Time-reversal and rotational symmetries in noncommutative phase space

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(Received 3 October 2018; published 15 January 2019)

Time-reversal symmetry is studied in the frame of quantum and classical mechanics in a space with noncommutativity of coordinates and noncommutativity of momenta of canonical type. The circular motion is examined as an apparent example of time-reversal symmetry breaking in the space. On the basis of an exact solution of the problem, we show that because of noncommutativity the period of the circular motion depends on its direction. We propose the way to recover the time-reversal and rotational symmetries in noncommutative phase space of canonical type. Namely, on the basis of the idea of generalization of parameters of noncommutativity to tensors, we construct noncommutative algebra which is rotationally invariant, invariant under time reversal, and equivalent to noncommutative algebra of canonical type.

DOI: 10.1103/PhysRevA.99.012114

I. INTRODUCTION

Much attention has been devoted to studies of a quantum space realized on the basis of the idea that coordinates can be noncommutative. The idea was suggested by Heisenberg and later formalized by Snyder [1]. Noncommutative phase space of canonical type has been studied intensively. It is characterized by the commutation relations for operators of coordinates and momenta

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \gamma_{ij}), \tag{2}$$

$$[P_i, P_j] = i\hbar\eta_{ij}, \tag{3}$$

with θ_{ij}, η_{ij}, and γ_{ij} elements of constant antisymmetric matrices.

The idea that coordinates and momenta may be noncommutative offers a possibility to construct quantum space (space with minimal length) at the same time noncommutativity causes fundamental problems. In noncommutative space of canonical type (1)–(3) with $\eta_{ij} = \gamma_{ij} = 0$, one faces the problems of rotational and time-reversal symmetries breaking [2–5]. The same problems appear in noncommutative phase space (1)–(3). In [6] it was proven that the CPT theorem holds and spin statics remains valid for noncommutative quantum field theories while C, P (in some cases), and T symmetries are broken. The authors of Ref. [7] examined the unitarity of noncommutative scalar field theories and showed that noncommutativity of space-time leads to violation of unitarity. In addition, the noncommutativity of space-time causes violation of causality [8].

To preserve rotational symmetry, the noncommutative algebra of canonical type was generalized in different ways. As a result, different types of algebras with noncommutativity of coordinates were proposed and examined [9–12]. Among these algebras rotationally invariant algebra with position-dependent noncommutativity (see, for example, [13–19]) and noncommutative algebras with spin noncommutativity (see, for example, [20,21]) were intensively studied. These algebras are rotationally invariant but they are not equivalent to noncommutative algebras of canonical type in the sense that the relation $[X_i, \theta_{ij}] = [P_i, \theta_{ij}] = 0$ does not hold in the frame of the algebras.

In the present paper we study time-reversal symmetry in noncommutative phase space of canonical type (1)–(3) in the frame of noncommutative quantum and classical mechanics. The circular motion is studied as an obvious example for observing the time-reversal symmetry breaking. On the basis of an exact solution of the problem, it is shown that because of noncommutativity, the period of the circular motion depends on its direction. So noncommutativity causes violation of the time-reversal symmetry. To recover the symmetry we propose noncommutative algebra which is time-reversal invariant, rotationally invariant, and equivalent to noncommutative algebra of canonical type.

The paper is organized as follows. In Sec. II the time-reversal symmetry is studied in noncommutative phase space of canonical type. The invariance of noncommutative algebra upon time reversal is analyzed. The influence of noncommutativity on the period of circular motion in different directions is examined. In Sec. III algebra with noncommutativity of coordinates and noncommutativity of momenta which is time-reversal and rotationally invariant is constructed. A summary is presented in Sec. IV.

