Crossed-laser-beam solutions for the Klein-Gordon equation

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We present analytic solutions of the Klein-Gordon equation for an electron of momentum p in N crossed laser beams $\mathbf{A}_j(\varphi_j)$ with phases $\varphi_j = \omega_j t - \mathbf{k}_j \cdot \mathbf{r} + \delta_j$. The solutions are of the form $\Psi_p = e^{ipx} \mathcal{F}_p(\mathbf{r},t)$. The determination of the distortion factor $\mathcal{F}_p$ is pursued within a method of "auxiliary variables," which uses the φ_j as variables, $\mathcal{F}_p(\mathbf{r},t) \equiv \mathcal{F}_p(\varphi_1, \dots, \varphi_N)$. The equation for $\mathcal{F}_p(\varphi_1, \dots, \varphi_N)$ is a linear second-order partial differential equation and it does not appear to be soluble analytically exactly. However, the second-order derivative terms are multiplied by coeffients that are small with respect to those of the first-order derivatives, if (A) the angles between all beams are small or (B) for all beams we have $\omega_j/mc^2 \ll 1$. In the second case, their neglect would be quite justified for frequencies up into the x-ray range. With this approximation the equation would reduce to a first-order derivatives "reduced equation." Mathematically, however, this cannot be done without further analysis. This is because we are in a situation typical of singular perturbations theory, in which the exact, perturbed solution might not be connected continuously to the unperturbed one. Nevertheless, on the basis of exactly soluble models (see the Appendix), we argue that the approximation is justified in our case and proceed to solve it. It turns out that the reduced equation can be solved exactly for certain crossed-beam geometries of interest. We consider first the case of laser pulses of arbitrary shape. The most general geometry we solve is that in which all beams have coplanar propagation directions and the fields are linearly polarized perpendicular to the propagation plane, except possibly for two that can be oblique and elliptically polarized. For two beams ($N = 2$) this covers the most general case possible. As an application, we calculate the closed-form solution for Gaussian-pulse crossed beams. Next, we treat the case of monochromatic beams as a limit of the laser-pulse case and derive closed-form solutions for some geometries, including standing waves. We then discuss the effect of the passage of crossed beams over an electronic wave packet and show that its momentum distribution is not modified and no pair production is possible (in the reduced equation approximation). We also show that the final wave packet displays the classical ponderomotive Lorentz shift, if low relativistic momenta are involved. The Appendix deals with soluble models of the exact equation for $\mathcal{F}_p$ which have the salient features of the original. For these models, the reduced equation solution is indeed a valid approximation to the exact one, if conditions similar to (A) or (B) above are met.

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I. INTRODUCTION

Relativistic solutions for an electron in a plane-wave radiation field have been found long ago for the Klein-Gordon (KG) equation by Gordon [1] and for the Dirac equation by Volkov [2]. These are single beam solutions (SBS), pertaining to radiation propagating in a given direction. They have proven to be seminal for the research on superintense lasers interacting with electrons and atoms (see [3,4]), and have been used either as asymptotic states in S-matrix calculations (see [3]) or as basis states in Dirac representation theory (see [5,6]).

There has also been long-standing interest in solutions for an electron in crossed beams and approximate results have been found for special configurations. Most of the work has been nonrelativistic, with retardation included. Thus crossed-beam solutions (CBS) have been found for an electron in a superposition of classical monochromatic plane waves by Rosenberg and Zhou [7] (see also [8]) and in a quantized multimode photon field by Guo *et al.* [9,10]. Work has also been done on the Dirac problem. The field was treated as a classical superposition of monochromatic plane waves at weak intensities [7] or as a superposition of quantized radiation modes propagating in the same direction [11]. The CBS work was in part motivated by standing-waves phenomena such as the Kapitza-Dirac effect (see [8]), atomic stabilization (see [12,13]), and attempts to suppress the Lorentz drift [14]. Renewed interest originates in the search for improved geometries for pair production in intense fields (see [15–17]). The need for solutions corresponding to tightly focused, high-intensity laser beams was emphasized in [18]. Besides, confocal laser beams are at the core of large facilities like NIF [19] and next generation facilities like ELI [20].

In this paper we derive CBS for a KG particle in an intense classical laser field, extending the well-known SBS [1]. This should be relevant also for a physical electron under these conditions because spin plays a minor role in intense laser-electron scattering; see Ehlotzky *et al.* [3], Secs. 3 and 5.

II. BASIC EQUATIONS: METHOD OF AUXILIARY VARIABLES

We write the KG equation for a field $A\,(\mathbf{A}, i\,0)$ as

$$\left[\left(P-\frac{e}{c}A\right)^2+m^2c^2\right]\Psi=0. \quad (1)$$

Here and in the following, four-vectors are denoted by $q(\mathbf{q}, iq_0)$ and scalar products by $(q\cdot q')=\mathbf{q}\cdot\mathbf{q}'-q_0q_0'$. We seek solutions describing the distortion by the radiation of an electron plane wave $\exp i(p\cdot x)$, where p is a free-particle momentum four-vector, which can be of the form $p_\pm(\mathbf{p}, iE_\pm/c)$ with energy $E_\pm=\pm\sqrt{m^2c^4+c^2\mathbf{p}^2}$; $p_\pm^2+m^2c^2=0$. The solutions are sought as $\Psi_p(\mathbf{r},t)=e^{ip\cdot x}\,\mathrm{F}_p(\mathbf{r},t)$, where $\mathrm{F}_p(\mathbf{r},t)$ is the distortion factor. Inserting Ψ_p into Eq. (1) leads to the partial differential equation (PDE):

$$-\frac{1}{2}\sum_{\mu=1}^{4}\frac{\partial^2\mathrm{F}_p}{\partial x_\mu^2}-i\sum_{\mu=1}^{4}\left(p_\mu-\frac{e}{c}A_\mu\right)\frac{\partial\mathrm{F}_p}{\partial x_\mu}+\left(-\frac{e}{c}\mathbf{A}\cdot\mathbf{p}+\frac{e^2}{2c^2}\mathbf{A}^2\right)\mathrm{F}_p=0, \quad (2)$$

where we have taken into account the Lorentz condition: $\sum_\mu \partial A_\mu/\partial x_\mu=\operatorname{div}\mathbf{A}=0$.

The physical situation we want to consider is the following. Initially, say at $t\to-\infty$, we have a free electron wave packet (wp), extended over an essentially finite region of space and spreading in time. The approaching radiation, also of finite spatial extension, has not yet overlapped the electron wp, i.e., the fields $\mathbf{A}_k$ vanish over the extension of the wp. This means that the individual momentum components of the wp are not yet distorted and, hence, $\mathrm{F}_p(\mathbf{r},t)\to 1$ at $t\to-\infty$ and finite distances. As time passes, the field overlaps the wp and eventually disappears to infinity, leaving at $t\to+\infty$ a modified free wp behind, whose structure we want to determine.

Equation (2) being of second order, to ensure the unicity of the solution we need an initial condition also for $(\partial\mathrm{F}_p/\partial t)$. We require that the change in F_p be induced only by the incoming radiation field and not by any other causes. We therefore impose as a second condition that $(\partial\mathrm{F}_p/\partial t)$ should vanish at $t\to-\infty$ (when there is no field): $(\partial\mathrm{F}_p/\partial t)\to 0$. Thus we have the two initial conditions for F_p:

$$\mathrm{F}_p(\mathbf{r},t)\to 1,\quad (\partial\mathrm{F}_p/\partial t)\to 0\quad \text{for } t\to-\infty \text{ at finite } \mathbf{r}. \quad (3)$$

This defines an initial value (Cauchy) problem for the determination of F_p, which has a unique solution. The conditions in Eq. (3) ensure that, in the absence of driving fields, Eq. (2) admits only the free particle solution $\mathrm{F}_p(\mathbf{r},t)\equiv 1$.

We shall represent the radiation field as a superposition of N beams propagating in discrete directions:

$$\mathbf{A}(\varphi_1,\varphi_2,\dots,\varphi_N)=\sum_{j=1}^{N}\mathbf{A}_j(\varphi_j), \quad (4)$$

where φ_j are the phases $\varphi_j\equiv-\kappa_j\cdot x+\delta_j=\omega_j t-\mathbf{k}_j\cdot\mathbf{r}+\delta_j$, with the possible dephasings δ_j. We have introduced here the four vectors $x(\mathbf{r}, ict)$ and $\kappa_j(\mathbf{k}_j, i\omega_j/c)$, with $\mathbf{k}_j\equiv\mathbf{n}_j(\omega_j/c)$, $\kappa_j^2=0$, and $\mathbf{k}_j\cdot\mathbf{A}_j=0$. The fields can be linearly polarized, of the type $\mathbf{A}_j(\varphi_j)=\mathbf{A}_{j0}(\varphi_j)\sin\varphi_j$, or elliptically polarized of the type $\mathbf{A}_j(\varphi_j)=\mathbf{A}_j^{(1)}(\varphi_j)+\mathbf{A}_j^{(2)}(\varphi_j+\delta_j)$, where $\mathbf{A}_j^{(1)}(\varphi_j)$ and $\mathbf{A}_j^{(2)}(\varphi_j+\delta_j)$ are linearly polarized and perpendicular to $\mathbf{k}_j$. They can be either laser pulses, in which case the amplitudes $\mathbf{A}_{j0}(\varphi_j)$ have finite spatial and temporal extension [$\mathbf{A}_{j0}(\varphi_j)\to 0$ for $\varphi_j\to\pm\infty$], or monochromatic plane waves, in which case $\mathbf{A}_{j0}$ are constants. We shall consider first the case of laser pulses and derive the monochromatic case as a limit of the former.

The choice for the fields, Eq. (4), is a simplification. It does not take into account the fact that the radiation is limited also in directions perpendicular to the propagation directions $\mathbf{k}_j$. To correct for this, one would need to use a full Fourier expansion for the total $\mathbf{A}$ in plane waves of continuously variable wave vectors $\mathbf{k}$.

