Uncertainty principle as a postquantum nonlocality witness for the continuous-variable multimode scenario

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The uncertainty principle is one of the central concepts in quantum theory. Different forms of this principle have been discussed in various foundational and information theoretic topics. Whereas in the discrete input-output scenario the limited nonlocal behavior of quantum theory has been explained by the *fine-grained* uncertainty relation, in the continuous-variable paradigm the Robertson-Schrödinger (RS) uncertainty relation has been used to detect multimode entanglement. Here we show that the RS uncertainty relation plays an important role in discriminating between quantum and postquantum nonlocal correlations in the multimode continuous outcome scenario. We provide a class of m-mode postquantum nonlocal correlations with a continuous outcome spectrum. While nonlocality of the introduced class of correlations is established through the Calvalcanti-Foster-Reid-Drummond class of Bell inequalities, the RS uncertainty relation detects their postquantum nature. Our result suggests a wider role of the uncertainty principle in the study of nonlocality in continuous-variable multimode systems.

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I. INTRODUCTION

Nonlocality is one of the most bizarre features of multipartite quantum systems. It was first established in the seminal paper by Bell [1]. Local outcomes of spatially separated quantum systems prepared in entangled states can produce correlations that cannot have any *local realistic* description—manifesting the nonlocal phenomena. Such nonlocal behavior can be witnessed through a violation of some local realistic inequality known as a Bell-type inequality. The Clauser-Horne-Shimony-Holt (CHSH) inequality is one such celebrated example [2]. The CHSH inequality considers the simplest $2-2-2$ scenario that involves two spatially separated parties, each performing one local measurement (out of a possible two), each measurement having two possible outcomes (the $2-2-2$ scenario is a special case of the general $m-n-k$ scenario, i.e., m parties, each with n possible measurements having k outcomes). While the local bound of the CHSH expression is 2, the maximum achievable value of this expression in quantum theory (QT) is $2\sqrt{2}$, known as the Cirel'son bound [3]. However, nonlocality is not a salient feature of QT alone. In 1994, Popescu and Rohrlich designed a correlation, famously known as PR correlation, which satisfies the relativistic causality or more broadly the no-signaling (NS) principle but at the same time depicts stronger nonlocal behavior as it achieves the algebraic maximum of the CHSH expression [4]. This observation brings up a very important question, namely whether there exists some other fundamental principle(s) (different from no-signaling) limiting the nonlocal strength of QT.

In the past few years, several information-theoretic as well as physical principles, viz., nontrivial communication complexity [5], information causality [6], macroscopic locality [7], relativistic causality [8], and local orthogonality [9], have been proposed that successfully explain the limited CHSH violation of QT. These principles also identify a part of the boundary between the set of quantum correlations and the postquantum NS correlations [10,11]. Furthermore, the applicability of these principles has also been proved useful in more general $m-n-k$ scenarios to witness postquantum correlations [12–17]. In a different approach, it has been shown that the limited CHSH nonlocality of QT can be connected to other fundamental features of the theory: Heisenberg's uncertainty principle [18,19], Bohr's complementarity principle [20,21], and preparation contextuality [22]. Not only do these connections hold true in QT, but they are also plausible in a larger class of theories.

A great deal of research has been done on quantum and postquantum nonlocal correlations in the discrete input-output scenario [23–26]. In the quantum domain, these studies mainly consider finite-input finite-output correlations arising from finite-dimensional quantum systems. Although nonlocal correlations have been studied for infinite-dimensional continuous-variable (CV) systems [27–32], those results are fundamentally not different from the finite input-output

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scenario as discrete binning of continuous outcomes has been considered there. A notable exception, in this regard, is the CV Bell inequalities introduced by Calvalcanti-Foster-Reid-Drummond (CFRD) [33]. There the authors derived a class of nonlinear Bell inequalities that apply for the continuous outcome spectrum without any need of discrete binning of the outcomes. A natural question of interest in this context will be the notion of postquantum nonlocal correlations. Very recently, Ketterer *et al.* developed a formalism to address this question for generic NS black-box measurement devices with continuous outputs, and they also provided a class of postquantum nonlocal correlations when only two sites or modes are involved [34].

A relevant question in the continuous outcome scenario is as follows: how can we certify postquantum nonlocality of a given correlation? The authors of Ref. [34] used the fact that for a two-mode scenario there is no quantum violation of the CFRD inequality [35], i.e., CFRD violation at the same time works as a nonlocal witness as well as a postquantumness witness. However, this is a very specific feature of the two-mode case that does not hold for a higher number of modes in general [36]. On the other hand, the principle-based methods [6,7,9] that have been proven to be useful for studying postquantum correlations in the discrete outcomes scenario have yet to be generalized for the continuous outcome spectrum.

Apart from the aforementioned foundational aspect of postquantum nonlocality in the CV scenario, there are motivations to explore it even from the perspective of applications. In the case of quantum information processing tasks, one of the most important notions is the device-independent (DI) scenario. In DI protocols, the experimenters do not possess the exact working knowledge of the apparatus and can only acquire the input-output statistics from the apparatus. In the discrete variable scenario, the use of nonlocality makes many DI tasks possible, such as DI quantum key distribution (QKD), DI randomness certification, etc. [37–44]. In general, the benefit of CV-QKD over discrete variable QKD manifests in a higher efficiency and key rate [45]. CV QKD also has the advantage of being compatible with standard telecommunication technology. Recently, long-distance CV QKD has been achieved for as much as 80 km [46]. In the CV scenario, measurement-device-independent (MDI) QKD has been recently introduced [47]. The advantage of MDI protocols over the standard quantum cases is that trust in the measurement devices is not needed for the former. However, MDI protocols need a trustworthy quantum state preparation device, which is not required in corresponding DI protocols. To further investigate various DI tasks in the CV scenario, the notion of nonlocality is of vital importance. Since nonlocal correlations can provide cryptographic security not achievable within classical theory, and they can be used to certify the presence of randomness and outperform classical communication at communication complexity problems, it is important to identify which nonlocal correlations are possible in a physical theory (more particularly, in quantum mechanics). Our study is thus significant in order to witness and rule out postquantum correlations in the continuous outcome scenario.

