definition of X-matrix states, one can conclude that the GHZ state $|\psi_G\rangle$ in Eq. (10) belongs to this family and, hence, $|\psi_G\rangle$ saturates the second inequality in Eq. (33) as shown in Sec. III. The saturation of the first inequality is realized by three states $\rho_d^{(3)} - \rho_d^{(5)}$ which refer to three different shapes of function $g(P)$, respectively. These states are given by

$$\begin{aligned}\rho_d^{(3)}(x) = {} & [a^-|000\rangle + a^+|111\rangle][a^-\langle 000| + a^+\langle 111|] \\ & + [a^+|010\rangle + a^-|101\rangle][a^+\langle 010| + a^-\langle 101|] \\ & + \tfrac{1}{2}|011\rangle\langle 011|, \end{aligned} \quad (35)$$

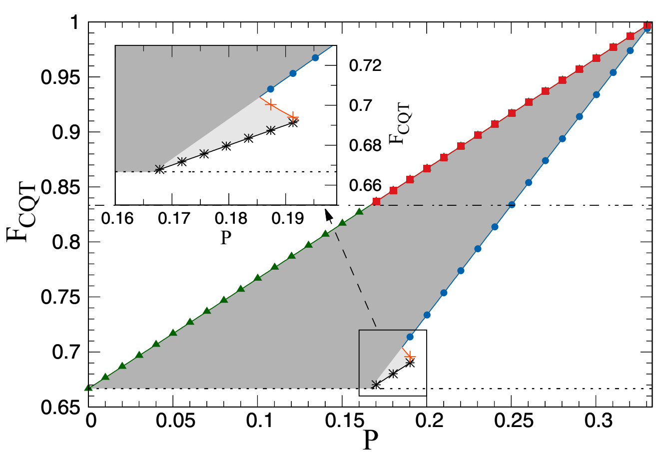

FIG. 4. Range of values of fidelity F_{CQT} for given control power P for three-qubit X-matrix states. Here, we use the same symbols as in Fig. 2. The light gray area stands for an additional range of attainable values of F_{CQT} versus P provided by the three-qubit X-matrix states. The dotted line corresponds to classical limit while the dashed-dotted line refers to no-cloning limit.

where $a^{\pm} = \sqrt{\frac{1}{6}(1 \pm \sqrt{1 - 27P^2})}$ and $\frac{1}{6} \leqslant P \leqslant \frac{1}{3\sqrt{3}}$;

$$\begin{aligned}\rho_d^{(4)}(x) = {} & \frac{3\mathcal{B} - 1}{10(2 - \mathcal{B})}[|000\rangle + \sqrt{\mathcal{B}}|111\rangle][\langle 000| + \sqrt{\mathcal{B}}\langle 111|] \\ & + \frac{3}{10}[\sqrt{\mathcal{B}}|010\rangle + |101\rangle][\sqrt{\mathcal{B}}\langle 010| + \langle 101|] \\ & + \frac{3(\mathcal{B} - 1)}{2(2 - \mathcal{B})}|011\rangle\langle 011|, \end{aligned} \quad (36)$$

where $\mathcal{B} = \frac{5P - 12P^2 - \sqrt{2P(6P-1)}}{2P(2-3P)}$ and $\frac{1}{24}(2 + \sqrt{6}) \leqslant P < \frac{1}{3\sqrt{3}}$; and

$$\begin{aligned}\rho_d^{(5)}(x) = {} & \frac{3P}{4}[|000\rangle + |111\rangle][\langle 000| + \langle 111|] \\ & + \frac{3P}{4}[|010\rangle + |101\rangle][\langle 010| + \langle 101|] \\ & + (1 - 3P)|100\rangle\langle 100|, \end{aligned} \quad (37)$$

where $\frac{1}{24}(2 + \sqrt{6}) \leqslant P \leqslant \frac{1}{3}$.