For fields like Eq. (4), the coefficients of Eq. (2) depend on $x(\mathbf{r}, ict)$ only via the phases $\{\varphi_j\}$. This suggests trying to use the latter as auxiliary variables for solving the problem. Writing the solutions we seek as $\mathrm{F}_p(\mathbf{r},t)\equiv\mathcal{F}_p(\varphi_1,\varphi_2,\dots,\varphi_N)$, the original Ψ becomes

$$\Psi_p\equiv e^{ip\cdot x}\,\mathcal{F}_p(\varphi_1,\varphi_2,\dots,\varphi_N), \quad (5)$$

and Eq. (2) for the distortion factor:

$$\sum_{1\leqslant k<}\sum_{j\leqslant N}\lambda_{jk}\frac{\partial^2\mathcal{F}_p}{\partial\varphi_j\partial\varphi_k}-i\sum_{j=1}^{N}X_j(\varphi_1,\dots,\varphi_N)\frac{\partial\mathcal{F}_p}{\partial\varphi_j}=f\langle\varphi_1,\dots,\varphi_N\rangle\mathcal{F}_p. \quad (6)$$

Here we have denoted

$$\lambda_{jk}\equiv-(\kappa_j\cdot\kappa_k)=(1-\mathbf{n}_j\cdot\mathbf{n}_k)\frac{\omega_j\omega_k}{c^2}, \quad (7)$$

$$X_j(\varphi_1,\dots,\varphi_N)\equiv-\kappa_j\cdot\left(p-\frac{e}{c}A\right)=a_j+\frac{e}{c}\mathbf{k}_j\cdot\sum_{k=1}^{N}\mathbf{A}_k(\varphi_k), \quad (8)$$

$$a_j\equiv-\kappa_j\cdot p=\frac{\omega_j}{c}\left(\frac{E}{c}-\mathbf{n}_j\cdot\mathbf{p}\right), \quad (9)$$

$$f\langle\varphi_1,\dots,\varphi_N\rangle\equiv\frac{e}{c}\mathbf{A}(\varphi)\cdot\mathbf{p}-\frac{e^2}{2c^2}\mathbf{A}^2(\varphi)=\sum_{j=1}^{N}\left[\frac{e}{c}[\mathbf{A}_j(\varphi_j)\cdot\mathbf{p}]-\frac{e^2}{2c^2}\mathbf{A}_j^2(\varphi_j)\right]-\frac{e^2}{c^2}\sum_{1\leqslant k<}\sum_{j\leqslant N}\mathbf{A}_j(\varphi_j)\cdot\mathbf{A}_k(\varphi_k). \quad (10)$$

Note that the matrix λ_{jk} has positive elements and no diagonal ones, as $\kappa_j^2=0$. The coefficients $-iX_j$ are purely imaginary and are proportional to ω_j/c, as seen from Eqs. (8) and (9).

Equations (6)–(10) obviously apply also to the SBS (Gordon) case for $N=1$. There is a substantial difference, however, between the SBS and CBS cases. The SBS equation does not contain second-order derivatives in Eq. (6) and has only one term in the sum over the j, $X_1(\varphi_1)$. Obviously, this greatly simplifies the integration.

There are two alternatives appearing in the calculation, depending on the charge of the particle: electron ($e < 0$), for which we choose $E \equiv E_+ > 0$ and thus $a_j \equiv a_j^+ > 0$ in Eq. (9), or positron ($e > 0$), for which $E \equiv E_- < 0$ and $a_j \equiv a_j^- < 0$. These correspond to the two types of solutions $\Psi_p^\pm = e^{ip_\pm \cdot x} \mathcal{F}_p^\pm$. The upper and lower indices $\pm$ will be generally ignored.

Equation (6) contains, indeed, only the auxiliary variables $\{\varphi_j\}$ and has no residual dependence on $(\mathbf{r}, t)$. Thus, for fields of the type of Eq. (4), it is possible to find solutions $\mathrm{F}_p(\mathbf{r}, t) \equiv \mathcal{F}_p(\varphi_1, \varphi_2, \dots, \varphi_N)$ depending only on the $\{\varphi_j\}$. However, if we want to solve the problem uniquely in the new variables, we need to formulate some conditions equivalent to Eq. (3). These conditions need to be of the *boundary conditions* (*bc*) type, as the variables $\{\varphi_j\}$ are on equal footing. Note that, in general, the auxiliary variables $\{\varphi_j\}$ are redundant and one cannot establish a one-to-one correspondence between them and the $(\mathbf{r}, t)$.

The initial conditions in Eq. (3) refer to the area of configuration space where t is large and negative (eventually $t \to -\infty$) and $\mathbf{r}$ is finite (the extension of the *wp*). According to the definition of auxiliary variables φ_j, this corresponds in the $\{\varphi_j\}$ space to a domain D_φ where all φ_j are very large and negative, where there are no fields as all $\mathbf{A}_j(\varphi_j) \to 0$. In D_φ, according to Eq. (3), we should have

$$\mathcal{F}_p(\varphi_1, \varphi_2, \dots, \varphi_N) \to 1, \quad \partial \mathcal{F}_p/\partial\varphi_j \to 0 \quad \text{for all } j. \tag{11}$$

Note that $\partial \mathrm{F}_p(\mathbf{r}, t)/\partial t \equiv \sum_{j=1}^N (\partial \mathcal{F}_p/\partial\varphi_j)\omega_j$. Then, in the absence of driving fields, Eq. (6) admits for a free particle only the solution $\mathcal{F}_p = 1$.

With Eq. (11) satisfied, it appears that $\mathcal{F}_p$ is uniquely defined, which would validate the method of auxiliary variables as independent of its configuration space counterpart. The advantage of the method is that, instead of having to solve Eq. (2) with the field $\mathbf{A}$ allowed to propagate in a continuum of directions, we need to solve only the simpler Eq. (6), in which it is restricted to N definite directions. Once the solution $\mathcal{F}_p(\varphi_1, \varphi_2, \dots, \varphi_N)$ has been found, the corresponding solution $\mathcal{F}_p(\mathbf{r}, t)$ is obtained by simply replacing the auxiliary variables by their values $\varphi_j = \omega_j t - \mathbf{k}_j \cdot \mathbf{r} + \delta_j$.

Returning to Eq. (6), this is a second-order PDE with complex coefficients for the complex-valued function $\mathcal{F}_p$ of the real variables φ_j. There is a large textbook literature on such equations for $N = 2$ and real-valued solutions. In this case Eq. (6) is classified as hyperbolic, and Eq. (11) ensures the unicity of its solution, because it allows one to determine unambiguously the two arbitrary functions contained in the general form of the solution; e.g., see [21], Chap. 4.1, and our Appendix. For $N > 2$, less is known, e.g., [21,22].

It does not appear possible to solve the equation exactly. In this respect, we signal a remarkable feature of Eq. (6), which appears as a consequence of introducing the auxiliary variables. This is that the second derivative terms it contains can be, in cases of physical interest, much smaller than the first derivative ones. This is because the λ_{jk} in Eq. (6) are proportional to $(\omega_j\omega_k/c^2)(1 - \mathbf{n}_j \cdot \mathbf{n}_k)$, while the X_j are proportional to (ω_j/c) and $f(\varphi_1; \dots; \varphi_N)$ is independent of the ω_j. Two cases should be considered. (A) If the angles between the beams $\mathbf{n}_j \cdot \mathbf{n}_k$ are small, the λ_{jk} terms could be neglected with respect to the sum over the X_j whatever the frequencies ω_j. (B) If $\omega_j/mc^2 \ll 1$ for *all* j, the λ_{jk} terms could again be neglected, whatever the angles between the beams $\mathbf{n}_j \cdot \mathbf{n}_k$. (This case can be controlled by the variation of a single parameter ω, if the frequencies are kept in a constant ratio as $\omega_j = c_j\omega$. Thereby the λ_{jk} become proportional to ω^2, the X_j to ω, while f is independent of ω.) We would then be left with a *reduced equation*, linear in the first-order derivatives, which can be written as

$$\sum_{j=1}^N X_j(\varphi_1, \dots, \varphi_N) \frac{\partial \Phi_p}{\partial \varphi_j} = i\, f\langle \varphi_1, \dots, \varphi_N\rangle \Phi_p. \tag{12}$$

Here we have denoted by Φ_p the approximate form of $\mathcal{F}_p$ satisfying it ($\mathcal{F}_p \simeq \Phi_p$). Coupled to the fact that Eq. (12) contains the nonrelativistic limit of the problem, this would be a quite valid approximation for ω_j up into the x-ray range. An improved $\mathcal{F}_p$ could then be obtained by perturbation theory [23].

However, there is the mathematical difficulty that we are dealing here with a case of "singular perturbation," as the perturbing terms in Eq. (6) contain higher- (second-) order derivatives than the unperturbed ones (first order). This creates difficulties with satisfying the initial conditions, as, in general, the solution of the unperturbed equation cannot satisfy all initial or boundary conditions required of the higher-order equation, and therefore cannot be an adequate approximation. Implicitly, the solution of the reduced equation cannot be used as a starting point of a regular perturbation scheme.

The approach to such a situation is given by "singular perturbation theory" (SPT). In SPT the exact equation is split into an unperturbed part, containing the first-order derivatives [e.g., Eq. (12)] and a perturbation, containing the second-order derivatives. The unperturbed, "reduced" equation is assumed to be soluble. A parameter ε is introduced to characterize the smallness of the second-order derivatives. [In case (A) above, this would be a common measure for the small angles of the beams, whereas in case (B) it would be the frequency parameter ω introduced above.] A procedure of successive approximations is developed in the parameter ε, to determine step by step the corrections in powers of ε to the unperturbed solution. The procedure is more intricate than regular perturbation theory, as one tries to correct at each step of the approximation for the unsatisfied initial conditions. This is achieved by introducing a system of "boundary layer functions" that are determined stepwise. An SPT expansion $\widetilde{\mathcal{F}}_p$ in ε is thus constructed to approximate the solution $\mathcal{F}_p$, which satisfies the exact equation and the initial condition to a desired order in ε. This formal expansion $\widetilde{\mathcal{F}}_p$ needs to be validated, i.e., to show that it is convergent in some sense and that it tends uniformly in the coordinates to the exact solution as $\varepsilon \downarrow 0$, i.e., $\widetilde{\mathcal{F}}_p \to \mathcal{F}_p$. It depends on the nature of the equation and on the initial or boundary conditions if this is possible. When possible, $\widetilde{\mathcal{F}}_p$ is an asymptotic approximation to the exact solution.

Although SPT for *two-variable* PDE (the case almost exclusively studied) is a well developed area of research, e.g., see [24,25], mathematical aspects we need here are lacking. For the two-variable hyperbolic PDE, which comes close to our interests, Geel [26] (see also [27]) has made important contributions. He has proven the validity of SPT for equations similar to our Eq. (6), but for real-valued functions, see [26], Chaps. V, VI and also [24], Chaps. 9 and 10. However, our $\mathcal{F}_p$ is complex. Besides, whereas case (A) mentioned above fits into the cases studied by Geel, case (B) does not, and extra considerations would be needed. Nevertheless, the extension of his proofs to the complex domain may be possible. This would prove rigorously the validity of our reduced-equation approximation in Eq. (12) for two variables. We shall not attempt to do this here. Instead, we shall give plausibility arguments to justify the fact, by analyzing the exact solutions of two-dimensional PDE models which have the salient features of Eq. (6); see the Appendix. We show there that the solution of the reduced equation (the analog of Φ_p) lies, indeed, at sufficiently small ε, within $O(\varepsilon)$ of the exact solution (the analog of $\mathcal{F}_p$), uniformly in the variables φ_1, φ_2. This supports our claim that the reduced-equation approximation should be an adequate approximation also for Eq. (6). Based on these considerations, we proceed to its solution.