Developing a systematic approach to study postquantum nonlocal correlations for the continuous outcome scenario in multimode cases is thus quite important. Interestingly, we find that the Robertson-Schrödinger (RS) uncertainty relation has a role to play in this regard. We construct a class of continuous outcome postquantum nonlocal correlations for the generic m-mode scenario. While the nonlocality of the proposed class of correlations is certified through violation of CFRD inequalities, the postquantum nature is guaranteed by violation of the RS uncertainty relation.

The rest of the paper is organized as follows. In Sec. II, we briefly review the framework introduced in [34] for CV outcomes. In Secs. III and IV, we review CV Bell's inequalities and the Robertson-Schrödinger uncertainty relation, respectively. Our main results are presented in Sec. V, and finally we draw our conclusions in Sec. VI. Some details of the calculations are given in the Appendixes.

II. THE FRAMEWORK

The standard $m-n-k$ Bell scenario considers m space-like separated observers or sites denoted as $\mathcal{A}_i$, with $i \in \{1, 2, \ldots, m\}$, each observer performs one out of n possible local measurements denoted by X_i, with $X_i \in \{0, 1, \ldots, n-1\}$, and each measurement has k distinct outcomes denoted by A_i^j, with $j \in \{0, 1, \ldots, k-1\}$. Now, we consider that the outcomes are continuum instead of k distinct values. As pointed out by Ketterer *et al.* in [34], it is convenient to adopt the language of probability measures while considering continuous outcomes.

A probability space consists of three elements: (i) a sample space (Ω), (ii) the Borel σ-algebra $[\mathcal{B}(\Omega)]$ of events on Ω, and (iii) a valid Borel probability measure $\xi : \mathcal{B}(\Omega) \to [0, 1]$. In our case, the sample space would be $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$, with each $\Omega_i = \mathbb{R}$ being the outcome sample space of the ith site. The probability measure satisfies the normalization condition $\xi(\mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}) = 1$, and it also satisfies the additivity property: $\xi(\cup_i \omega_i) = \sum_i \xi(\omega_i)$, for all countable sequences $\{\omega_i\}_i$ of disjoint events $\omega_i \in \mathcal{B}(\Omega)$. The relation between a probability measure ξ and a probability density p is given by

$$\xi(A_1 \times \cdots \times A_m) := \int_{A_1 \times \cdots \times A_m} d\xi(a_1', \ldots, a_m') = \int_{A_1} \cdots \int_{A_m} p(a_1', \ldots, a_m') da_1' \cdots da_m'. \tag{1}$$

Here $A_1 \times \cdots \times A_m \in \mathcal{B}(\Omega)$, each $A_i \in \mathcal{B}(\mathbb{R})$, and $p(a_1', \ldots, a_m')$ denotes the corresponding probability density to ξ. We will denote the set of all probability measures on $\mathcal{B}(\Omega)$ as $\mathcal{M}_{\mathbb{R}^m}$.

From now on we consider that one of two possible local measurements will be performed on each site, i.e., $X_i \in \{0, 1\}, \forall\ i$. In such a scenario, an m-mode Bell behavior is defined as the collection of joint conditional probability measures $\{\xi^{\mathcal{A}_1 \cdots \mathcal{A}_m}_{X_1 \cdots X_m} | X_1, \ldots, X_m = 0, 1\}$, where each $\xi^{\mathcal{A}_1 \cdots \mathcal{A}_m}_{X_1 \cdots X_m} \in \mathcal{M}_{\mathbb{R}^m}$. Whenever there is no confusion we will avoid the superscript notation denoting the modes. The collection of all m-mode Bell behavior will be denoted as $\mathcal{M}^{2^m}_{\mathbb{R}^m}$. Consider any arbitrary grouping of m modes into two disjoint (nonempty) sets $\mathcal{K}$, $\mathcal{K}^c$ with $\mathcal{K} \cup \mathcal{K}^c = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$. The NS condition puts the restrictions that measurement choice of one set does not determine the outcome probability of another set for any of the above groupings. In measure theoretic language, these conditions read

$$\xi_{\{X_i\}_{i\in\mathcal{K}}\cup\{X_j\}_{j\in\mathcal{K}^c}}\left(\prod_{i\in\mathcal{K}}A_i\times\prod_{j\in\mathcal{K}^c}\mathbb{R}_j\right)$$
$$=\xi_{\{X_i\}_{i\in\mathcal{K}}\cup\{X_j\oplus 1\}_{j\in\mathcal{K}^c}}\left(\prod_{i\in\mathcal{K}}A_i\times\prod_{j\in\mathcal{K}^c}\mathbb{R}_j\right) \quad (2)$$