II. TIME-REVERSAL SYMMETRY IN NONCOMMUTATIVE PHASE SPACE OF CANONICAL TYPE

An obvious example for observing violation of the time-reversal symmetry in noncommutative phase space is a circular motion. The effect of noncommutativity on the motion

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depends on its direction. To show this let us consider noncommutative algebra of canonical type in the two-dimensional case

$$[X_1, X_2] = i\hbar\theta, \tag{4}$$

$$[X_1, P_1] = [X_2, P_2] = i\hbar(1+\gamma), \tag{5}$$

$$[P_1, P_2] = i\hbar\eta \tag{6}$$

and study the Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - \frac{k}{X}. \tag{7}$$

Here θ, η, and γ are constants and $X = \sqrt{X_1^2 + X_2^2}$. In the classical limit $\hbar \to 0$ from (4)–(6) we have the Poisson brackets

$$\{X_1, X_2\} = \theta, \tag{8}$$

$$\{X_1, P_1\} = \{X_2, P_2\} = 1+\gamma, \tag{9}$$

$$\{P_1, P_2\} = \eta. \tag{10}$$

Taking into account that X_i and P_i in (7) satisfy (8)–(10), one obtains equations of motion

$$\dot{X}_1 = \frac{P_1}{m}(1+\gamma) + \frac{k\theta X_2}{X^3}, \tag{11}$$

$$\dot{X}_2 = \frac{P_2}{m}(1+\gamma) - \frac{k\theta X_1}{X^3}, \tag{12}$$

$$\dot{P}_1 = \frac{\eta P_2}{m} - \frac{kX_1}{X^3}(1+\gamma), \tag{13}$$

$$\dot{P}_2 = -\frac{\eta P_1}{m} - \frac{kX_2}{X^3}(1+\gamma). \tag{14}$$

The obtained equations have the solution

$$X_1(t) = R_0\cos(\omega t), \quad X_2(t) = R_0\sin(\omega t), \tag{15}$$

$$P_1(t) = -P_0\sin(\omega t), \quad P_2(t) = P_0\cos(\omega t). \tag{16}$$

The solution corresponds to circular motion with radii R_0, momentum

$$P_0 = \frac{m\omega R_0^3 + km\theta}{R_0^2(1+\gamma)}, \tag{17}$$

and frequency

$$\omega = \frac{1}{2}\left[\sqrt{\frac{4k}{mR_0^3}[(1+\gamma)^2 - \theta\eta] + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} - \frac{\eta}{m} - \frac{k\theta}{R_0^3}\right]. \tag{18}$$

The period of the motion reads

$$T = 4\pi\left[\sqrt{\frac{4k}{mR_0^3}[(1+\gamma)^2 - \theta\eta] + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} - \frac{\eta}{m} - \frac{k\theta}{R_0^3}\right]^{-1}. \tag{19}$$

Let us consider the circular motion in the opposite direction with radii R_0. The corresponding solution reads

$$X_1(t) = R_0\cos(\omega t), \quad X_2(t) = -R_0\sin(\omega t), \tag{20}$$

$$P_1(t) = P_0'\sin(\omega t), \quad P_2(t) = P_0'\cos(\omega t). \tag{21}$$

Expressions (20) and (21) correspond to (15) and (16) with $-t$. We put P_0' in (21) to distinguish momentum which corresponds to the motion in the opposite direction. Substituting (20) and (21) into (11)–(14), we obtain

$$\omega' = \frac{1}{2}\left[\sqrt{\frac{4k}{mR_0^3}[(1+\gamma)^2 - \theta\eta] + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} + \frac{\eta}{m} + \frac{k\theta}{R_0^3}\right], \tag{22}$$

$$T' = 4\pi\left[\sqrt{\frac{4k}{mR_0^3}[(1+\gamma)^2 - \theta\eta] + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} + \frac{\eta}{m} + \frac{k\theta}{R_0^3}\right]^{-1}, \tag{23}$$

and

$$P_0' = -\frac{m\omega' R_0^3 - km\theta}{R_0^2(1+\gamma)}. \tag{24}$$

Note that the obtained frequency (22) and period (23) do not coincide with (18) and (19). We have

$$\Delta\omega = \omega' - \omega = \frac{\eta}{m} + \frac{k\theta}{R_0^3}. \tag{25}$$

The period and the frequency are different for motions in a circle of radius R_0 in different directions. In comparison to (18) and (19), expressions for ω' and T' contain parameters of noncommutativity with opposite signs. One also has that $P_0' \neq -P_0$ (in the ordinary space one has that $P_0' = -P_0$, which corresponds to the motion in the opposite direction). The discrepancy between expressions (18) and (19) and expressions (22) and (23) is because of noninvariance of noncommutative algebra (4) and (6) upon time reversal, because of time-reversal symmetry breaking in noncommutative phase space.