The reduced Eq. (12) is a first-order quasilinear partial derivatives equation for Φ_p of the variables φ_j; e.g., see [28], Chap. II, Sec. III, and Chap. V, Sec. I, and [29], Sec. 3. It is shown that the integration of Eq. (12) is equivalent to that of the system of N differential ("characteristic") equations:

$$\frac{d\varphi_1}{X_1} = \cdots = \frac{d\varphi_N}{X_N} = \frac{d\Phi_p}{i\, f\langle\varphi_1, \varphi_2, \ldots, \varphi_N\rangle\, \Phi_p}. \tag{13}$$

This is a system for the $N+1$ variables $\varphi_1, \ldots, \varphi_N$, Φ_p, written in a symmetric form which allows the treatment of the variables on equal footing. The system has a maximum of N functionally independent "first integrals," i.e., functions $g(\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi_p)$ that reduce to constants along any solution $\{\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi_p\}$ of Eq. (13): $g_j(\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi_p) = C_j$, with $j = 1, 2, \ldots, N$. The solutions of the system Eq. (13), called "characteristics," can be obtained by solving the N equations $g_j(\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi_p) = C_j$ in terms of one of the variables (e.g., φ_1) and the constants C_j. This gives the general solution of the differential system. The "general integral" of the partial derivatives equation, Eq. (12), is obtained by taking an arbitrary function $G(g_1, g_2, \ldots, g_N)$ of the N first integrals $g_1, g_2, \ldots, g_N$ of Eq. (13), assumed functionally independent, and equating it to zero $G[g_1(\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi_p), \ldots, g_N(\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi_p)] = 0$. Solving this for Φ_p gives the most general solution of Eq. (12). The function $G(g_1, g_2, \ldots, g_N)$ convenient to the problem is determined with the help of the first initial condition in Eq. (3).

III. CBS FOR LASER-PULSE BEAMS

In some physically relevant cases it is possible to solve the reduced equation, Eq. (12), exactly. The prime difficulty is finding the first integrals of the associated characteristic system, Eq. (13). These depend critically on the geometrical configuration of the laser pulses $\mathbf{A}_j(\varphi_j)$, i.e., their directions of the propagation and their polarizations, which determine the form of the X_j. Whereas SBS pertain to a single geometry, with crossed beams there are infinite possibilities.

The most general case for which we could find all the first integrals and obtain the exact solution is the following.

(A) *All N beams have their propagation vectors $\mathbf{k}_j$ in the same plane, with the fields linearly polarized perpendicular to it, except possibly for two, which are allowed to be oblique to the plane and elliptically polarized.* Let us denote the oblique fields by $k = 1, 2$, and the perpendicular ones by $k = 3, \ldots, N (N \geqslant 3)$. If the oblique fields $\mathbf{A}_1(\varphi_1)$ and $\mathbf{A}_2(\varphi_2)$ are elliptically polarized, they can be described each as superpositions of two dephased linearly polarized ones, as $\mathbf{A}_1(\varphi_1) = \mathbf{A}_1^{(1)}(\varphi_1) + \mathbf{A}_1^{(2)}(\varphi_1 + \delta_1)$, with both $\mathbf{A}_1^{(1)}$, $\mathbf{A}_1^{(2)}$ perpendicular to their direction of propagation $\mathbf{n}_1$, and similarly $\mathbf{A}_2(\varphi_2) = \mathbf{A}_2^{(1)}(\varphi_2) + \mathbf{A}_2^{(2)}(\varphi_2 + \delta_2)$, with both $\mathbf{A}_2^{(1)}$, $\mathbf{A}_2^{(2)}$ perpendicular to $\mathbf{n}_2$. For $k = 1, 2$, we have, because of transversality, $\mathbf{n}_k \cdot \mathbf{A}_k = 0$. For $k = 3, \ldots, N$ and all j, we have by assumption $\mathbf{n}_j \cdot \mathbf{A}_k = 0$. The X_j of Eq. (8) are then given by

$$X_1(\varphi_2) = a_1 + \frac{e}{c}\mathbf{k}_1 \cdot \mathbf{A}_2(\varphi_2),$$
$$X_2(\varphi_1) = a_2 + \frac{e}{c}\mathbf{k}_2 \cdot \mathbf{A}_1(\varphi_1), \tag{14}$$

$$X_j(\varphi_1, \varphi_2) = a_j + \frac{e}{c}\mathbf{k}_j \cdot [\mathbf{A}_1(\varphi_1) + \mathbf{A}_2(\varphi_2)] \quad (j = 3, \ldots, N). \tag{15}$$

We now proceed to the integration of the characteristic system Eq. (13), i.e., the determination of its first integrals $g_j(\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi_p)$. The equation obtained combining $d\varphi_1$ and $d\varphi_2$ in Eq. (13) gives the first integral:

$$g_1(\varphi_1, \varphi_2) \equiv \int_{q_2}^{\varphi_2} X_1(\xi)d\xi - \int_{q_1}^{\varphi_1} X_2(\xi)d\xi = C_1. \tag{16}$$

Equation (16) expresses the fact that the function $g_1(\varphi_1, \varphi_2)$ reduces to a constant C_1 for any solution $\{\varphi_1, \varphi_2, \ldots, \varphi_N, \Phi\}$ of Eq. (13). Here, q_1, q_2 are arbitrary constants; their choice affects only the value of C_1. The equation $g_1(\varphi_1, \varphi_2) = C_1$ defines φ_2 as an implicit function of φ_1. Let us denote this by u:

$$\varphi_2 = u(\varphi_1, C_1). \tag{17}$$

Next, we consider the equation obtained combining $d\varphi_1$ and $d\varphi_j$ in Eq. (14):

$$X_1(\varphi_2)d\varphi_j - X_j(\varphi_1, \varphi_2)d\varphi_1 = 0 \quad (j = 3, \ldots, N). \tag{18}$$

For the solutions we are interested in, φ_2 is connected to φ_1 via Eq. (17). Inserting φ_2 into Eq. (18) gives

$$X_1(u(\varphi_1, C_1))d\varphi_j - X_j(\varphi_1, u(\varphi_1, C_1))d\varphi_1 = 0 \quad (j = 3, \ldots, N). \tag{19}$$

Integrating for all j with respect to φ_1, we find the $N-2$ first integrals:

$$g_{j-1}(\varphi_1, \varphi_j, C_1) \equiv \varphi_j - s_j(\varphi_1, C_1) = C_{j-1} \quad (j = 3, \ldots, N), \tag{20}$$

where

$$s_j(\varphi_1, C_1) \equiv \int_{q_j}^{\varphi_1} \frac{X_j(\xi, u(\xi, C_1))}{X_1(u(\xi, C_1))}d\xi \quad (j = 3, \ldots, N). \tag{21}$$

Equations (17) and (20) define the variables $\varphi_2, \dots, \varphi_N$ in terms of φ_1 and the constants $C_1, \dots, C_{N-1}$. Introducing the $\varphi_2, \dots, \varphi_N$ into the equation connecting $d\varphi_1$ and $d\Phi$ in Eq. (13)

$$i\, f\,\langle \varphi_1, \varphi_2, \dots, \varphi_N\rangle \frac{d\varphi_1}{X_1(\varphi_2)} = \frac{d\Phi_p}{\Phi_p}, \tag{22}$$

the left-hand side depends only on φ_1 and the constants $C_1, \dots, C_{N-1}$, while the right-hand side only on Φ_p. We find thus for the Nth first integral:

$$g_N(\varphi_1, \Phi_p, C_1, \dots, C_{N-1}) \equiv \Phi_p - \exp i \int_{\varphi_1^0}^{\varphi_1} \frac{1}{X_1(u(\zeta, C_1))} f\langle \zeta, u(\zeta, C_1), \dots, s_j(\zeta, C_1) + C_{j-1}, \dots\rangle d\zeta = C_N, \tag{23}$$

where φ_1^0 is a constant and the variables φ_j $(3 \leqslant j \leqslant N)$ of $f(\varphi_1, \varphi_2, \dots, \varphi_N)$ are replaced by $s_j + C_{j-1}$. We need to have the explicit dependence of the first integrals $g_1, g_2, \dots, g_N$, on the variables $\varphi_1, \varphi_2, \dots, \varphi_N, \Phi_p$. These are expressed by Eqs. (16), (20), and (23), which, however, contain also the integration constants $C_1, C_2, \dots, C_N$. The constants can now be expressed in terms of the combination of variables they represent, i.e., C_1 by Eq. (16) and C_{j-1} (for $3 \leqslant j \leqslant N$) by Eq. (20). For g_N, for example, we get from Eq. (23):

$$\begin{aligned} &g_N(\varphi_1, \varphi_2, \dots, \varphi_N, \Phi_p) \\ &\quad \equiv \Phi_p - \exp i \int_{\varphi_1^0}^{\varphi_1} \frac{1}{X_1(u(\zeta, g_1(\varphi_1, \varphi_2)))} f\langle \zeta, u(\zeta, g_1(\varphi_1, \varphi_2)), \dots, \varphi_j + \mathrm{v}_j(\zeta, \varphi_1, g_1(\varphi_1, \varphi_2)), \dots\rangle d\zeta, \end{aligned} \tag{24}$$

where we have denoted

$$\mathrm{v}_j(\zeta, \varphi_1, g_1) \equiv s_j(\zeta, g_1) - s_j(\varphi_1, g_1) = \int_{\varphi_1}^{\zeta} \frac{X_j(\xi, u(\xi, g_1))}{X_1(u(\xi, g_1))} d\xi, \tag{25}$$

and similarly for the other g_j. They are functionally independent (have nonvanishing Jacobian), as required.

To apply the general theory further, we consider an arbitrary function G of the $g_1, g_2, \dots, g_N$ and solve the equation $G(g_1, g_2, \dots, g_N) = 0$ for the variable Φ_p. As Φ_p is contained solely in g_N, we solve first the equation for g_N, to get $g_N = H(g_1, \dots, g_{N-1})$, where H is another arbitrary function. Inserting here g_N from Eq. (24), we immediately have the expression of Φ_p in terms of the unknown function $H(g_1, \dots, g_{N-1})$. Now, Φ_p also depends on the unspecified lower integration limit φ_1^0 in g_N. In this respect, we note that a modification of φ_1^0 results in a term depending on the first integrals $g_1, \dots, g_{N-1}$, which can be absorbed in $H(g_1, \dots, g_{N-1})$. At this point we take $\varphi_1^0 \to -\infty$ in Eq. (24) as the integral is convergent. With this, if $\varphi_1 \to -\infty$, $g_N \to \Phi_p - 1 = H(g_1, g_2, \dots, g_N)$. By taking $H \equiv 0$, we satisfy the first initial condition Eq. (3), as should be. Thus we have finally for *geometry* (A)

$$\Phi_p(\varphi_1, \varphi_2, \dots, \varphi_N) = \exp i \int_{-\infty}^{\varphi_1} \frac{1}{X_1(u(\zeta, g_1(\varphi_1, \varphi_2)))} f\langle \zeta, u(\zeta, g_1(\varphi_1, \varphi_2)), \dots, \varphi_j + \mathrm{v}_j(\zeta, \varphi_1, g_1(\varphi_1, \varphi_2)), \dots\rangle d\zeta, \tag{26}$$

with v_j defined by Eq. (25). Here, the variables φ_j of $f\,\langle \varphi_1, \varphi_2, \dots, \varphi_N\rangle$ are to be replaced for $j \geqslant 3$ by $\varphi_j + \mathrm{v}_j$. The fact that Eq. (26) satisfies Eq. (12) can be checked by direct calculation.