We note that the state $\rho_d^{(3)}$ yields $|\Delta_1| = 0$, i.e., $F_{\mathrm{NC}} = \frac{1}{2}$, while for $\rho_d^{(4)}$ and $\rho_d^{(5)}$ one has $0 < |\Delta_1| < 1$. For $\rho_d^{(3)}$ the maximal attainable value of controlled fidelity is $F_{\mathrm{CQT}}(\rho_d^{(3)}) = \frac{9+2\sqrt{3}}{18}$ which is greater than $2F_{\mathrm{NC}} - \frac{1}{3} = \frac{2}{3}$ [cf. Eq. (25)] and it is also a maximal violation of $F_{\mathrm{CQT}} \leqslant 2F_{\mathrm{NC}} - \frac{1}{3}$ provided by X-matrix states. A natural question arises as to whether this relation can be violated stronger if one takes an arbitrary mixed state. Due to the large number of parameters which characterize three-qubit mixed states, we are not able to analyze this task analytically. However, one can perform such analysis numerically in order to find (at least) a partial answer for that question. We have found that the inequalities in Eq. (33) hold for arbitrary mixed states, as we have numerically verified using a large ensemble of randomly generated density matrices.

V. CONCLUSIONS

We have investigated the problem of controlled quantum teleportation protocol via nonmaximally entangled pure states. By testing the Karush-Kuhn-Tucker extremality conditions within a generalized Lagrange multiplier method, we have derived a tight upper and lower bound for the fidelity and control power for given values of the three-tangle and genuine concurrence, and we have identified all states for which these bounds are saturated. Interestingly, we have shown that the controlled teleportation cannot be performed through all W-class states. In particular, the standard W state is unsuitable for controlled teleportation, which is in contrast to Ref. [15]. As the control power takes nonzero values also for biseparable mixed states, all these results imply that the (minimal) control power cannot be interpreted as an appropriate measure of a degree of tripartite entanglement as suggested in Ref. [15]. Nonetheless, the conditioned fidelity seems to be useful for experimental quantification of three-tangle, at least for the pure states or in the presence of small levels of noise.

Finding relations between the fidelity (control power) and tripartite entanglement for general three-qubit mixed states is impeded by the lack of an analytical formula for all discussed quantities. However, using the class of symmetric mixed states, namely, the X-matrix states, we have shown that the above-described relations are no longer satisfied. Remarkably, three-qubit pure states are not extreme ones, and perfect controlled teleportation can be reached for any value of tripartite entanglement if one uses, for instance, the statistical mixture of GHZ states.

Finally, we have analyzed the localizable FEF which measures, on average, the distance between maximally entangled state and the reduced bipartite state obtained by local measurements on the rest of the particles. Since the FEF provides an upper bound for the entanglement of distillation, the localizable FEF leads to the upper limit for the entanglement of distillation which can be concentrated in two particular qubits. We have found that for three-qubit states, the localizable FEF can be at most twice as large as the FEF of the reduced bipartite state where the rest of the particles have been traced out if the last one is greater than approximately 0.278.

ACKNOWLEDGMENTS

The authors were supported by GA ČR (Project No. 17-23005Y). A.B. also thanks MŠMT ČR for support by the Project No. CZ.02.1.01/0.0/0.0/16_019/0000754 and J.S. acknowledges financial support from the Postdoctoral program DGAPA-UNAM 2018. Numerical calculations were performed in the Wroclaw Centre for Networking and Supercomputing, Poland.

APPENDIX: PROOF 1

Let $|W\rangle$ be the standard W state written in a form [30]

$$|W\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |101\rangle + |110\rangle). \tag{A1}$$

Then, it is straightforward to notice that the correlation tensor T_{13} in Eq. (8) is a diagonal matrix (cf. [15])

$$T_{13} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

and the trace norm which is given by $||T_{13}||_1 = \mathrm{Tr}\sqrt{T_{13}^{\dagger}T_{13}} = \frac{5}{3}$. Therefore, the nonconditioned fidelity in Eq. (7)

$$F_{\mathrm{NC}}(|W\rangle) = \frac{||T_{13}||_1 + 3}{6} = \frac{7}{9} \geqslant \frac{2}{3}. \tag{A2}$$

Note that the same result can be obtained for T_{12} and T_{23}, which is cased by the permutation symmetry of $W\rangle$. In other words, regardless of who (Alice, Bob, or Charlie) plays a role of controller, not all W-class states are suitable for CQT. This result is in contrast to [15].