Ordinary commutation relations for coordinates and momenta ($\theta = \eta = \gamma = 0$) are invariant under the time-reversal transformation [22]. In analogy to the ordinary case ($\theta = \eta = \gamma = 0$), considering transformations of coordinates and momenta upon time reversal as $X_i \to X_i$ and $P_i \to -P_i$ and taking into account that in quantum mechanics the time-reversal operation involves complex conjugation [22], for (4)–(6) we obtain commutation relations

$$[X_1, X_2] = -i\hbar\theta, \tag{26}$$

$$[X_1, P_1] = [X_2, P_2] = i\hbar(1+\gamma), \tag{27}$$

$$[P_1, P_2] = -i\hbar\eta, \tag{28}$$

which in the classical limit correspond to the Poisson brackets $\{X_1, X_2\} = -\theta$, $\{X_1, P_1\} = \{X_2, P_2\} = 1 + \gamma$, and $\{P_1, P_2\} = \eta$. From (26)–(28) it follows that the algebra (4)–(6) is not invariant upon time reversal.

Note that the results (22) and (23) can be obtained by taking into account that the motion in the opposite direction corresponds to the time-reversal transformation and upon time reversal one has (26)–(28). Therefore, expressions (22) and (23) for ω' and T' can be found by changing the signs of parameters of noncommutativity in (18) and (19) (changing θ to $-\theta$ and η to $-\eta$). We also have that by changing signs of parameters of noncommutativity in P_0' [Eq. (24)] (changing θ to $-\theta$ and η to $-\eta$), one obtains $-P_0$ [Eq. (17)].

III. RECOVERING TIME-REVERSAL AND ROTATIONAL SYMMETRIES IN NONCOMMUTATIVE PHASE SPACE

To recover the time-reversal and rotational symmetries in noncommutative phase space we consider the idea to construct tensors of noncommutativity involving additional coordinates and additional momenta. On the basis of studies presented in the preceding section, we can conclude that in order to preserve the time-reversal symmetry the tensors θ_{ij} and η_{ij} have to transform under the time reversal as

$$\theta_{ij} \to -\theta_{ij}, \quad \eta_{ij} \to -\eta_{ij}. \tag{29}$$

Expressions for the tensors of noncommutativity in which (29) hold in terms of simplicity can be written as

$$\theta_{ij} = \frac{c_\theta}{\hbar} \sum_k \varepsilon_{ijk} p_k^a, \tag{30}$$

$$\eta_{ij} = \frac{c_\eta}{\hbar} \sum_k \varepsilon_{ijk} p_k^b, \tag{31}$$

where c_θ and c_η are constants and p_i^a and p_i^b are additional momenta.

To preserve the rotational symmetry, additional coordinates a_i and b_i and momenta p_i^a and p_i^b conjugate of them are supposed to be governed by rotationally symmetric systems. For simplicity, the systems are considered to be harmonic oscillators

$$H_{\rm osc}^a = \frac{(\mathbf{p}^a)^2}{2m_{\rm osc}} + \frac{m_{\rm osc}\omega_{\rm osc}^2 \mathbf{a}^2}{2}, \tag{32}$$

$$H_{\rm osc}^b = \frac{(\mathbf{p}^b)^2}{2m_{\rm osc}} + \frac{m_{\rm osc}\omega_{\rm osc}^2 \mathbf{b}^2}{2}, \tag{33}$$

with $\sqrt{\hbar}/\sqrt{m_{\rm osc}\omega_{\rm osc}} = l_P$ and very large frequency $\omega_{\rm osc}$. So the distance between the energy levels is large and harmonic oscillators put into the ground states remain in the states [23]. So we propose the noncommutative algebra

$$[X_i, X_j] = ic_\theta \sum_k \varepsilon_{ijk} p_k^a, \tag{34}$$

$$[X_i, P_j] = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar^2}(\mathbf{p}^a \cdot \mathbf{p}^b)\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar^2} p_j^a p_i^b\right), \tag{35}$$

$$[P_i, P_j] = ic_\eta \sum_k \varepsilon_{ijk} p_k^b. \tag{36}$$

where we take into account (30) and (31) and consider $\gamma_{ij} = \sum_k \theta_{ik}\eta_{jk}/4$, as was considered in [24].