We note that geometry (A) covers the *most general configuration for two elliptically (or linearly) polarized beams.* The result is obtained from Eq. (26) by considering only the two beams allowed to be oblique to the propagation plane ($k = 1, 2$) and ignoring the rest ($k \geqslant 3$). Thus, with Eq. (10) for f and Eqs. (16), (17) for u,

$$\begin{aligned} \Phi_p(\varphi_1, \varphi_2) = \exp i \int_{-\infty}^{\varphi_1} &\frac{1}{X_1(u(\zeta, g_1(\varphi_1, \varphi_2)))} \left\langle \left[\frac{e}{c}[\mathbf{A}_1(\zeta) \cdot \mathbf{p}] - \frac{e^2}{2c^2}\mathbf{A}_1^2(\zeta)\right] \right. \\ &\left. + \left[\frac{e}{c}\mathbf{A}_2(u(\zeta, g_1(\varphi_1, \varphi_2))) \cdot \mathbf{p} - \frac{e^2}{2c^2}\mathbf{A}_2^2(u(\zeta, g_1(\varphi_1, \varphi_2)))\right] - \frac{e^2}{c^2}\mathbf{A}_1(\zeta) \cdot \mathbf{A}_2(u(\zeta, g_1(\varphi_1, \varphi_2))) \right\rangle d\zeta. \end{aligned} \tag{27}$$

We shall now consider a special case of geometry (A), namely:

(B) *only one field (possibly elliptically polarized) is oblique to the plane of the beams*, say $k = 1$, the rest $k = 2, \dots, N$ being perpendicular. Equations (14) and (15) become

$$X_1 = a_1, \quad X_j(\varphi_1) = a_j + \frac{e}{c}\mathbf{k}_j \cdot \mathbf{A}_1(\varphi_1) \quad (j = 2, \dots, N). \tag{28}$$

To apply Eq. (26) to this case, we need the following ingredients; see Eqs. (16), (17), and (25):

$$g_1(\varphi_1, \varphi_2) \equiv a_1\varphi_2 - \int_{q_1}^{\varphi_1} X_2(\xi) d\xi = C_1, \tag{29}$$

$$u(\varphi_1, C_1) \equiv \frac{1}{a_1}\left[\int_{q_1}^{\varphi_1} X_2(\xi) d\xi + C_1\right], \tag{30}$$

$$u(\zeta, g_1(\varphi_1, \varphi_2)) = \varphi_2 + \frac{1}{a_1}\int_{\varphi_1}^{\zeta} X_2(\xi)d\xi, \tag{31}$$

$$\varphi_j + \mathrm{v}_j(\zeta) = \varphi_j + \frac{1}{a_1}\int_{\varphi_1}^{\zeta} X_j(\xi)d\xi \quad (j = 3, \ldots, N). \tag{32}$$

We have adjusted the notation so as to take into account that $X_j(\xi)$ and $\mathrm{v}_j(\xi)$ in Eqs. (28) and (32) do not depend on φ_2, like in Eqs. (15) and (25). Note that, by taking $j = 2$ in Eq. (32), it becomes equal to the expression in Eq. (31) for u. We can write therefore Eq. (26) as

$$\Phi_p(\varphi_1, \varphi_2, \ldots, \varphi_N) = \exp \frac{i}{a_1}\int_{-\infty}^{\varphi_1} f\left\langle \zeta, \ldots, \varphi_j + \frac{1}{a_1}\int_{\varphi_1}^{\zeta} X_j(\xi)d\xi, \ldots \right\rangle d\zeta, \tag{33}$$

where subscript j now runs from 2 to N. In fact, this is the same as

$$\Phi_p(\varphi_1, \varphi_2, \ldots, \varphi_N) = \exp \frac{i}{a_1}\int_{-\infty}^{\varphi_1} f\left\langle \ldots, \varphi_j + \frac{1}{a_1}\int_{\varphi_1}^{\zeta} X_j(\xi)d\xi, \ldots \right\rangle d\zeta, \tag{34}$$

where j runs from 1 to N, because X_1 is a constant, $X_1 = a_1$. Changing here the integration variable $\zeta = a_1\zeta' + \varphi_1$ gives

$$\Phi_p(\varphi_1, \varphi_2, \ldots, \varphi_N) = \exp i\int_{-\infty}^{0} f\left\langle \ldots, \varphi_j + \frac{1}{a_1}\int_{\varphi_1}^{a_1\zeta'+\varphi_1} X_j(\xi)d\xi, \ldots \right\rangle d\zeta'. \tag{35}$$

Concerning the lower limit of the integration over ζ' in Eq. (35), recall that the lower limit in Eq. (33) resulted from taking $\varphi_1^0 \to -\infty$ in Eq. (26). In terms of φ_1^0, the lower limit in Eq. (35) would have been $(a_j/a_1)(\varphi_1^0 - \varphi_1) + \varphi_j$. Allowing here $\varphi_1^0 \to -\infty$ (at finite φ_1, φ_j), gives $-\infty$.

Equations (34) and (35) give the solution for *geometry* (B). We want to write the solution in an alternative way, introducing the explicit expression of f, Eq. (10). Upon inserting the latter into Eq. (34), we get, for the first sum of f in Eq. (10), the expression

$$\exp i\frac{1}{a_1}\left\{\sum_{j=1}^{N}\int_{-\infty}^{\varphi_1} D_j\left(\varphi_j + \frac{1}{a_1}\int_{\varphi_1}^{\zeta} X_j(\xi)d\xi\right)d\zeta\right\}, \tag{36}$$

where we have denoted

$$D_j(\varphi) \equiv \frac{e}{c}[\mathbf{A}_j(\varphi)\cdot\mathbf{p}] - \frac{e^2}{2c^2}\mathbf{A}_j^2(\varphi). \tag{37}$$

In term j of Eq. (36) we change the integration variable, from ζ to ζ':

$$\zeta' = \varphi_j + \frac{1}{a_1}\int_{\varphi_1}^{\zeta} X_j(\xi)d\xi. \tag{38}$$

Thereby Eq. (36) becomes

$$\exp i\left\{\sum_{j=1}^{N}\int_{-\infty}^{\varphi_j} D_j(\zeta')\frac{d\zeta'}{X_j(\zeta)}\right\}. \tag{39}$$

Here, under the integrals, we have $X_j(\zeta)$ from Eq. (28), and ζ' and ζ are related by Eq. (38).

The double sum in Eq. (10) we write in the form it appears in Eq. (35). Combining with Eq. (39), we obtain the alternative result for geometry (B):

$$\Phi_p(\varphi_1, \varphi_2, \ldots, \varphi_N) = \exp i\left\{\sum_{j=1}^{N}\int_{-\infty}^{\varphi_j}\left[\frac{e}{c}[\mathbf{A}_j(\zeta')\cdot\mathbf{p}] - \frac{e^2}{2c^2}\mathbf{A}_j^2(\zeta')\right]\frac{d\zeta'}{X_j(\zeta)} - \frac{e^2}{c^2}\sum_{1\leqslant k<j\leqslant N}\int_{-\infty}^{0}\mathbf{A}_j\left(\varphi_j + \frac{1}{a_1}\int_{\varphi_1}^{a_1\zeta'+\varphi_1}X_j(\xi)d\xi\right)\cdot\mathbf{A}_k\left(\varphi_k + \frac{1}{a_1}\int_{\varphi_1}^{a_1\zeta'+\varphi_1}X_k(\xi)d\xi\right)d\zeta'\right\}. \tag{40}$$

We shall now specialize the beam geometry still further, to the following.

(C) *All fields are perpendicular to the plane of the beams* (linear polarization only), i.e., $\mathbf{n}_j\cdot\mathbf{A}_k = 0$, for all j and k, and all the X_j in Eq. (28) reduce to constants: $X_j = a_j$. The result for Φ_p can immediately be obtained from Eq. (40). This gives for *geometry* (C)

$$\Phi_p = \exp i\left\{\sum_{j=1}^{N}\frac{1}{a_j}\int_{-\infty}^{\varphi_j}\left[\frac{e}{c}[\mathbf{A}_j(\zeta'')\cdot\mathbf{p}] - \frac{e^2}{2c^2}\mathbf{A}_j^2(\zeta'')\right]d\zeta'' - \frac{e^2}{c^2}\sum_{1\leqslant k<j\leqslant N}\int_{-\infty}^{0}\mathbf{A}_j(a_j\zeta'+\varphi_j)\cdot\mathbf{A}_k(a_k\zeta'+\varphi_k)d\zeta'\right\}. \tag{41}$$

Note that, in geometry (C), $\mathbf{A}_j\cdot\mathbf{A}_k = \pm A_jA_k$.

Thus, beside the additive contribution of the individual beams contained in the first sum of the exponential, the CBS contains also the specific interference terms in the double sum. In the SBS case $N = 1$, Eq. (41) reduces to the Gordon solution [1]. Of course, the results for geometries (B) and (C) can be derived directly, without passing through geometry (A).

Equation (41) also covers the case of parallel beams, propagating in the same or opposite sense, of linear or elliptic polarizations. Here, too, we have $\mathbf{n}_j\cdot\mathbf{A}_k = 0$, for all j and k, although the fields $\mathbf{A}_i$ may not be parallel as in geometry (C).

This includes the case of standing waves, to be discussed in more detail in Sec. V.

Low intensities. The distinction between various geometries is a high intensity one, as at low intensities the X_j reduce to constants, $X_j \simeq a_j$, independent of geometry. The calculations for this case formally coincide with those for geometry (C) and Eq. (41) applies. Thus, at high intensity, Eq. (41) is valid only for geometry (C), but at low intensities it covers all geometries.