for all $A_i\in\mathcal{B}(\mathbb{R})$, where $\oplus$ denotes modulo 2 sum. The set of all no-signaling correlations $\mathcal{M}_{\rm NS}$ is naturally a strict subset of $\mathcal{M}^{2^m}_{\mathbb{R}^m}$. A behavior will be called quantum *iff* it can be obtained according to the Born probability rule, i.e., $\xi_{X_1\cdots X_m}(A_1\times\cdots\times A_m)=\mathrm{Tr}[\otimes_{i=1}^m M_{X_i}(A_i)\rho], \forall A_i\in\mathcal{B}(\mathbb{R})$, where ρ is a density operator acting on some tensor product Hilbert space $\otimes_{i=1}^m\mathcal{H}_i$, with $\mathcal{H}_i$ being the ith site's Hilbert space (in this case infinite-dimensional), and $M_{X_i}(A_i):\mathcal{B}(\mathbb{R})\mapsto\mathcal{L}_+(\mathcal{H}_i)$ are the positive operator valued measures on $\mathcal{H}_i$. A behavior $\{\xi_{X_1\cdots X_m}\}_{X_i=0,1}$ will be called postquantum if $\{\xi_{X_1\cdots X_m}\}_{X_i=0,1}\in\mathcal{M}_{\rm NS}$ but $\{\xi_{X_1\cdots X_m}\}_{X_i=0,1}\notin\mathcal{M}_Q$, the set of quantum behaviors. Local-realistic correlations are those where the outputs are locally generated from local inputs and some preestablished classical correlations encoded in some shared variable $\lambda\in\Lambda$. Such behaviors are of the form $\xi_{X_1\cdots X_m}=\int_\Lambda\delta_{a_1(X_1,\lambda),\dots,a_m(X_m,\lambda)}\,d\eta(\lambda)$, where $\eta:\mathcal{B}(\Lambda)\to\mathbb{R}_{\geqslant 0}$ is a probability measure and $\delta_{a_1(x_1,\lambda),\dots,a_m(X_m,\lambda)}$ is the CV version of the λth local deterministic response function: $\delta_{a_1,\dots,a_m}(A_1\times\cdots\times A_m):=1$ if $a_i\in A_i$ and 0, otherwise. The set of all local behaviors $\mathcal{M}_L$ is a strict subset of $\mathcal{M}_Q$, and behaviors not belonging to $\mathcal{M}_L$ manifest nonlocal features.

III. CONTINUOUS-VARIABLE BELL INEQUALITIES

The initial study of the Bell test for CV systems was based upon coarse graining of the continuous outcome spectrum into discrete domains [30–32,48]. One of the main motivations of studying the CV Bell scenario is to achieve better detection efficiency as the homodyne detection method is a highly efficient detection technique [48–50]. Another way to increase the detection efficiency is to use the idea of continuous realizations of outcomes instead of discrete ones. The idea was initially motivated by the CV version of the EPR paradox [51]. In Ref. [33], Cavalcanti, Foster, Reid, and Drummond (CFRD) derived a class of local realistic inequalities without any assumption on the number of measurement outcomes and therefore their inequalities are directly applicable to CV systems with no need of discrete binning of the outcomes. They have focused on the correlation functions of observables for m sites or observers, each equipped with n possible local measurement settings, and considered any real, complex, or vector function $F(\mathbf{X}^1,\mathbf{X}^2,\dots)$ of the local observables. All such functions, in a local hidden variable (LHV) theory, are functions of hidden variables $\lambda\in\Lambda$. The average over the LHV ensemble $P(\lambda)$ is given by $\langle F\rangle=\int_\Lambda P(\lambda)F(\mathbf{X}^1,\mathbf{X}^2,\dots)d\lambda$. Using the fact that any function of random variables has non-negative variance, the class of CRFD local realistic inequalities read $|\langle F\rangle|^2\leqslant\langle|F|^2\rangle$. For the two-site scenario it was first shown that it is impossible to violate the CFRD inequality with quantum phase-space quadrature operators [35]. Subsequently, this result has been generalized for arbitrary quantum measurements [29]. However it is possible to obtain violation of CFRD inequalities in QT with higher number of modes, in particular, explicit violation has been shown for multipartite GHZ like states [36]. We will use this particular class of inequalities to establish nonlocal feature of a continuous outcome correlation. Before arriving at our result, let us digress to Robertson-Schrödinger (RS) uncertainty relation a bit which plays a crucial role for our purpose.

IV. ROBERTSON-SCHRÖDINGER UNCERTAINTY RELATION

A proper mathematical formulation of Heisenberg's preparation uncertainty relation was first introduced by Kennard [52]. Schrödinger rederived this idea for two observable correlations in a more refined way [53], which was further extended for more than two observables by Robertson [54]. For an m-mode quantum state denoted by ρ, the noncommutativity of the canonical operators and the positive semidefiniteness of the state leads to the famous restriction—the RS uncertainty relation: $\mathbf{V}+\iota\mathbf{\Omega}\geqslant 0$ [55], where $\mathbf{V}$ is a $2m\times 2m$ real symmetric matrix, namely the covariance matrix (CM), and $\mathbf{\Omega}$ is the *symplectic form* and $\iota=\sqrt{-1}$. CM is calculated from the second moments of position ($\hat{q}_i$) and momentum ($\hat{p}_i$) operators, which we denote as elements of a vector $\hat{\alpha}=(\hat{q}_1,\hat{p}_1,\dots\hat{q}_m,\hat{p}_m)^\intercal$. Then we have $V_{ij}:=\frac{1}{2}\langle\{\Delta\hat{\alpha}_i,\Delta\hat{\alpha}_j\}\rangle_\rho$, where $\Delta\hat{\alpha}_i:=\hat{\alpha}_i-\langle\hat{\alpha}_i\rangle$, $\{.,.\}$ denotes anticommutator, $\langle.\rangle_\rho$ is the expectation value with respect to the state ρ, and $\mathbf{\Omega}$ is defined as $2\iota\Omega_{ij}=[\hat{\alpha}_i,\hat{\alpha}_j]$. Whether any given real symmetric matrix corresponds to a *bona fide* quantum CM can be verified by the RS uncertainty relation. This criterion is necessary and sufficient for Gaussian states, while for more general non-Gaussian states it is only a necessary criterion.

V. ROBERTSON SCHRÖDINGER UNCERTAINTY RELATION AS A WITNESS OF POSTQUANTUMNESS

Equipped with all the required tools, we now introduce continuous outcome postquantum nonlocal correlations for the m-mode scenario.