Additional coordinates and additional momenta are supposed to satisfy the ordinary commutation relations

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = \left[p_i^a, p_j^a\right] = \left[p_i^b, p_j^b\right] = \left[p_i^a, p_j^b\right] = 0, \tag{37}$$

$$\left[a_i, p_j^a\right] = \left[b_i, p_j^b\right] = i\hbar\delta_{ij}, \tag{38}$$

$$\left[a_i, p_j^b\right] = \left[b_i, p_j^a\right] = 0. \tag{39}$$

Also, the relations

$$[a_i, X_j] = [a_i, P_j] = \left[p_i^b, X_j\right] = \left[p_i^b, P_j\right] = 0 \tag{40}$$

hold. So operators X_i and P_i, and θ_{ij} and η_{ij}, satisfy the same commutation relations as in the case of noncommutative phase space of canonical type (1)–(3). We have $[\theta_{ij}, X_k] = [\theta_{ij}, P_k] = [\eta_{ij}, X_k] = [\eta_{ij}, P_k] = [\gamma_{ij}, X_k] = [\gamma_{ij}, P_k] = 0$. In this sense noncommutative algebra (34)–(36) is equivalent to (1)–(3).

The coordinates and momenta a_i, b_i, p_i^a, and p_i^b can be treated as internal coordinates and momenta of a particle. Quantum fluctuations of these coordinates lead effectively to a nonpointlike particle with size of the order of the Planck scale.

As a result of involving additional coordinates and additional momenta, one has to consider the total Hamiltonian defined as

$$H = H_s + H_{\rm osc}^a + H_{\rm osc}^b, \tag{41}$$

where H_s is the Hamiltonian of a system under consideration and $H_{\rm osc}^a$ and $H_{\rm osc}^b$ are given by (32) and (33). Taking into account that coordinates and momenta upon time reversal transform as $X_i \to X_i$, $P_i \to -P_i$, $a_i \to a_i$, $p_i^a \to -p_i^a$, $b_i \to b_i$, and $p_i^b \to -p_i^b$ and the time-reversal operation involves complex conjugation, one obtains that the algebra (34)–(36) and the Hamiltonian (41) are invariant under the time reversal. So the time-reversal symmetry is preserved in a space with (34)–(36).

We would like to note here that because of invariance of noncommutative algebra (34)–(36) on time reversal, independently of representation, one can find that upon time reversal $X_i \to X_i$ and $P_i \to -P_i$. For example, the coordinates and momenta which satisfy (34)–(36) can be represented as

$$X_i = x_i + \frac{c_\theta}{2\hbar}[\mathbf{p}^a \times \mathbf{p}]_i, \tag{42}$$

$$P_i = p_i - \frac{c_\eta}{2\hbar}[\mathbf{x} \times \mathbf{p}^b]_i, \tag{43}$$

with x_i and p_i satisfying the ordinary commutation relations $[x_i, x_j] = [p_i, p_j] = 0$ and $[x_i, p_j] = i\hbar\delta_{ij}$. After time reversal one has $x_i \to x_i$, $p_i \to -p_i$, $p_i^a \to -p_i^a$, and $p_i^b \to -p_i^b$ and taking into account (42) and (43), noncommutative coordinates and noncommutative momenta transform as $X_i \to X_i$ and $P_i \to -P_i$. We would like to mention here that in the case of noncommutative algebra of canonical type the transformation of noncommutative coordinates and noncommutative momenta upon time reversal depends on their representation (see the Appendix). This is a consequence of noninvariance of noncommutative algebra of canonical type on the time reversal.