IV. CBS FOR GAUSSIAN-PULSE BEAMS IN GEOMETRY (C)

As an example of application of Eq. (41) for geometry (C), let us consider CBS for Gaussian-pulse beams of linear polarization of the form $\mathbf{A}_j(\varphi_j) = \mathbf{A}_{j0} f_j(\varphi_j)\sin\varphi_j$, $f_j(\varphi_j) \equiv \exp(-\alpha_j\varphi_j^2)$, and $\varphi_j = \omega_j t - \mathbf{k}_j\cdot\mathbf{r} + \delta_j$. In this case the result for Φ_p can be obtained in analytical closed form as

$$\Psi_p \simeq e^{ip\cdot x}\Phi_p$$

$$= \exp i\left\{p\cdot x + \sum_{j=1}^{N}\left[\frac{e}{c}(\mathbf{A}_{j0}\cdot\mathbf{p})S_j - \frac{e^2}{2c^2}\mathbf{A}_{j0}^2 T_j\right] - \frac{e^2}{c^2}\sum_{1\leqslant j<}\sum_{k\leqslant N}(\mathbf{A}_{j0}\cdot\mathbf{A}_{k0})U_{jk}\right\}, \tag{42}$$

$$S_j \equiv e^{-s_j}\left[I_j^{(1)}\cos\varphi_j + I_j^{(2)}\sin\varphi_j\right], \tag{43}$$

$$T_j \equiv J_j + \frac{1}{2}e^{-2s_j}\left[I_j^{(3)}\cos 2\varphi_j + I_j^{(4)}\sin 2\varphi_j\right], \tag{44}$$

$$U_{jk} \equiv \frac{1}{2}e^{-s_{jk}}\left[I_{jk}^{(1)}\cos(\varphi_j-\varphi_k) + I_{jk}^{(2)}\sin(\varphi_j-\varphi_k) + I_{jk}^{(3)}\cos(\varphi_j+\varphi_k) + I_{jk}^{(4)}\sin(\varphi_j+\varphi_k)\right]. \tag{45}$$

Let us explain the notation. We first define the following integrals:

$$I_{\binom{s}{c}}\langle q,r,a\rangle \equiv \int_0^\infty \exp(-q\zeta^2 + r\zeta)\begin{pmatrix} i\ \sin a\zeta \\ \cos a\zeta\end{pmatrix}d\zeta, \tag{46}$$

where the subscripts $\binom{s}{c}$ refer to $i\ \sin a\zeta$ or $\cos a\zeta$ on the right-hand side. Equation (46) was calculated as

$$I_{\binom{s}{c}}\langle q,r,a\rangle = \frac{1}{4}\sqrt{\frac{\pi}{q}}\,[e^{z^2}(1-\operatorname{erf} z)\mp \text{c.c.}], \tag{47}$$

see [30], 3.897, where

$$z = -\frac{r+ia}{2\sqrt{q}} \tag{48}$$

and the subscripts $\binom{s}{c}$ of I correspond to the $\mp$ signs on the right-hand side. Here, erf z is the "error function," see [31], Chap. 7, also called "probability integral" $\Phi(z)$, see [30], 8.250.1; erf $z \equiv \Phi(z)$. This is a tabulated function.

The $I_j^{(k)}$ appearing in Eqs. (43) and (44) are defined in terms of Eqs. (46)–(48) as

$$I_j^{(1)} = +iI_s\langle q_j, r_j, a=a_j\rangle,$$
$$I_j^{(2)} = I_c\langle q_j, r_j, a=a_j\rangle \quad (\text{for } S_j);$$
$$I_j^{(3)} = -I_c\langle 2q_j, 2r_j, a=2a_j\rangle,$$
$$I_j^{(4)} = iI_s\langle 2q_j, 2r_j, a=2a_j\rangle \quad (\text{for } T_j), \tag{49}$$

where q_j, r_j stand for

$$q_j = \alpha_j {a_j}^2, \quad r_j = 2\alpha_j a_j\varphi_j. \tag{50}$$

The $I_{jk}^{(k)}$ appearing in Eq. (45) are

$$I_{jk}^{(1)} = I_c\langle q_{jk}, r_{jk}, a=a_j-a_k\rangle,$$
$$I_{jk}^{(2)} = -iI_s\langle q_{jk}, r_{jk}, a=a_j-a_k\rangle,$$
$$I_{jk}^{(3)} = -I_c\langle q_{jk}, r_{jk}, a=a_j+a_k\rangle,$$
$$I_{jk}^{(4)} = iI_s\langle q_{jk}, r_{jk}, a=a_j+a_k\rangle \quad (\text{for } U_{jk}), \tag{51}$$

where q_{jk}, r_{jk} stand for

$$q_{jk} = \alpha_j a_j^2 + \alpha_k a_k^2, \quad r_{jk} = 2(\alpha_j a_j\varphi_j + \alpha_k a_k\varphi_k). \tag{52}$$

The s_j and s_{jk} in Eqs. (43)–(45) are given by

$$s_j = \alpha_j\varphi_j^2, \quad s_{jk} = \alpha_j\varphi_j^2 + \alpha_k\varphi_k^2. \tag{53}$$

J_j in Eq. (44) can be expressed as (use [31], Eqs. 7.7.6. and 7.4.1)

$$J_j \equiv \frac{1}{2a_j}\int_{-\infty}^{\varphi_j} e^{-2\alpha_j\zeta^2}d\zeta = \frac{1}{4a_j}\sqrt{\frac{\pi}{2\alpha_j}}[1+\operatorname{erf}(\sqrt{2\alpha_j}\,\varphi_j)]. \tag{54}$$

Note that J_j originates in the term $\mathbf{A}_j^2(\zeta'')$ of Eq. (41), which contains $\sin^2\zeta'' = (1/2) - (1/2)\cos 2\zeta''$. J_j represents the contribution of the (1/2) term. Although the variable z in Eq. (48) is complex, S_j, T_j, U_{jk} in Eqs. (43)–(45) are real.

We are interested in the behavior of the quantities S_j, T_j, U_{jk} and of Φ_p at $\varphi_j \to \mp\infty$ for the following reasons. At $\varphi_j \to -\infty$ we check the initial conditions, Eq. (3), and at $\varphi_j \to +\infty$, we want to find the final form of Φ_p in configuration space (recall that $\varphi_j \to \mp\infty$ corresponds to $t \to \mp\infty$ at finite distances). We give only the results. At $\varphi_j \to -\infty$ the functions S_j, T_j, U_{jk} all vanish and $\Phi_p \to 1$, as should be. At $\varphi_j \to +\infty$, S_j vanishes and so does U_{jk}, but not T_j. The latter tends to a **p**-dependent constant. With $\Phi_p = e^{i\chi_p}$, we find

$$\Phi_p \to e^{i\chi_p^\infty},$$

$$\chi_p^\infty \equiv -\frac{e^2}{4c^2}\sum_{j=1}^{N}\frac{1}{a_j}\mathbf{A}_{j0}^2\left(\frac{\pi}{2\alpha_j}\right)^{1/2}\times\left[1-\exp\left(-\frac{1}{2\alpha_j}\right)\right] < 0. \tag{55}$$

Thus Φ_p^∞ reduces to a constant phase factor.

V. CBS FOR MONOCHROMATIC PLANE WAVES IN GEOMETRY (C)

We consider next the case of a superposition of monochromatic plane waves of linear polarization, $\mathbf{A}_j(\varphi_j) = \mathbf{A}_{j0}\sin\varphi_j$ with constant amplitudes $\mathbf{A}_{j0}$, in geometry (C). As the direct application of Eq. (41) leads to ambiguities on the integration limits, we shall obtain the monochromatic case as the limit of a realistic laser pulse for which the envelope broadens indefinitely and tends to a constant. It can be shown that the limit is independent of the specific form of the envelope. In the following, we shall take the limit of the Gaussian-pulse CBS, already calculated in Eqs. (42)–(45). To this end, we shall let the shape parameters $\alpha_j \to 0$, while keeping the other variables $\mathbf{p}$, $\mathbf{A}_{j0}$ fixed.

In Eqs. (42)–(45) the parameters α_j are contained in the quantities q, r, s; see Eqs. (50), (52), and (53). Let us first analyze the behavior of the integrals $I_{\binom{s}{c}}$, Eq. (46), as $\alpha_j \to 0$. For the integrals appearing in S_j, T_j, the variable z in Eq. (48) tends to infinity along the negative imaginary axis, $z_S \to -i/(2\sqrt{\alpha_j})$, $z_T \to -i/\sqrt{2\alpha_j}$, respectively. In the case of U_{jk}, we have $z_U \to -ia/2\sqrt{q_{jk}}$, with either $a = a_j + a_k > 0$ or $a = a_j - a_k$; the latter quantity can be positive or negative but, for the moment, let us assume that it is different from zero. Thus z_U tends to large imaginary (positive or negative) values, depending on the sign of a. We need, in all cases, the asymptotic behavior of Eq. (47). This is given by

$$e^{z^2}[1 - \operatorname{erf} z] \sim \frac{1}{\sqrt{\pi}\, z}, \quad \text{for } |z| \to \infty, \ |\arg z| < \frac{3\pi}{4}, \tag{56}$$

see [31], 7.2.2 and 7.12,1. With this, we find that the $I_{\binom{s}{c}}$ contained in S_j behave in the limit as $I_s \sim i/a_j$ and $I_c \sim 0$, those contained in T_j behave as $I_s \sim i/2a_j$ and $I_c \sim 0$, and those contained in U_{jk} as $I_s \sim i/a$ and $I_c \sim 0$. For T_j, we need also the limit of J_j, Eq. (54). The first term of J_j is a constant c_j, independent of φ_j. For the second term, we need erf z for small values of z, erf $z \simeq 2z/\sqrt{\pi}$; see [31], 7.6.1; this yields $\varphi_j/2a_j$. Thus, in the $\alpha_j \to 0$ limit,

$$S_j \simeq -\frac{1}{a_j}\cos\varphi_j, \quad T_j \simeq c_j + \frac{\varphi_j}{2a_j} - \frac{1}{4a_j}\sin 2\varphi_j,$$
$$U_{jk} \simeq \frac{1}{2}\left[\frac{\sin(\varphi_j - \varphi_k)}{a_j - a_k} - \frac{\sin(\varphi_j + \varphi_k)}{a_j + a_k}\right]. \tag{57}$$

We now return to the expression of $\Psi_p \simeq e^{ip\cdot x}\Phi_p$ in Eq. (42). The terms $\varphi_j/2a_j = (-\kappa_j \cdot x + \delta_j)/2a_j$ of T_j we combine with the $p \cdot x$ at the exponent of $e^{ip\cdot x}$ to form the term $\widetilde{p} \cdot x$, where $\widetilde{p}$ is the four-vector:

$$\widetilde{p}_\mu \equiv p_\mu - \frac{e^2}{4c^2}\sum_{j=1}^{N}\mathbf{A}_{j0}^2\frac{\kappa_{j\mu}}{(\kappa_j \cdot p)}. \tag{58}$$

We omit further the terms with the constants c_j and δ_j as they contribute irrelevant constant phases to Ψ_p. By inserting Eq. (57) in Eq. (42), we find for geometry (C) the result

$$\Psi_p \simeq \exp i\left\{\widetilde{p}\cdot x + \sum_{j=1}^{N}\frac{1}{a_j}\left[-\frac{e}{c}(\mathbf{A}_{j0}\cdot\mathbf{p})\cos\varphi_j + \frac{e^2}{8c^2}\mathbf{A}_{j0}^2\sin 2\varphi_j\right]\right.$$
$$\left. - \frac{e^2}{2c^2}\sum_{1\leqslant k<}\sum_{j\leqslant N}(\mathbf{A}_{j0}\cdot\mathbf{A}_{k0})\left[\frac{\sin(\varphi_j - \varphi_k)}{a_j - a_k} - \frac{\sin(\varphi_j + \varphi_k)}{a_j + a_k}\right]\right\}. \tag{59}$$

A somewhat similar result was obtained by Rosenberg and Zhou in [7], Appendix B, using an approximation in which they treated the mixed-mode term on the second line of our Eq. (59) perturbatively [see their Eqs. (B6)–(B9)]. Being a low-field approximation, their result does not contain any reference to the geometry of the beams (see end of our Sec. III).