A. Three-mode scenario

First, we give an example in the three-mode case. Consider the following Bell behavior:

$$\xi^{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}_{111}=\tfrac{1}{4}[\mathcal{N}_{(l,l,-l),\sigma}+\mathcal{N}_{(l,-l,l),\sigma}+\mathcal{N}_{(-l,l,l),\sigma}+\mathcal{N}_{(-l,-l,-l),\sigma}], \quad (3a)$$

$$\xi^{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}_{\rm rest}=\tfrac{1}{4}[\mathcal{N}_{(l,l,l),\sigma}+\mathcal{N}_{(l,-l,-l),\sigma}+\mathcal{N}_{(-l,l,-l),\sigma}+\mathcal{N}_{(-l,-l,l),\sigma}], \quad (3b)$$

where rest $\in\{0,1\}^3\setminus\{111\}$, with 0 and 1 denoting position and momentum measurements, respectively. $\mathcal{N}_{\mathbf{a},\sigma}$ is the normal (Gaussian) probability measure defined through (1) with probability density centered around $\mathbf{a}:=(a_1,a_2,a_3)$ with width σ, i.e., $p_{\mathbf{a},\sigma}(\mathbf{a}')=1/(\sigma\sqrt{2\pi})^3\exp[-\sum_{i=1}^3(a_i-a_i')^2/(2\sigma^2)]$. It is straightforward to show

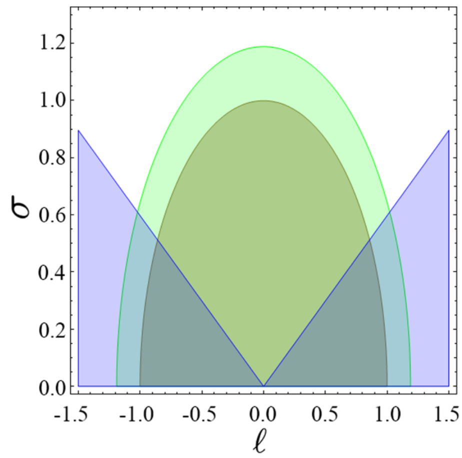

FIG. 1. The blue region denotes the values of l and σ for which three-mode CFRD inequality (5) is violated. The region bounded by the smaller green half-circle denotes the values of l and σ of (3), which violates the RS uncertainty relation with the product choice of joint probability distribution (i.e., $c = 0$). The region bounded by the larger green half-circle represents the RS violation for a nonproduct choice of marginal distribution (here, we have considered $c = 1$). The overlap of the blue region with the region bounded by the half-circles indicates three-mode postquantum nonlocal correlations. Clearly, $c = 0$ corresponds to the minimum region violating the RS relation. The units of l and σ correspond to the measurements chosen. Refer to Appendix A for details.

that the above behavior is indeed a NS behavior. The CFRD inequality for three modes is defined as [33]

$$\langle\tilde{X}_3\rangle^2 + \langle\tilde{Y}_3\rangle^2 \leqslant \left\langle \prod_{k=1}^{3} \left(\left(X_0^k\right)^2 + \left(X_1^k\right)^2\right) \right\rangle, \tag{4}$$

where $\tilde{X}_3$ and $\tilde{Y}_3$ are obtained from $\tilde{X}_3 + \iota\tilde{Y}_3 = \prod_{k=1}^{3}(X_0^k + \iota X_1^k)$. Here k denotes the mode. For the correlation (3), the CFRD expression (4) turns out to be

$$5l^6 \leqslant 2(l^2 + \sigma^2)^3. \tag{5}$$

For suitable choices of (l, σ), correlation (3) can violate inequality-(5), as shown in Fig. 1, and hence establishes nonlocality of those correlations. Naturally the question arises whether such nonlocal correlations are quantum realizable or whether they are postquantum in nature. One way is to find the two-mode marginal correlations and check whether the two-mode marginals violate the two-mode CFRD inequality. But in this case, the two-mode marginals being a local correlation satisfy the corresponding CFRD inequality (see Appendix A).

So, at this point we utilize the RS uncertainty relation, which puts necessary conditions on a distribution to be quantum realizable: if the RS uncertainty relation is violated, then the given distribution cannot be a quantum realizable one. To calculate the CM from (3), we require the single-mode marginals $\xi_{X_i}^{\mathcal{A}_i}$ as well as the 2-mode marginals $\xi_{X_iX_j}^{\mathcal{A}_i\mathcal{A}_j}$, which can be readily calculated by integrating out the appropriate mode(s). But calculation of CM also requires single-mode position-momentum joint distribution $\xi_{(X_i=0,X_i=1)}^{\mathcal{A}_i}$. Note that given marginal probability distributions $\xi_{X_i=0}^{\mathcal{A}_i}$ and $\xi_{X_i=1}^{\mathcal{A}_i}$, the choice of joint distribution $\xi_{(X_i=0,X_i=1)}^{\mathcal{A}_i}$ is not unique. With a (trivial) product choice of joint distribution $\xi_{(X_i=0,X_i=1)}^{\mathcal{A}_i} = \xi_{X_i=0}^{\mathcal{A}_i} \times \xi_{X_i=1}^{\mathcal{A}_i}$, we have $\langle\hat{q}_i\hat{p}_i\rangle = 0$, which in turn gives that the RS uncertainty relation will be violated if $l^2 + \sigma^2 < 1$ (see Appendix A), i.e., describing a half-circle region on the l-σ plane (see Fig. 1). In Fig. 1 the overlapping region of a blue curve and the inner half-circle violates both the CFRD inequality and the RS relation (calculated with a product choice of distribution) and hence establishes postquantum nonlocality of those correlations. At this point one can ask whether the values of l and σ lying outside the inner circle but within the blue region denote quantum realizable probability distribution. However, it is not straightforward to answer this question. First of all, if we calculate CM with some nonproduct distribution $\xi_{(X_i=0,X_i=1)}^{\mathcal{A}_i} \neq \xi_{X_i=0}^{\mathcal{A}_i} \times \xi_{X_i=1}^{\mathcal{A}_i}$, we have $\langle\hat{q}^i\hat{p}^i\rangle = c$, with c being a real number ($c = 0$ corresponding to the product choice), and consequently the RS uncertainty relation will be violated if $l^2 + \sigma^2 < \sqrt{1 + c^2}$. Therefore, the area of the postquantum region increases, as shown in Fig. 1 by the larger green half-circle (for $c = 1$). Even if one can specify the value of c, it will not be possible in general to guarantee quantumness of the correlations outside the green half-circle region as the RS relation is a sufficient criterion for bona-fide CM only for Gaussian distribution. However, this calculation asserts the existence of postquantum nonlocal correlations independent of whether we take a product or a nonproduct form of joint position-momentum distribution for each of the modes.