Besides being time-reversal invariant, the algebra (34)–(36) is rotationally invariant. After rotation $X_i' = U(\varphi)X_iU^+(\varphi)$, $P_i' = U(\varphi)P_iU^+(\varphi)$, $a_i' = U(\varphi)a_iU^+(\varphi)$, $p_i^{b\prime} = U(\varphi)p_i^bU^+(\varphi)$, and the commutation relations (34)–(36) remain the same,

$$[X_i', X_j'] = ic_\theta \sum_k \varepsilon_{ijk} p_k^{a\prime}, \tag{44}$$

$$[X_i', P_j'] = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar}(\mathbf{p}^{a\prime}\cdot\mathbf{p}^{b\prime})\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar}p_j^{a\prime}p_i^{b\prime}\right), \tag{45}$$

$$[P_i', P_j'] = ic_\eta \sum_k \varepsilon_{ijk} p_k^{b\prime}. \tag{46}$$

The operator of rotation has the form $U(\varphi) = \exp[i\varphi(\mathbf{n}\cdot\mathbf{L^t})/\hbar]$, with $\mathbf{L^t} = [\mathbf{x}\times\mathbf{p}] + [\mathbf{a}\times\mathbf{p}^a] + [\mathbf{b}\times\mathbf{p}^b]$ and $U^+(\varphi) = \exp[-i\varphi(\mathbf{n}\cdot\mathbf{L^t})/\hbar]$ [23].

So the noncommutative algebra (34)–(36) is rotationally and time-reversal invariant and is equivalent to noncommutative algebra of canonical type. Note that the proposed algebra is consistent. The Jacobi identity is satisfied and can be easily checked for all possible triplets of operators because of explicit representation (42) and (43).

We would like to mention that in our previous paper [23], in order to preserve rotational symmetry, we proposed noncommutative algebra (34)–(36) with tensors of noncommutativity defined as

$$\theta_{ij} = \frac{l_0}{\hbar}\sum_k \varepsilon_{ijk} a_k, \tag{47}$$

$$\eta_{ij} = \frac{p_0}{\hbar}\sum_k \varepsilon_{ijk} p_k^b. \tag{48}$$

with l_0 and p_0 constants and a_k and p_k^b additional coordinates and momenta governed by harmonic oscillators. In the case when θ_{ij} is defined as (47), the commutation relations (1) are not invariant under the time reversal. Upon time reversal one has

$$[X_i, X_j] = -il_0\sum_k \varepsilon_{ijk} a_k = -i\hbar\theta_{ij}. \tag{49}$$

We would like also to note here that instead of examining the total Hamiltonian (41), one can study an effective Hamiltonian

$$H_0 = \langle H_s\rangle_{ab} + H_{\rm osc}^a + H_{\rm osc}^b, \tag{50}$$

up to second order in

$$\Delta H = H - H_0 = H_s - \langle H_s\rangle_{ab}. \tag{51}$$

This is because the corrections to the energy levels of the total Hamiltonian H [Eq. (41)] caused by terms ΔH vanish up to second order in the perturbation theory [25]. Here the notation $\langle\cdots\rangle_{ab}$ is used for averaging over degrees of freedom of harmonic oscillators $H_{\rm osc}^a$ and $H_{\rm osc}^b$ in the ground states

$$\langle\cdots\rangle_{ab} = \langle\psi_{0,0,0}^a\psi_{0,0,0}^b|\ldots|\psi_{0,0,0}^a\psi_{0,0,0}^b\rangle, \tag{52}$$

where $\psi_{0,0,0}^a$ and $\psi_{0,0,0}^b$ are eigenstates of $H_{\rm osc}^a$ and $H_{\rm osc}^b$. Note that H_0 does not contain terms linear over parameters of noncommutativity. After averaging over $\psi_{0,0,0}^a$ and $\psi_{0,0,0}^b$ the terms in the Hamiltonian H_s in the first order in θ_{ij} and η_{ij} vanish because of $\langle a_i\rangle_{ab} = \langle p_i^b\rangle_{ab} = 0$. The Hamiltonian H_0 depends only on

$$\langle\theta_i^2\rangle = \frac{l_0^2}{\hbar^2}\langle\psi_{0,0,0}^a|a_i^2|\psi_{0,0,0}^a\rangle = \frac{l_0^2 l_P^2}{2\hbar^2} = \frac{\langle\theta^2\rangle}{3}, \tag{53}$$

$$\langle\eta_i^2\rangle = \frac{p_0^2}{\hbar^2}\langle\psi_{0,0,0}^b|(p_i^b)^2|\psi_{0,0,0}^b\rangle = \frac{p_0^2}{2\hbar^2 l_P^2} = \frac{\langle\eta^2\rangle}{3}. \tag{54}$$