Note that, from Eq. (58), we have

$$\widetilde{p}^2 + \left(m^2c^2 + \frac{e^2}{2c^2}\sum_{j=1}^{N}\mathbf{A}_{j0}^2\right)$$
$$= \frac{1}{8}\left(\frac{e^2}{c^2}\right)^2\sum_{1\leqslant k<}\sum_{j\leqslant N}\mathbf{A}_{j0}^2\mathbf{A}_{k0}^2 f_{jk}, \tag{60}$$

with $f_{jk} = -\lambda_{jk}/a_j a_k$; see Eqs. (7) and (9). For both $E_\pm$ cases, $f_{jk} < 0$. It follows that now it is not possible to define an exact "effective mass" $\widetilde{m}$ (such that $\widetilde{p}^2 + \widetilde{m}^2c^2 = 0$, with $\widetilde{m}$ depending only on the field amplitudes), because the right-hand side of Eq. (60) depends on p via f_{jk}. For $p \lesssim mc$, $f_{jk} \simeq (1/m^2c^2)$ and these terms can become large at high intensities. For the relevance of the effective mass, see [32].

Equation (59) is well defined only if $a_j - a_k \neq 0$. For configurations $\mathbf{p}$, $\mathbf{A}_j(\varphi_j)$ for which $a_j = a_k$, Eq. (59) contains a singularity at the exponent. This could be a matter of concern, as the role of the CBS, Eq. (59), is to form electron wave packets by integration over $\mathbf{p}$ [see Eq. (63) below], just as in the case of SBS. However, a closer analysis reveals that there are no difficulties, as the singularities along the hypersurfaces $a_j = a_k$ are integrable in $\mathbf{p}$ space.

Let us now specialize Eq. (59) to the case of an electron in standing waves of linear polarization. Taking $\mathbf{k} \equiv \mathbf{k}_1 = -\mathbf{k}_2$, $\omega \equiv \omega_1 = \omega_2$, and $\mathbf{A}_0 \equiv \mathbf{A}_{10} = -\mathbf{A}_{20}$, the phases become $\varphi_1 = (\omega t - \mathbf{k}\cdot\mathbf{r})$, $\varphi_2 = (\omega t + \mathbf{k}\cdot\mathbf{r})$ and the total field is $\mathbf{A} = -2\mathbf{A}_0\cos\omega t\ \sin\mathbf{k}\cdot\mathbf{r}$. We find

$$\Psi_p \simeq \exp i\left\{\widetilde{p}\cdot x - 2\frac{e}{c}\frac{(\mathbf{A}_0\cdot\mathbf{p})}{(\omega E/c^2)^2 - (\mathbf{k}\cdot\mathbf{p})^2}[(\mathbf{k}\cdot\mathbf{p})\cos\omega t\ \cos\mathbf{k}\cdot\mathbf{r} + (\omega E/c^2)\sin\omega t\ \sin\mathbf{k}\cdot\mathbf{r}]\right.$$
$$\left. + \frac{e^2}{4c^2}\frac{\mathbf{A}_0^2}{(\omega E/c^2)^2 - (\mathbf{k}\cdot\mathbf{p})^2}[(\omega E/c^2)\sin 2\omega t\ \cos 2\mathbf{k}\cdot\mathbf{r} - (\mathbf{k}\cdot\mathbf{p})\cos 2\omega t\ \sin 2\mathbf{k}\cdot\mathbf{r}] + \frac{e^2}{4c^2}\mathbf{A}_0^2\left[\frac{\sin 2\mathbf{k}\cdot\mathbf{r}}{(\mathbf{k}\cdot\mathbf{p})} - \frac{\sin 2\omega t}{(\omega E/c^2)}\right]\right\}, \tag{61}$$

and the expression for $\widetilde{p}$ resulting from Eq. (58). The singularity at $\mathbf{k}\cdot\mathbf{p}=0$ derives from that at $a_1-a_2=0$ in Eq. (59) and the previous remark applies.

VI. WAVE PACKETS OF CBS

An important issue is the effect of the passage of crossed beams over a free electronic (positive energy) wave packet (wp) [33]. We write the wp at some time t_i, before the arrival of the radiation, as

$$\Psi(\mathbf{r},t_i)=\int c(\mathbf{p})e^{ip_+\cdot x_i}d\mathbf{p}\equiv\int c(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{r}-E_+t_i)}d\mathbf{p}. \quad (62)$$

Initially, the wp undergoes the usual quantum-mechanical spreading due to the term $(-iE_+t)$ at the exponent. When the crossed beams cover the area where the wp is located, the plane waves $e^{ip_+\cdot x_i}$ become, in the reduced-equation approximation, $\Psi_p^{(+)}(\mathbf{r},t_i)\simeq e^{ip_+x_i}\Phi_p(\varphi_1,\ldots,\varphi_N)$ and the expression of the wp is

$$\Psi(\mathbf{r},t)=\int c(\mathbf{p})\Psi_p^{(+)}(\mathbf{r},t)d\mathbf{p}\equiv\int c(\mathbf{p})e^{i[\mathbf{p}\cdot\mathbf{r}-E_+t+\chi_p(\varphi)]}d\mathbf{p}, \quad (63)$$

where the coefficients $c(\mathbf{p})$ are the same as in Eq. (62) and we have introduced the phase χ_p of $\Phi_p(\varphi)\equiv e^{i\chi_p(\varphi)}$. At the end of the pulse t_f, when the fields have vanished and the electron is again free, the final form of the wp is given by Eq. (63), with $\chi_p(\varphi)$ replaced by χ_p^∞, the limit of $\chi_p(\varphi)$ when all $\varphi_j\to+\infty$ (this is their limit for $t_f\to\infty$ and the finite distances imposed by the wp). We have calculated this limit for a Gaussian pulse in geometry (C) in Eq. (55), and have seen that the original expansion coefficients $c(\mathbf{p})$ got multiplied by a $\mathbf{p}$-dependent phase factor $e^{i\chi_p^\infty}$. Let us now show that this result is general, valid for any beam geometry. Referring to Eq. (63), the exponential in $\Psi(\mathbf{r},t_f)$ acquires an extra phase compared to that in the initial $\Psi(\mathbf{r},t_i)$, Eq. (62), equal to

$$\Theta_p=\chi_p^\infty-E_+(t_f-t_i). \quad (64)$$

[Note that $\chi_p(t_f)\equiv\chi_p^\infty$, as the radiation vanishes for $t>t_f$.] The phase $\chi_p(\varphi)$ stays real from the beginning to the end of the pulse and for any beam geometry. Indeed, by inserting $\Phi_p(\varphi)\equiv e^{i\chi_p(\varphi)}$ into Eq. (12), the equation for $\chi_p(\varphi)$ has real coefficients, and with the initial condition $\chi_p=0$ at $\varphi_j\to-\infty$ [corresponding to Eq. (3)], the function $\chi_p(\varphi)$ stays real while propagating. Thus the final form of the wp Eq. (63) contains the same $e^{ip_+\cdot x_i}$ as initially, but with the different coefficients, $c(\mathbf{p})e^{i\Theta_p}$. This means that we are still dealing with an electronic wave packet and that crossed laser beams cannot induce transitions to negative energy states (create pairs), no matter how intense the field is (just as in the single beam case). Moreover, as $|c(\mathbf{p})e^{i\Theta_p}|^2=|c(\mathbf{p})|^2$, the momentum distribution remains unchanged. Whereas these conclusions are exact in the SBS case, as they are based on the exact SBS form of the equation for the distortion factor F_p, Eq. (6), in the CBS case they are based on the reduced-equation approximation, Eq. (12).

Next, let us look more closely at the phase at the end of the pulse Θ_p, Eq. (64). To fix the ideas we shall consider the case of *geometry (C)* for which the phase χ_p^∞ can be obtained, at any intensity, by taking the limit $\varphi_j,\varphi_k\to\infty$ in the exact Eq. (41) with $a\equiv a_j^+>0$. As noted, at lesser intensities, the formula is valid for an arbitrary geometry. Let us first show that the last integrals of Eq. (41) vanish in the limit. Indeed, for the type of pulses we have chosen, $\mathbf{A}_j(\varphi_j)$ has a maximum at $\varphi_j=0$ and tends to zero for $\varphi_j\to\pm\infty$. The maxima of $\mathbf{A}_j(a_j^+\zeta'+\varphi_j)$ and $\mathbf{A}_k(a_k^+\zeta'+\varphi_k)$ will occur at $\zeta_j'=-(\varphi_j/a_j^+)$ and $\zeta_k'=-(\varphi_k/a_{k,}^+)$, respectively. For large t and finite $\mathbf{r}$, this means $\zeta_j'\simeq-(\omega_j/a_j^+)t,\ \zeta_k'\simeq-(\omega_k/a_k^+)t$. When $t\to\infty$, the separation of two points ζ_j' and ζ_k' grows indefinitely. As the shape of the pulses stays the same, their overlap tends to zero and so does the respective integral. (This result is explicitly displayed in the special case of Gaussian pulses by the terms U_{jk}; see end of Sec. IV.)

We shall now also assume that we are dealing with a wp of low relativistic momenta. Expanding the a_j^+ to first order in p/mc gives

$$a_j^+\simeq\frac{\omega_j}{c}mc\left(1-\frac{1}{mc}\mathbf{n}_j\cdot\mathbf{p}\right). \quad (65)$$

Inserting this into the first line of Eq. (41), we can write $\chi_p^\infty\simeq\chi_0^\infty+\chi_{p1}^\infty$, where χ_0^∞ and χ_{p1}^∞ are the contributions of the p-independent and first-order p terms, respectively. The χ_{p1}^∞ term can be written as

$$\chi_{p1}^\infty=-\mathbf{p}\cdot\mathbf{\Delta},\quad \mathbf{\Delta}=\sum_{j=1}^N\mathbf{\Delta}_j, \quad (66)$$

$$\mathbf{\Delta}_j\equiv-\frac{c}{\omega_j}\frac{e}{mc^2}\int_{-\infty}^{\infty}\mathbf{A}_j(\zeta)d\zeta+\mathbf{n}_j\frac{c}{\omega_j}\frac{e^2}{2m^2c^4}\int_{-\infty}^{\infty}\mathbf{A}_j^2(\zeta)d\zeta. \quad (67)$$

Concerning the first term in Eq. (67), see [34]. When inserting further χ_p^∞ into Eq. (63), we get for the wp at the end of the pulse $t_{f,}$ at low relativistic momenta:

$$\Psi(\mathbf{r},t_f)\simeq e^{i\chi_0^\infty}\int c(\mathbf{p})\,e^{i[\mathbf{p}(\mathbf{r}-\mathbf{\Delta})-E_+t_f]}d\mathbf{p}\equiv e^{i\chi_0^\infty}\widetilde{\Psi}(\mathbf{r}-\mathbf{\Delta},t_f). \quad (68)$$

Here, $e^{i\chi_0^\infty}$ is a physically irrelevant constant phase factor which can be omitted and $\widetilde{\Psi}(\mathbf{r},t)$ represents the wave function of the initial wave packet Eq. (62) at time t, had it been spreading in the absence of the field. With the field present, a spatial shift $\mathbf{\Delta}$ is superposed on the spreading. The value of $\mathbf{\Delta}_j$ in Eq. (67) agrees with the classical displacement of a charge e under the effect of a single beam; e.g., see [35], Eq. (27), with $\mathbf{r}_0=0,\ \boldsymbol{\beta}_0=0,\ \gamma_0=1,\varphi=\eta$, and note that the $(-e)$ it contains stands for e in our notation; see also [6], Sec. II. The second term of Eq. (67) represents the Lorentz displacement of a classical electron under the effect of the beam. As opposed to SBS, for CBS $\mathbf{\Delta}$ is the resultant of N such displacements.