B. *m*-mode scenario

We now generalize the above three-mode example to m number of modes. Consider a vector $\mathbf{P}_i \in \mathbb{R}^m$ with the first i number of elements being $-l$ and the following $(m - i)$ number of elements being $+l$. Denote by $\mathcal{P}_i$ the set of all vectors obtained from $\mathbf{P}_i$ by permuting its elements. Consider now an m-mode Bell behavior defined as

$$\xi_{11\cdots1}^{\mathcal{A}_0\mathcal{A}_1\cdots\mathcal{A}_m} = \frac{1}{2^{m-1}} \sum_{\substack{i\,\in\,\mathbb{N}_o \\ i\,\leqslant\,m}} \sum_{\mathbf{P}_i\in\mathcal{P}_i} \mathcal{N}_{\mathbf{P}_i,\sigma}, \tag{6a}$$

$$\xi_{\text{rest}}^{\mathcal{A}_0\mathcal{A}_1\cdots\mathcal{A}_m} = \frac{1}{2^{m-1}} \sum_{\substack{i\,\in\,\mathbb{N}_e \\ i\,\leqslant\,m}} \sum_{\mathbf{P}_i\in\mathcal{P}_i} \mathcal{N}_{\mathbf{P}_i,\sigma}. \tag{6b}$$

Here, $\mathbb{N}_o(\mathbb{N}_e)$ denotes the set of odd (even) integers, and $\mathcal{N}_{\mathbf{a},\sigma}$ is the normal (Gaussian) probability measure defined through (1) with probability density centered around $\mathbf{a} \equiv (a_1, \ldots, a_m)$ with widths σ, i.e., $p_{\mathbf{a},\sigma}(\mathbf{a}') = 1/(\sigma\sqrt{2\pi})^m \exp\{-[\sum_{i=1}^{m}(a_i - a_i')^2]/(2\sigma^2)\}$. The expression of the m-mode CFRD inequality with this probability measure takes the following form (see Appendix B): when m is even, we get

$$\left[\left[2^{m/2}\cos\left(\frac{m\pi}{4}\right) + (-1)^{\frac{m}{2}+1}2\right]^2 + 2^m\sin^2\left(\frac{m\pi}{4}\right)\right]l^{2m} \leqslant 2^m(l^2 + \sigma^2)^m. \tag{7}$$

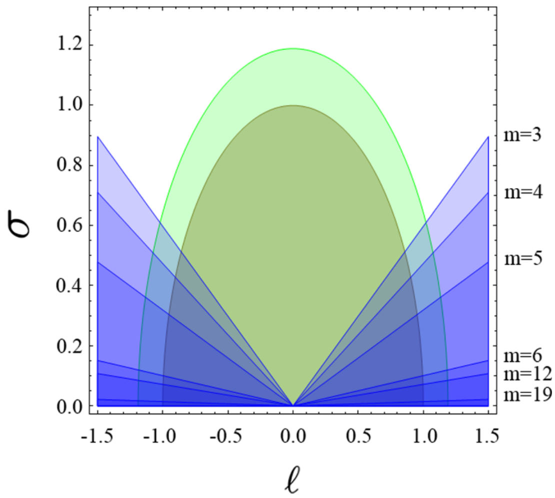

FIG. 2. For different numbers of modes ($m = 3, 4, 5, 6, 12, 19$), the corresponding CFRD inequalities violation has been depicted by different shades of blue, as shown. The smaller and larger half-circular regions denote RS uncertainty violations for $c = 0$ and 1 as in Fig. 1. Units of l and σ are the same as in Fig. 1.

For odd m, it is found to be

$$\left[\left(2^{m/2}\sin\left(\frac{m\pi}{4}\right)+(-1)^{\frac{m-1}{2}+1}2\right)^2+2^m\cos^2\left(\frac{m\pi}{4}\right)\right]l^{2m} \leqslant 2^m(l^2+\sigma^2)^m. \quad (8)$$

For suitable choices of l and σ, the m-mode probability measure of Eq. (6) violates the corresponding CFRD inequality (see Fig. 2). A similar calculation to that given in the three-mode example shows that the RS uncertainly relation, calculated with single-mode product [non-product] joint distribution, will be violated by the probability measure Eq. (6) if $l^2+\sigma^2 < 1(l^2+\sigma^2 < \sqrt{1+c^2})$. Correspondingly, the choices of l and σ that violate both the CFRD inequality and the RS uncertainty relation give the m-mode postquantum nonlocal correlations.