Therefore, H_0 is invariant on replacement of θ_{ij} by $-\theta_{ij}$ and taking into account that (49) is invariant under the time-reversal transformation. Note that this statement holds for different definitions of the tensors of noncommutativity on which $\langle\theta_i\rangle_{ab} = \langle\eta_i\rangle_{ab} = 0$. We would also like to mention that the Hamiltonian H_0 depends on the mean values $\langle\theta^2\rangle$ and $\langle\eta^2\rangle$ and does not depend explicitly on the way the tensors of noncommutativity θ_{ij} and η_{ij} are defined and on the rotationally invariant system which governs a_i, b_i, p_i^a, and p_i^b. So, independently of the definition of the tensors of noncommutativity (only one condition has to be satisfied, $\langle\theta_{ij}\rangle_{ab} = \langle\eta_{ij}\rangle_{ab} = 0$) the effective Hamiltonian H_0 is invariant upon time reversal.

So the idea to define tensors of noncommutativity, introducing additional coordinates and additional momenta, offers a possibility to construct noncommutative algebra which is rotationally invariant, invariant under the time-reversal transformation, and equivalent to noncommutative algebra of canonical type.

IV. CONCLUSION

Time-reversal symmetry has been studied in the frame of quantum and classical mechanics in a space with noncommutativity of coordinates and noncommutativity of momenta of canonical type. It has been shown that noncommutative algebra (4)–(6) is not time-reversal invariant. Upon time reversal one obtains noncommutative algebra with opposite signs of parameters of noncommutativity (26) and (28). We have also concluded that because of noninvariance of algebra (4)–(6), transformations for noncommutative coordinates and noncommutative momenta upon time reversal depend on their representation (A8)–(A11).

Circular motion has been examined in noncommutative phase space as an evident example for studying the time-reversal symmetry breaking. The frequency and period of the motion have been found exactly in noncommutative phase space of canonical type. We have concluded that because of noncommutativity, the frequency and the period of the circular motion depend on its direction [Eqs. (18), (19), (22), and (23)]. The effect of noncommutativity on the motion in a circle of radius R_0 depends on its direction.

To recover the time-reversal symmetry in noncommutative phase space we have considered the idea to generalize the parameters of noncommutativity to tensors. We have shown that the time-reversal symmetry is preserved if the tensors of noncommutativity transform as $\theta_{ij}\to-\theta_{ij}$ and $\eta_{ij}\to-\eta_{ij}$ under time reversal. So, on the basis of this statement and in terms of simplicity, we proposed the tensors of noncommutativity to be defined as (30) and (31). To construct these tensors additional coordinates and additional momenta have been considered. To preserve the rotational symmetry the coordinates and the momenta have been supposed to be governed by rotationally invariant systems, which for simplicity are considered to be harmonic oscillators. As a result, we proposed noncommutative algebra (34)–(36) which is rotationally invariant, time-reversal invariant, and equivalent to noncommutative algebra of canonical type.

ACKNOWLEDGMENTS

This work was partly supported by the projects ΦΦ-63Hp (Project No. 0117U007190) and ΦΦ−30Φ (Project No. 0116U001539) from the Ministry of Education and Science of Ukraine.

APPENDIX

Because of noninvariance of noncommutative algebra of canonical type (4)–(6) under time reversal, the transformation of X_i and P_i upon time reversal depends on representation. It is known that the noncommutative coordinates and noncommutative momenta which satisfy (4)–(6) can be represented by coordinates and momenta x_i and p_i with

$$[x_i, x_j] = 0, \tag{A1}$$

$$[x_i, p_j] = i\hbar\delta_{ij}, \tag{A2}$$

$$[p_i, p_j] = 0. \tag{A3}$$

Namely, for coordinates and momenta which satisfy (4)–(6) we can write

$$X_1 = \varepsilon(x_1 - \theta_1' p_2), \tag{A4}$$

$$X_2 = \varepsilon(x_2 + \theta_2' p_1), \tag{A5}$$

$$P_1 = \varepsilon(p_1 + \eta_1' x_2), \tag{A6}$$

$$P_2 = \varepsilon(p_2 - \eta_2' x_1), \tag{A7}$$

with ε, θ_1', θ_2', η_2', and η_2' constants. Upon time reversal, considering transformations $x_i \to x_i$ and $p_i \to -p_i$, we have