VII. CONCLUSION

Analytic solutions of the Klein-Gordon equation have been found for crossed laser beams acting on an electron plane wave, in the form $\Psi_p=e^{ip\cdot x}\,\mathrm{F}_p(\mathbf{r},t)$. They generalize the

Gordon-Volkov single beam solutions. The field was considered to be a superposition of N beams $\mathbf{A}_j(\varphi_j)$ of given phases $\varphi_j = \omega_j t - \mathbf{k}_j\mathbf{r}+\delta_j$. For this type of field, we have used an alternative method of solution to that in configuration space, based on the "auxiliary variables" $\{\varphi_1, \ldots, \varphi_N\}$. By setting $\mathrm{F}_p(\mathbf{r}, t)\equiv F(\varphi_1, \ldots, \varphi_N)$, the function $F(\varphi_1, \ldots, \varphi_N)$ should satisfy Eq. (6) and is shown to be equivalent to the solution in configuration space if the boundary conditions Eq. (11) are satisfied.

Finding exact solutions for Eq. (6) does not appear to be possible. However, Eq. (6) displays the remarkable fact that its second-order derivatives are multiplied by coefficients λ_{ij} that are small with respect to the coefficients of the first-order derivatives X_j, in two cases: (A) the angles between the beams are small; (B) if for all ω_j we have $\omega_j/mc^2 \ll 1$. This suggests treating the second-order derivatives in Eq. (6) as a perturbation of the part of the equation containing the first-order derivatives, which we designate as the "reduced equation," Eq. (12). From the physical point of view, case (B) is a valid approximation, as the ω_j/mc^2 are quite small quantities up into the x-ray range and at this time there are no intense lasers in sight to violate the approximation.

However, there is the mathematical difficulty that whenever the perturbing terms contain higher-order derivatives than the unperturbed ones this gives rise to difficulties which can invalidate regular perturbation theory. In such cases a special form of perturbation theory needs to be applied — "singular perturbation theory" (SPT). This endeavors one to construct by successive approximations a solution which is indeed infinitesimally close to the exact solution when the perturbation parameters λ_{ij} tend to zero. We have not attempted to do this and thereby rigorously justify the validity of the "reduced equation approximation," Eq. (12). Instead, we have shown on typical soluble models of Eq. (6) that the assumption is justified (see the Appendix) and have passed to the solution of the reduced equation. We note, however, that the solution of the exact equation will probably manifest new physical aspects at high frequencies $\omega_j \gtrsim mc^2$, requiring further analysis.

The reduced equation can be solved exactly for certain geometries (propagation directions of the beams; polarization of the fields) by applying known mathematical methods for first-order PDE. We have started with the case of pulsed beams. The most general geometry solved is that in which the propagation vectors of the beams are coplanar and the fields are linearly polarized perpendicular to the propagation plane, with the possible exception of two, which can be oblique to it and can be elliptically polarized. This covers all configurations discussed in connection with physical applications. In particular, it covers the most general configuration for two crossed beams. An explicit solution was given for Gaussian-shape pulses in one of the geometries considered. We have considered further the case of crossed monochromatic plane waves, treating it as a limiting case of the laser-pulse case. Explicit solutions were given for some geometries, including standing waves.

Finally, we have discussed the effect of the passage of crossed beams over a free-electron wave packet. We have found that the beams cannot change the momentum distribution of the wave packet or induce transitions to negative energy states (create pairs). However, this conclusion may need to be revised upon consideration of the solutions of the exact Eq. (6). We have also indicated how the classical free particle Lorentz displacement shows up in the evolution of a quantum-mechanical wave packet of CBS.

The Appendix contains a soluble model of the exact Eq. (6). It illustrates the connection between the solutions of the exact and the reduced equations, and justifies the reduced-equation approximation.

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APPENDIX: MODEL PDE FOR ASSESSING THE VALIDITY OF THE REDUCED-EQUATION APPROXIMATION

In view of assessing the validity of the "reduced-equation approximation" (12), with respect to the exact equation (6), we have developed exactly soluble two-dimensional models of hyperbolic PDE. Let us first explain how we have chosen them.

Consider the physical case of geometry (C), Sec. III, for two rays. The exact equation for $\mathcal{F}_p$, Eq. (6), has the form

$$\lambda \frac{\partial^2 F}{\partial x \partial y} - ia\frac{\partial F}{\partial x} - i\, b\frac{\partial \mathcal{F}}{\partial y} = f(x, y)\mathcal{F}, \qquad \text{(A1)}$$

where x, y stand for the auxiliary variables φ_1, φ_2 and $\lambda > 0,\ a > 0,\ b > 0$ are constants and $f(x, y)$ is a real function. In order to find a soluble model, it is preferable to work with the equation for the *phase* ζ of $\mathcal{F} \equiv e^{i\zeta}$:

$$\lambda\left(\frac{\partial^2 \zeta}{\partial x \partial y} + i\frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\right) - i\, a\frac{\partial \zeta}{\partial x} - i\, b\frac{\partial \zeta}{\partial y} = -i\, f(x, y). \qquad \text{(A2)}$$

This includes the term $(\partial\zeta/\partial\xi)(\partial\zeta/\partial\eta)$ which, although nonlinear, contains only first-order derivatives that can be handled by regular perturbation theory. As it is not interesting for our purposes, it shall not be considered. Note that, as opposed to Eq. (A1), the driving term $f(x, y)$ appears in Eq. (A2) as an inhomogeneity, which eases the solution. Although we have considered the general case $a \neq 0,\ b \neq 0$, we shall present in the following only the case $a = 0$. The general case is more intricate and leads to the same conclusions.

The model we shall present is thus

$$\lambda \frac{\partial^2 \zeta_{II}}{\partial x \partial y} - i\, b\frac{\partial \zeta_{II}}{\partial y} = -i\, f(x, y). \qquad \text{(A3)}$$

The subscript II refers to the fact that $\zeta_{II}(\lambda; x, y)$ is a solution of a second-order PDE. Further, let us take the domain of definition of the solution to be Δ_L $(-L \leqslant x < \infty,\ -L \leqslant y < \infty)$, where eventually the limit $L \to \infty$ will be taken. We designate a subdomain in the vicinity of the border along $(x = -L,\ y = \text{variable})$ and $(x = \text{variable},\ y = -L)$ by $\mathcal{D}_L$. The inhomogeneity $f(x, y)$ is assumed to vanish in $\mathcal{D}_L$, together with its derivatives, as in the physical case; hence $f(x, -L) = f(-L, y) = 0$.

Let us now consider the properties we should assign to the quantities λ and b in order to emulate the physical situation;

see Sec. II, cases (A) and (B). In case (A) λ is variable and can be small, whereas b is a constant. In case (B), both λ and b depend on a frequency parameter ω, as $\lambda = c\,\omega^2$ and $b = c'\omega$, where c, c' are constants.

Under the change of unknown, $\zeta_{II}(\lambda\,; x, y) = e^{i\,(b/\lambda)\,x}\,u(\lambda\,; x, y)$, Eq. (A1) becomes

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{i}{\lambda} e^{-i\frac{b}{\lambda}x} f(x, y). \tag{A4}$$

The most general solution of Eq. (A4) is easy to find:

$$u(\lambda; x, y) = \phi(x) + \psi(y) - \frac{i}{\lambda}\int_{-L}^{y} d\eta \int_{-L}^{x} d\xi\, e^{-i\frac{b}{\lambda}\xi} f(\xi, \eta), \tag{A5}$$

and is expressed in terms of the two arbitrary functions $\phi(x)$, $\psi(y)$; see also [21], Chap. 4, Sec 1.

The corresponding solution $\zeta_{II}(\lambda; x, y)$ is then

$$\zeta_{II}(\lambda; x, y) = e^{i(b/\lambda)x}[\phi(x) + \psi(y)] - \frac{i}{\lambda} e^{i\frac{b}{\lambda}x}\int_{-L}^{y} d\eta \int_{-L}^{x} d\xi\, e^{-i\frac{b}{\lambda}\xi} f(\xi, \eta). \tag{A6}$$

The solution takes at the boundaries of Δ_L the values

$$\zeta_{II}(\varepsilon; x, -L) = e^{i(b/\varepsilon\lambda)x}[\phi(x) + \psi(-L)],$$
$$\zeta_{II}(\varepsilon; -L, y) = e^{-i(b/\varepsilon\lambda)L}[\phi(-L) + \psi(y)]. \tag{A7}$$

As we want our model to simulate the physical situation, we have to choose the functions $\phi(x), \psi(y)$ appropriately. With $\digamma \equiv e^{i\zeta}$, the analog of the bc in Eq. (11) is that $\zeta_{II}(\lambda; x, y)$ should vanish with its derivatives in $\mathcal{D}_L$. This requires that both $\phi(x), \psi(y)$ be identically zero: $\phi(x) \equiv 0$, $\psi(y) \equiv 0$. Thus

$$\zeta_{II}(\varepsilon; x, -L) \equiv 0, \quad \zeta_{II}(\varepsilon; -L, y) \equiv 0. \tag{A8}$$

Moreover, since we have required that $f(x, y)$ and its derivatives vanish sufficiently rapidly in $\mathcal{D}_L$, we can allow $L \to \infty$ and write the solution of interest in whole space as

$$\zeta_{II}(\lambda; x, y) = -\frac{i}{\lambda} e^{i\frac{b}{\lambda}x}\int_{-\infty}^{y} d\eta \int_{-\infty}^{x} d\xi\, e^{-i\frac{b}{\lambda}\xi} f(\xi, \eta). \tag{A9}$$

Integrating here by parts and taking into account that $f(-\infty, \eta) = 0$, we get

$$\zeta_{II}(\lambda; x, y) = \frac{1}{b}\int_{-\infty}^{y} f(x, \eta) d\eta - \frac{1}{b} e^{i\frac{b}{\lambda}x}\int_{-\infty}^{y} d\eta \int_{-\infty}^{x} d\xi \times e^{-i\frac{b}{\lambda}\xi}\frac{\partial f(\xi, \eta)}{\partial \xi}. \tag{A10}$$

The "reduced equation" associated with Eq. (A3) is

$$-i\,b\,\frac{\partial \zeta_I(x, y)}{\partial y} + i\,f(x, y) = 0, \tag{A11}$$

differential in y, which should hold for all x. This gives for any x, y

$$\zeta_I(x, y) = \frac{1}{b}\int_{-\infty}^{y} f(x, \eta) d\eta + h(x), \tag{A12}$$

where $h(x)$ is an arbitrary function. As we want $\zeta_I(x, y)$ to be able to approximate $\zeta_{II}(\lambda; x, y)$, we take $h(x)$ so as to satisfy the first bc for ζ_{II} in Eq. (A8); hence $h(x) \equiv 0$. Thus

$$\zeta_I(x, y) = \frac{1}{b}\int_{-\infty}^{y} f(x, \eta) d\eta, \tag{A13}$$

which is the first term of Eq. (A10). It is important to note that consequently $\zeta_I(x, y)$ also satisfies the second bc in Eq. (A8). Indeed, the integral in Eq. (A13) vanishes at $x \to -\infty$, as $f(x, y) \to 0$ when $x \to -\infty$, whatever y. The situation, in which the solution of the reduced, first-order equation $\zeta_I(x, y)$ can satisfy *both* bc of the exact second-order equation is exceptional, typical of our physical problem.