C. Two-mode scenario

So far, we have shown that the RS uncertainty relation plays a crucial role in certifying postquantumness for m-mode CV correlations, with $m \geqslant 3$. What will be the implication of our approach for the two-mode case? We find that for the two-mode case, the probability measure (6), originally considered in [34], yields the CFRD expression as $2l^4-(l^2+\sigma^2)\leqslant 0$. In this case, the RS uncertainty relation, calculated with product and nonproduct single mode joint distribution, will be satisfied if $(l^2+\sigma^2)\geqslant\sqrt{1+2l^4}$ and $(l^2+\sigma^2)\geqslant\sqrt{1+l^4+(l^2+c^2)^2}$, respectively. From these expressions, it is evident that any such correlation-violating CFRD inequality indeed violates the RS uncertainty relation (see Appendix C). Therefore, the postquantumness of those correlations can be asserted from the RS uncertainty relation even without referring to the results of [35].

VI. CONCLUSIONS

The usefulness of the Robertson-Schrödinger uncertainty relation in detecting multimode entanglement has already been demonstrated in [56]. On the other hand, the work by Oppenheim and Wehner [18] is also quite worthy of mention in the context of the present work. In the $2-2-2$ scenario, they have shown that quantum mechanics cannot be more nonlocal with measurements that respect the uncertainty principle in *fine-grained* form. In the continuous outcome scenario, the role of the uncertainty principle in certifying postquantumness has been explored in the present study. Our work also raises several interesting questions. Will it be the case that any postquantum nonlocal correlation violates some form(s) of the uncertainty principle? Another curious avenue to explore is to construct a *g*enuine nonlocal inequality for the m-mode scenario with a continuous outcome, which can be used to certify the inbuilt genuineness of the correlation presented here. Although our work employs the uncertainty relation and the CFRD inequality to detect postquantum nonlocal correlation in the continuous outcome scenario, it is worthwhile to determine whether one could do the same with some operational task.

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APPENDIX A: THREE-MODE SCENARIO

1. Calculation of the CFRD inequality

Given a probability measure ξ with probability density p, the expectation $\langle\prod_k (X^k_{i_k})^{n_k}\rangle$ can be calculated according to the following:

$$\left\langle\prod_k (X^k_{i_k})^{n_k}\right\rangle := \int\prod_k (A^k_{i_k})^{n_k}d\xi, \quad (A1)$$

where $k\in\{1,2,\dots,m\}$, $i_k\in\{0,1\}$, $n_k\in\{1,2\}$.

For the three-mode case, given the probability measure (3), we find

$$\left\langle(X^1_{i_1})^2(X^2_{i_2})^2(X^3_{i_3})^2\right\rangle = (l^2+\sigma^2)^3, \forall\, i_1,i_2,i_3=0,1;$$
$$\left\langle X^1_{i_1}X^2_{i_2}X^3_{i_3}\right\rangle = l^3 \quad \text{when } i_1i_2i_3=0;$$
$$\left\langle X^1_1X^2_1X^3_1\right\rangle = -l^3.$$

Finally using the above expressions, the CFRD inequality is calculated as (5).

2. Calculation of the RS uncertainty relation

Denoting the position and momentum observable for the ith mode as $(\hat{q}_i,\hat{p}_i)$, the vector $\vec{\alpha}$ for three modes looks like

$$\hat{\alpha} = (\hat{q}_1,\hat{p}_1,\hat{q}_2,\hat{p}_2,\hat{q}_3,\hat{p}_3)^T \equiv \hat{\alpha}_i\,|_{i=1,\dots,6}. \quad (A2)$$

The covariance matrix (CM) V is defined as $V_{ij} = \frac{1}{2}\langle\{\Delta\hat{\alpha}_i, \Delta\hat{\alpha}_j\}\rangle$, where, $\Delta\hat{\alpha}_i = \hat{\alpha}_i - \langle\hat{\alpha}_i\rangle$, and $\{.,.\}$ is an anticommutator. We can find the single-mode and two-mode marginals from (3) by integrating out the appropriate modes, and they turn out to be

$$\xi^{\mathcal{A}_i}_{X_i} = \tfrac{1}{2}[\mathcal{N}_{l,\sigma} + \mathcal{N}_{-l,\sigma}], \forall\, i = \{1, 2, 3\}, X_i = \{0, 1\},$$
$$\xi^{\mathcal{A}_i,\mathcal{A}_j}_{X_i,X_j} = \tfrac{1}{4}[\mathcal{N}_{(l,l),\sigma} + \mathcal{N}_{(l,-l),\sigma} + \mathcal{N}_{(-l,l),\sigma} + \mathcal{N}_{(-l,-l),\sigma}],$$

where $\forall\, i, j = \{1, 2, 3\}; i \neq j; X_i, X_j = \{0, 1\}$.

While calculating terms like $\langle\hat{q}^i\hat{p}^i\rangle$, we require the position-momentum joint probability distribution $\xi^{\mathcal{A}_i}_{(X_i=0,X_i=1)}$ for the ith mode. But given $\xi^{\mathcal{A}_i}_{X_i=0}$ and $\xi^{\mathcal{A}_i}_{X_i=1}$, the choice of $\xi^{\mathcal{A}_i}_{(X_i=0,X_i=1)}$ is not unique. First we consider a (trivial) product choice $\xi^{\mathcal{A}_i}_{(X_i=0,X_i=1)} = \xi^{\mathcal{A}_i}_{X_i=0} \times \xi^{\mathcal{A}_i}_{X_i=1}$. In this case, $\langle\hat{q}_i\hat{p}_i\rangle = 0$ and the CM becomes

$$\mathbf{V^p} = \bigoplus_{i=1}^{3}\begin{bmatrix} l^2+\sigma^2 & 0 \\ 0 & l^2+\sigma^2\end{bmatrix}. \tag{A3}$$