$$X_1 \to X_1' = \varepsilon(x_1 + \theta_1' p_2), \tag{A8}$$

$$X_2 \to X_2' = \varepsilon(x_2 - \theta_2' p_1), \tag{A9}$$

$$P_1 \to -P_1' = \varepsilon(-p_1 + \eta_1' x_2), \tag{A10}$$

$$P_2 \to -P_2' = \varepsilon(-p_2 - \eta_2' x_1). \tag{A11}$$

So transformations (A8)–(A11) depend on the parameters ε, $\theta_1', \theta_2', \eta_2'$, and η_2', therefore they depend on the representation.

The parameters ε, θ_1', θ_2', η_2', and η_2', can be chosen in different ways. Taking into account (A1)–(A7), we have

$$[X_1, X_2] = i\hbar\varepsilon^2(\theta_1' + \theta_2'), \tag{A12}$$

$$[X_1, P_1] = i\hbar\varepsilon^2(1 + \theta_1'\eta_1') \tag{A13}$$

$$[X_2, P_2] = i\hbar\varepsilon^2(1 + \theta_2'\eta_2'), \tag{A14}$$

$$[P_1, P_2] = i\hbar\varepsilon^2(\eta_1' + \eta_2'). \tag{A15}$$

On the basis of comparison of (A12)–(A15) with (4)–(6) we can write the equations

$$\varepsilon^2 = 1, \quad \theta_1'\eta_1' = \theta_2'\eta_2' = \gamma, \tag{A16}$$

$$\theta_1' + \theta_2' = \theta, \tag{A17}$$

$$\eta_1' + \eta_2' = \eta, \tag{A18}$$

from which we obtain

$$\theta_1' = \frac{1}{2}\left(\theta \pm \sqrt{\theta^2 - 4\frac{\theta\gamma}{\eta}}\right), \tag{A19}$$

$$\theta_2' = \frac{1}{2}\left(\theta \mp \sqrt{\theta^2 - 4\frac{\theta\gamma}{\eta}}\right), \tag{A20}$$

$$\eta_1' = \frac{1}{2}\left(\eta \mp \sqrt{\eta^2 - 4\frac{\eta\gamma}{\theta}}\right), \tag{A21}$$

$$\eta_2' = \frac{1}{2}\left(\eta \pm \sqrt{\eta^2 - 4\frac{\eta\gamma}{\theta}}\right), \tag{A22}$$

and $\gamma \leqslant \theta\eta/4$. So, choosing the signs in (A19)–(A22), we obtain two different representations for noncommutative coordinates and noncommutative momenta and therefore two different transformations upon time reversal (A8)–(A11).

Symmetric representation with $\varepsilon = 1$, $\theta_1' = \theta_2' = \theta/2$, and $\eta_1' = \eta_2' = \eta/2$ is well known. In this case coordinates and momenta X_i and P_i satisfy (1)–(3) with $\gamma = \theta\eta/4$ [26].

For $\gamma = 0$ in (5) the commutator of coordinates and momenta is equal to $i\hbar$ as in the ordinary space. Comparing (A12)–(A15) and (4)–(6) with $\gamma = 0$, we can write