The solution of Eq. (A3) we are interested in is finally

$$\zeta_{II}(\lambda; x, y) = \zeta_I(x, y) + R(\lambda; x, y), \tag{A14}$$

where

$$R(\lambda; x, y) \equiv -\frac{1}{b} e^{i\frac{b}{\lambda}x}\int_{-\infty}^{y} d\eta \int_{-\infty}^{x} d\xi\, e^{-i\frac{b}{\lambda}\xi}\frac{\partial f(\xi, \eta)}{\partial \xi}. \tag{A15}$$

The solution is unique, satisfying the bc in Eq. (A8). It is valid for any constant values of λ, b ($\lambda \neq 0$).

We are interested in the possibility of approximating $\zeta_{II}(\lambda; x, y)$ by $\zeta_I(x, y)$ in cases (A) and (B). To assess this possibility in case (A) we evaluate the magnitude of $R(\lambda\,; x, y)$ at small $\lambda > 0$, with b constant. By performing the change of variable $t = -\xi + x$, R can be written:

$$R(\lambda; x, y) = -\frac{1}{b}\int_{-\infty}^{y} d\eta \int_{0}^{\infty} dt\, e^{i\frac{b}{\lambda}t}\left[\frac{\partial f(\xi, \eta)}{\partial \xi}\right]_{\xi=-t+x}. \tag{A16}$$

$R(\lambda; x, y)$ is expressible as

$$R(\lambda; x, y) = -\frac{1}{b}\int_{-\infty}^{y} I(\zeta) d\eta, \tag{A17}$$

in terms of the integral

$$I(\zeta) \equiv \int_{0}^{\infty} e^{i\zeta t} q(t) dt, \tag{A18}$$

if we take

$$\zeta \equiv \frac{b}{\lambda}, \quad q(t) \equiv \left[\frac{\partial f(\xi, \eta)}{\partial \xi}\right]_{\xi=-t+x}. \tag{A19}$$

Equation (A18) is a Fourier integral, and its large ζ (small λ) behavior is well known. One can write the expansion (see [36], Chap. 3, Sec. 5.2 and also Chap. 4, Sec. 1.3)

$$I(\zeta) = \sum_{s=0}^{n-1}\left(\frac{i}{\zeta}\right)^{s+1} q^{(s)}(0) + T_n(\zeta), \quad \zeta \to \infty, \tag{A20}$$

where we have denoted $q^{(s)}(t) \equiv d^s q(t)/dt^s$. It is assumed that the individual terms of the series expansion of $q(t)$, when inserted in Eq. (A18), give rise to convergent integrals [37]. Moreover, it is shown that $T_n(\zeta) = O(\zeta^{-n-1})$ and hence we are dealing with the "asymptotic expansion" of $I(\zeta)$ for $\zeta \to \infty$ (for definitions, see, e.g., Ref. [36], Chap. 1, Sec. 7; Ref. [24], Chap. 2).

In our case $q(t)$, Eq. (A19), also depends on x, η, and we shall write $q^{(s)}(0) \equiv q^{(s)}(0; x, \eta)$. Inserting Eq. (A20) into

Eq. (A17) gives

$$R(\lambda; x, y) = -\frac{1}{b}\sum_{s=0}^{n-1}\left(i\,\frac{\lambda}{b}\right)^{s+1} \times \int_{-\infty}^{y} q^{(s)}(0; x, \eta)d\eta + \widetilde{T}_n(\lambda; x, y). \quad \text{(A21)}$$

As

$$q^{(s)}(0; x, \eta) = (-1)^s \frac{\partial^{s+1}}{\partial x^{s+1}} f(x, \eta), \quad \text{(A22)}$$

Eq. (A21) can be expressed as

$$R(\lambda; x, y) = \left[\sum_{s=1}^{n}\left(-i\,\frac{\lambda}{b}\right)^{s}\frac{\partial^s}{\partial x^s}\right]\zeta_I(x, y) + O\left(\left(\frac{\lambda}{b}\right)^{n+1}\right), \quad \text{(A23)}$$

in terms of $\zeta_I(x, y)$, the solution of the reduced equation, Eq. (A13). The expansion holds uniformly in x, y. Inserting Eq. (A23) into Eq. (A14) we get the asymptotic expansion:

$$\zeta_{II}(\lambda; x, y) = \left[\sum_{s=0}^{n}\left(-i\,\frac{\lambda}{b}\right)^{s}\frac{\partial^s}{\partial x^s}\right]\zeta_I(x, y) + O\left(\left(\frac{\lambda}{b}\right)^{n+1}\right). \quad \text{(A24)}$$

It follows from Eq. (A24)

$$|R(\lambda; x, y)| = |\zeta_{II}(\lambda; x, y) - \zeta_I(x, y)| = O\left(\frac{\lambda}{b}\right), \quad \lambda > 0. \quad \text{(A25)}$$

Thus, in case (A) at arbitrarily small λ, $\zeta_{II}(\lambda; x, y)$ can lie as close to $\zeta_I(x, y)$ as desired, uniformly in x, y, and $\zeta_I(x, y)$ is a good approximation [38]. Case (A) is the typical situation considered in STP studies.

Case (B) is somewhat different. Now both λ and b depend on the underlying parameter ω. The exact solution ζ_{II}, Eqs. (A14), (A15), as well as the reduced equation solution in Eq. (A13), are proportional to $1/b = 1/(c\,\omega)$, and are singular at $\omega = 0$. However, $b\,\zeta_{II}$ is finite for $\omega \downarrow 0$ and one can write for it an expansion similar to Eq. (A24). Thereby, $b\,\zeta_{II}$ can be brought as close as desired to $b\,\zeta_I$ as $\omega \downarrow 0$, or, equivalently, ζ_{II} coincides with ζ_I to dominant order $O(1/\omega)$.

For both cases we can draw the *Conclusion*: the solution $\zeta_{II}(\lambda; x, y)$ of the exact equation Eq. (A3) can be brought to be arbitrarily close, to dominant order, to the solution $\zeta_I(x, y)$ of the reduced equation (A11), uniformly in x, y. In case (A) the dominant order with respect to the parameter λ is $O(1)$ and in case (B) the dominant with respect to ω is $O(1/\omega)$. This supports the assumption made in Sec. II that the solution of the reduced equation, Eq. (6), is a valid approximation and a good starting point for a successive approximation procedure.

It is interesting to consider what SPT would give for our model, in case (A). SPT seeks to construct a formal solution as an expansion in powers of λ (for convenience we take $i\,\lambda/b$):

$$\zeta_{II}^{\rm SPT}(\lambda; x, y) = \sum_{k=0}^{n}\left(i\,\frac{\lambda}{b}\right)^{k}\chi^{(k)}(x, y) + V_n(\varepsilon; x, y), \quad \text{(A26)}$$

starting with $\chi^{(0)} \equiv \zeta_I(x, y)$, the solution of the reduced equation, Eq. (A13). In general, the expansion Eq. (A26) should contain also "boundary layer functions" to accommodate for the initial condition of $\zeta_{II}(\lambda; x, y)$ that $\zeta_I(x, y)$ cannot satisfy. In our case these are not needed, as we are in the exceptional situation that $\zeta_I(x, y)$ can satisfy *both* initial conditions for ζ_{II}. Inserting the expansion Eq. (A26) in Eq. (A3), we get the sequence of equations

$$\frac{\partial\chi^{(k)}}{\partial y} \simeq -\frac{\partial^2\chi^{(k-1)}}{\partial x\partial y} \quad (k \geqslant 1). \quad \text{(A27)}$$

To integrate them we need to apply the *bc* for the exact ζ_{II}, Eq. (A8). The latter require that, at $x, y \to -\infty$ (when $\chi^{(0)} \to 0$), the solution should vanish too, i.e., $\chi^{(k)} \to 0$ for all k. As a consequence, all arbitrary functions of x appearing in the course of the integration should be taken zero. This leads to

$$\chi^{(k)} = (-1)^k \frac{\partial^k}{\partial x^k}\chi_0(x, y) \quad \text{(A28)}$$

and the formal SPT expansion becomes

$$\zeta_{II}^{\rm SPT}(\lambda; x, y) = \left[\sum_{k=0}^{n}\left(-i\frac{\lambda}{b}\right)^{k}\frac{\partial^k}{\partial x^k}\right]\chi_I(x, y) + V_n(\lambda; x, y). \quad \text{(A29)}$$

By comparing Eqs. (A24) and (A29) we see that SPT reproduces the asymptotic expansion of the exact result (with no need for layer functions). On the basis of Eq. (A24), we conclude that $V_n = O((\lambda/b)^{n+1})$, uniformly in the variables. (Of course, an independent SPT calculation would have to prove this.) Because of the similarities of the model with the physical case of Sec. II, we can expect this to happen also in the latter case, i.e., the corrections to the reduced equation approximation Eq. (12) can be obtained by successive approximations.

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[38] The smooth behavior for $\lambda \downarrow 0$ in Eq. (A24) is misleading, because it might give the impression that $\lambda = 0$ is a regular point of $\zeta_{II}(\lambda; x, y)$. To understand what happens in reality one must allow λ to become complex. In this case $R(\lambda; x, y)$, Eq. (A15), is an analytic function throughout the λ plane whatever x, y (see [36], Chap. 2, Theorem 1.1), with the exception of the point $\lambda = 0$, where it has an *essential singularity*. Thus, in the complex λ plane, $R(\lambda; x, y)$ does not have a well defined value at $\lambda = 0$, albeit it may have definite limits along rays of approach towards $\lambda = 0$, as we have seen for the real positive axis. A closer analysis leads to our extended conclusion for case (A): by letting $\lambda \to 0$ along any ray contained in a specified sector **S** of the plane containing the real axis, the exact solution $\zeta_{II}(\lambda; x, y)$ approaches the solution of the reduced equation $\zeta_I(x, y)$, uniformly in x, y. This is not true for rays outside **S**.