For a nonproduct choice $\langle\hat{q}_i\hat{p}_i\rangle = c$, where c is some nonzero real number. In this case, CM becomes

$$\mathbf{V^{np}} = \bigoplus_{i=1}^{3}\begin{bmatrix} l^2+\sigma^2 & c \\ c & l^2+\sigma^2\end{bmatrix}. \tag{A4}$$

A bona fide CM needs to satisfy the RS uncertainty relation $\mathbf{V} + i\mathbf{\Omega} \geqslant 0$, where $\Omega = \bigoplus_{i=1}^{3}\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right]$. Respectively, for product and nonproduct choices the RS uncertainty relation will be violated if

$$l^2 + \sigma^2 < 1 \quad \text{(for product)}, \tag{A5}$$

$$l^2 + \sigma^2 < \sqrt{1+c^2} \quad \text{(for non-product)}. \tag{A6}$$

By comparing Eqs. (A5) and (A6), it is obvious that the region of (l, σ) violating the RS uncertainty relation for a product choice is strictly inscribed by the region of (l, σ) violating the RS uncertainty relation for a nonproduct choice.

APPENDIX B: *m*-MODE SCENARIO

1. Calculation of the CFRD inequality

For m-mode CFRD, inequality was defined in terms of the kth mode local observables $\{X^k_0, X^k_1\}$ as

$$\langle\tilde{X}_m\rangle^2 + \langle\tilde{Y}_m\rangle^2 \leqslant \left\langle\prod_{k=1}^{m}\left(\left(X^k_0\right)^2 + \left(X^k_1\right)^2\right)\right\rangle, \tag{B1}$$

where $\tilde{X}_m$ and $\tilde{Y}_m$ can be obtained from

$$\tilde{X}_m + \iota\tilde{Y}_m = \prod_{k=1}^{m}\left(X^k_0 + \iota X^k_1\right). \tag{B2}$$

The key point while calculating the m-mode CFRD inequality for correlation (6) is

$$\left\langle X^1_{i_1}X^2_{i_2}\cdots X^m_{i_m}\right\rangle = l^m \text{ if} \prod_{k=1}^{m} i_k = 0, \tag{B3}$$

$$\left\langle X^1_1X^2_1\cdots X^m_1\right\rangle = -l^m, \tag{B4}$$

$$\left\langle\left(X^1_{i_1}\right)^2\left(X^2_{i_2}\right)^2\cdots\left(X^m_{i_m}\right)^2\right\rangle = (l^2+\sigma^2)^m, \quad \forall \quad i_1, \cdots, i_m \in \{0, 1\}. \tag{B5}$$

Thus, the right-hand side of (B1) is readily seen as

$$\left\langle\prod_{k=1}^{m}\left(\left(X^k_0\right)^2 + \left(X^k_1\right)^2\right)\right\rangle = 2^m(l^2+\sigma^2)^m. \tag{B6}$$

Calculation of the left-hand side of (B1) requires us to know the number of terms with negative signatures in $\tilde{X}_m$ and $\tilde{Y}_m$ which we define as a_m and b_m respectively. a_m and b_m follow recursion relations that can be specified from the following expressing:

$$\begin{aligned}
\tilde{X}_m + \iota\tilde{Y}_m &= \prod_{k=1}^{m}\left(X^k_0 + \iota X^k_1\right) \\
&= \prod_{k=1}^{m-1}\left(X^k_0 + \iota X^k_1\right)\left(X^m_0 + \iota X^m_1\right) \\
&= (\tilde{X}_{m-1} + \iota\tilde{Y}_{m-1})\left(X^m_0 + \iota X^m_1\right) \\
&= \left(\tilde{X}_{m-1}X^m_0 - \tilde{Y}_{m-1}X^m_1\right) \\
&\quad + \iota\left(\tilde{X}_{m-1}X^m_1 + \tilde{Y}_{m-1}X^m_0\right), \\
\Rightarrow \tilde{X}_m &= \left(\tilde{X}_{m-1}X^m_0 - \tilde{Y}_{m-1}X^m_1\right), \\
\tilde{Y}_m &= \left(\tilde{X}_{m-1}X^m_1 + \tilde{Y}_{m-1}X^m_0\right).
\end{aligned}$$

Thus we have the following coupled recursion relations,

$$a_m = 2^{m-2} + a_{m-1} - b_{m-1}, \tag{B7}$$

$$b_m = a_{m-1} + b_{m-1}. \tag{B8}$$

Closed-form expressions for a_m and b_m turn out to be

$$a_m = \frac{1}{2}\left[2^{m-1} - 2^{m/2}\cos\left(\frac{m\pi}{4}\right)\right], \tag{B9}$$
$$b_m = \frac{1}{2}\left[2^{m-1} - 2^{m/2}\sin\left(\frac{m\pi}{4}\right)\right]. \tag{B10}$$

We also need to know the signature of the term $X^1_1X^2_1\cdots X^m_1$ as well as whether it is included in $\tilde{X}_m$ or $\tilde{Y}_m$. We notice that,

$$(-1)^{m/2}X^1_1X^2_1\cdots X^m_1 \in \tilde{X}_m, \quad \text{if m is even},$$
$$(-1)^{(m-1)/2}X^1_1X^2_1\cdots X^m_1 \in \tilde{Y}_m, \quad \text{if m is odd}.$$

The required expectation values of $\tilde{X}_m$ and $\tilde{Y}_m$ thus become:

$$\begin{aligned}\langle\tilde{X}_m\rangle &= \left[2^{m-1} - 2a_m + (-1)^{\frac{m}{2}+1}2\right]l^m, \\ \langle\tilde{Y}_m\rangle &= [2^{m-1} - 2b_m]l^m \quad \text{for even } m;\end{aligned} \tag{B11}$$

$$\begin{aligned}\langle\tilde{X}_m\rangle &= \left[2^{m-1} - 2a_m\right]l^m, \\ \langle\tilde{Y}_m\rangle &= \left[2^{m-1} - 2b_m + (-1)^{\frac{m-1}{2}+1}2\right]l^m \quad \text{for odd } m.\end{aligned} \tag{B12}$$