$$\varepsilon^2 = \frac{1}{1 + \theta_1'\eta_1'}, \tag{A23}$$

$$\theta_1'\eta_1' = \theta_2'\eta_2', \tag{A24}$$

$$\varepsilon^2(\theta_1' + \theta_2') = \theta, \tag{A25}$$

$$\varepsilon^2(\eta_1' + \eta_2') = \eta. \tag{A26}$$

Note that we have four equations (A23)–(A26) and five parameters ε, θ_1', θ_2', η_1', and η_2'. So by choosing one of them one can obtain different representations of coordinates and momenta which satisfy (4)–(6) with $\gamma = 0$ and different transformations (A8)–(A11). For instance, one can choose $\theta_2' = 0$. As a result, from (A23)–(A26) one obtains $\varepsilon = 1$, $\eta_1' = 0$, $\eta_2' = \eta$, and $\theta_1' = \theta$ and the representation reads $X_1 = x_1 - \theta p_2$, $X_2 = x_2$, $P_1 = p_1$, and $P_2 = p_2 - \eta x_1$. In this case, upon time reversal the coordinate X_2 and momentum P_1 transform in the traditional way, $X_2 \to X_2$ and $P_1 \to -P_1$. However, for X_1 and P_1 one has $X_1 \to X_1' = x_1 + \theta p_2$ and $P_2 \to -P_2' = -p_2 - \eta x_1$. It is also possible to write two symmetric representations (A4)–(A7) with the parameters $\varepsilon = (1 + \theta'\eta')^{-1/2}$, $\theta_1' = \theta_2' = (1 \pm \sqrt{1 - \theta\eta})/\eta$, and $\eta_1' = \eta_2' = (1 \pm \sqrt{1 - \theta\eta})/\theta$ [26,27], which lead to different transformations under time reversal.

[1] H. Snyder, Phys. Rev. **71**, 38 (1947).
[2] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001).
[3] A. P. Balachandran and P. Padmanabhan, J. High Energy Phys. 12 (2010) 001.
[4] J. Ben Geloun, S. Gangopadhyay, and F. G. Scholtz, Europhys. Lett. **86**, 51001 (2009).
[5] F. G. Scholtz, L. Gouba, A. Hafver, and C. M. Rohwer, J. Phys. A: Math. Theor. **42**, 175303 (2009).
[6] M. Chaichian, K. Nishijima, and A. Tureanu, Phys. Lett. B **568**, 146 (2003).
[7] J. Gomis and T. Mehen, Nucl. Phys. B **591**, 265 (2000).
[8] N. Seiberg, L. Susskind, and N. Toumbas, J. High Energy Phys. 06 (2000) 044.
[9] E. F. Moreno, Phys. Rev. D **72**, 045001 (2005).
[10] V. Gáliková and P. Presnajder, J. Phys: Conf. Ser. **343**, 012096 (2012).
[11] R. Amorim, Phys. Rev. Lett. **101**, 081602 (2008).
[12] Kh. P. Gnatenko and V. M. Tkachuk, Phys. Lett. A **378**, 3509 (2014).
[13] M. Daszkiewicz, J. Lukierski, and M. Woronowicz, Phys. Rev. D **77**, 105007 (2008).
[14] M. Daszkiewicz, J. Lukierski, and M. Woronowicz, J. Phys. A: Math. Theor. **42**, 355201 (2009).
[15] A. Borowiec, K. S. Gupta, S. Meljanac, and A. Pachol, Europhys. Lett. **92**, 20006 (2010).
[16] A. Borowiec, J. Lukierski, and A. Pachol, J. Phys. A: Math. Theor. **47**, 405203 (2014).
[17] A. Borowiec and A. Pachol, SIGMA **10**, 107 (2014).
[18] M. Gomes and V. G. Kupriyanov, Phys. Rev. D **79**, 125011 (2009).
[19] V. G. Kupriyanov, J. Phys. A: Math. Theor. **46**, 245303 (2013).
[20] H. Falomir, J. Gamboa, J. Lopez-Sarrion, F. Mendez, and P. A. G. Pisani, Phys. Lett. B **680**, 384 (2009).
[21] A. F. Ferrari, M. Gomes, V. G. Kupriyanov, and C. A. Stechhahn, Phys. Lett. B **718**, 1475 (2013).
[22] J. M. Domingos, Int. J. Theor. Phys. **18**, 213 (1979).
[23] Kh. P. Gnatenko and V. M. Tkachuk, Int. J. Mod. Phys. A **32**, 1750161 (2017).
[24] O. Bertolami and R. Queiroz, Phys. Lett. A **375**, 4116 (2011).
[25] Kh. P. Gnatenko and V. M. Tkachuk, Int. J. Mod. Phys. A **33**, 1850037 (2018).
[26] O. Bertolami, J. G. Rosa, C. M. L. de Aragao, P. Castorina, and D. Zappala, Mod. Phys. Lett. A **21**, 795 (2006).
[27] Kh. P. Gnatenko, Mod. Phys. Lett. A **32**, 1750166 (2017).