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).

[3] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).

[4] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 5932 (1998).

[5] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Rev. Mod. Phys. **83**, 33 (2011).

[6] O. Landry, J. A. W. van Houwelingen, A. Beveratos, H. Zbinden, and N. Gisin, J. Opt. Soc. Am. B **24**, 398 (2007).

[7] R. Valivarthi, M. G. Puigibert, Q. Zhou, G. H. Aguilar, V. B. Verma, F. Marsili, M. D. Shaw, S. W. Nam, D. Oblak, and W. Tittel, Nat. Photonics **10**, 676 (2016).

[8] Q.-C. Sun, Y.-L. Mao, S.-J. Chen, W. Zhang, Y.-F. Jiang, Y.-B. Zhang, W.-J. Zhang, S. Miki, T. Yamashita, H. Terai *et al.*, Nat. Photonics **10**, 671 (2016).

[9] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).

[10] S. Popescu, Phys. Rev. Lett. **72**, 797 (1994).

[11] N. Gisin, Phys. Lett. A **210**, 157 (1996).

[12] A. Karlsson and M. Bourennane, Phys. Rev. A **58**, 4394 (1998).

[13] H. Yonezawa, T. Aoki, and A. Furusawa, Nature (London) **431**, 430 (2004).

[14] X.-H. Li and S. Ghose, Phys. Rev. A **90**, 052305 (2014).

[15] K. Jeong, J. Kim, and S. Lee, Phys. Rev. A **93**, 032328 (2016).

[16] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).

[17] P. Badziąg, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **62**, 012311 (2000).

[18] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).

[19] F. Verstraete and H. Verschelde, Phys. Rev. A **66**, 022307 (2002).

[20] M.-J. Zhao, Z.-G. Li, S.-M. Fei, and Z.-X. Wang, J. Phys. A: Math. Gen. **43**, 275203 (2010).

[21] A. Barasiński, I. I. Arkhipov, and J. Svozilík, Sci. Rep. **8**, 15209 (2018).

[22] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).

[23] M. Walter, B. Doran, D. Gross, and M. Christandl, Science **340**, 1205 (2013).

[24] G. H. Aguilar, S. P. Walborn, P. H. Santo Ribeiro, and L. C. Céleri, Phys. Rev. X **5**, 031042 (2015).

[25] Z.-H. Ma, Z.-H. Chen, J.-L. Chen, C. Spengler, A. Gabriel, and M. Huber, Phys. Rev. A **83**, 062325 (2011).

[26] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[27] S. Lee, J. Joo, and J. Kim, Phys. Rev. A **72**, 024302 (2005).

[28] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. **92**, 027901 (2004).

[29] M. Popp, F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac, Phys. Rev. A **71**, 042306 (2005).

[30] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Phys. Rev. Lett. **85**, 1560 (2000).

[31] R. Horodecki, M. Horodecki, and P. Horodecki, Phys. Lett. A **222**, 21 (1996).

[32] O. Cohen and T. A. Brun, Phys. Rev. Lett. **84**, 5908 (2000).

[33] T. A. Brun and O. Cohen, Phys. Lett. A **281**, 88 (2001).

[34] A. Barasiński, Sci. Rep. **8**, 12305 (2018).

[35] P. Agrawal and A. Pati, Phys. Rev. A **74**, 062320 (2006).

[36] V. Scarani, S. Iblisdir, N. Gisin, and A. Acín, Rev. Mod. Phys. **77**, 1225 (2005).

[37] A. Barasiński and M. Nowotarski, Phys. Rev. A **94**, 062319 (2016).

[38] A. Barasiński and M. Nowotarski, Phys. Rev. A **95**, 042333 (2017).

[39] T. Yu and J. H. Eberly, Quantum Inf. Comput. **7**, 459 (2007).

[40] S. M. Hashemi Rafsanjani, M. Huber, C. J. Broadbent, and J. H. Eberly, Phys. Rev. A **86**, 062303 (2012).