Finally, when m is even the CFRD inequality is given by

$$\left[\left[2^{m/2}\cos\left(\frac{m\pi}{4}\right)+(-1)^{\frac{m}{2}+1}2\right]^2+2^m\sin^2\left(\frac{m\pi}{4}\right)\right]l^{2m} \leqslant 2^m(l^2+\sigma^2)^m, \tag{B13}$$

and for odd m it is found to be

$$\left[\left(2^{m/2}\sin\left(\frac{m\pi}{4}\right)+(-1)^{\frac{m-1}{2}+1}2\right)^2+2^m\cos^2\left(\frac{m\pi}{4}\right)\right]l^{2m} \leqslant 2^m(l^2+\sigma^2)^m. \tag{B14}$$

2. Calculation of the RS uncertainty relation

As in the three-mode case, for the general m-mode case also we have single-mode and two-mode marginals of the following forms:

$$\xi^{\mathcal{A}_i}_{X_i}=\tfrac{1}{2}[\mathcal{N}_{l,\sigma}+\mathcal{N}_{-l,\sigma}],\ \forall\, i=\{1,\dots,m\},\ X_i=\{0,1\},$$

$$\xi^{\mathcal{A}_i,\mathcal{A}_j}_{X_i,X_j}=\tfrac{1}{4}[\mathcal{N}_{(l,l),\sigma}+\mathcal{N}_{(l,-l),\sigma}+\mathcal{N}_{(-l,l),\sigma}+\mathcal{N}_{(-l,-l),\sigma}],$$

where $\forall i, j=\{1,\dots,m\};\quad i\neq j;\quad X_i,X_j=\{0,1\}$.

As in the three-mode case, the product and nonproduct choices of single-mode position-momentum joint distributions give the respective CM matrices: $\mathbf{V^p}=\bigoplus_{i=1}^m\left[\begin{smallmatrix} l^2+\sigma^2 & 0\\ 0 & l^2+\sigma^2\end{smallmatrix}\right]$ and $\mathbf{V^{np}}=\bigoplus_{i=1}^m\left[\begin{smallmatrix} l^2+\sigma^2 & c\\ c & l^2+\sigma^2\end{smallmatrix}\right]$. And consequently the RS uncertainty will be violated if $l^2+\sigma^2<1$ and $l^2+\sigma^2<\sqrt{1+c^2}$, respectively.

APPENDIX C: TWO-MODE SCENARIO

Consider the two-mode correlation introduced in Ref. [34]:

$$\xi^{\mathcal{A}_0\mathcal{A}_1}_{00}=\xi^{\mathcal{A}_0\mathcal{A}_1}_{01}=\xi^{\mathcal{A}_0\mathcal{A}_1}_{10}=\tfrac{1}{2}[\mathcal{N}_{(l,l),\sigma}+\mathcal{N}_{(-l,-l),\sigma}], \tag{C1}$$

$$\xi^{\mathcal{A}_0\mathcal{A}_1}_{11}=\tfrac{1}{2}[\mathcal{N}_{(l,-l),\sigma}+\mathcal{N}_{(-l,l),\sigma}]. \tag{C2}$$

In this case, the CFRD inequality turns out to be $8l^4\leqslant 4(l^2+\sigma^2)^2$. With product and nonproduct choices of single-mode position-momentum joint distribution, the CM becomes

$$\mathbf{V^p}=\begin{bmatrix} l^2+\sigma^2 & 0 & l^2 & l^2\\ 0 & l^2+\sigma^2 & l^2 & -l^2\\ l^2 & l^2 & l^2+\sigma^2 & 0\\ l^2 & -l^2 & 0 & l^2+\sigma^2\end{bmatrix},$$

$$\mathbf{V^{np}}=\begin{bmatrix} l^2+\sigma^2 & c & l^2 & l^2\\ c & l^2+\sigma^2 & l^2 & -l^2\\ l^2 & l^2 & l^2+\sigma^2 & c\\ l^2 & -l^2 & c & l^2+\sigma^2\end{bmatrix}.$$

Respectively, the RS uncertainty relation will be violated if

$$(l^2+\sigma^2)<\sqrt{(1+2l^4)}\quad\text{(for product)}, \tag{C3}$$

$$(l^2+\sigma^2)<\sqrt{1+l^4+(l^2+c^2)^2}\quad\text{(for nonproduct)}. \tag{C4}$$

From the expressions of the CFRD inequality and the RS uncertainty relation, it is evident that any (l,σ) that violates the CFRD inequality also violates the RS uncertainty relation (both the product and nonproduct forms), as shown in Fig. 3.

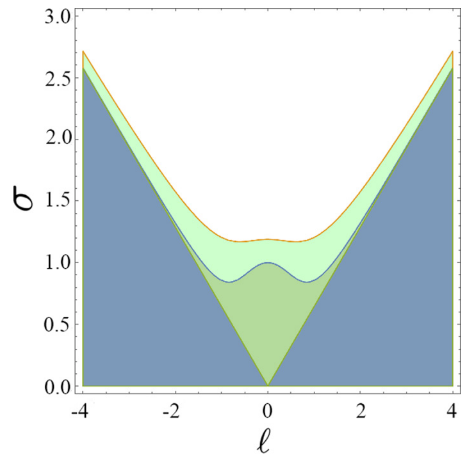

FIG. 3. The blue region denotes violation of the CFRD inequality. Deep and light green regions correspond to violation of the RS uncertainty relation for product and nonproduct (with $c=1$) choices of single-mode position-momentum joint distribution. Units are as in the previous figures.

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