Current in an open tight-binding system

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We propose a revised observable for the current in an open tight-binding system. Assuming a conserved number of electrons and within Markovian dynamics, we first derive the general formula using local charge conservation, then a simple expression of the open system current operator is recovered under translation invariance and single-particle approximation. Compared with the established result for a closed system, this expression contains an extra term for each Lindblad operator. In the demonstration with a two-dimensional two-band topological insulator at zero temperature, we show that such an extra term is nontrivial in maintaining the robustness of quantized Hall conductivity against pure dephasing and cooling in an energy basis. Moreover, we show that site-local noise described in two different ways is trivial to the formalism of the current operator, which nevertheless by no means prevents it from affecting conductivity.

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I. INTRODUCTION

Probability current is fundamental to the study of transport property in quantum mechanics. To study this current in complicated systems, a current operator is required. That is, an observable whose mean value yields the probability current (see Fig. 1). In various quantum systems and under various conditions, the formalism of the corresponding current operators have long been a focus of research [1–3]. Particularly, how to account for the effect of environment on the formalism of the current operator remains an open question [4–6]. Here, we focus on the tight-binding (TB) model [7–10]. Although its current observable has been well established [9,11] as an isolated system, a formalism for the open system counterpart has yet to be fully addressed.

The transport property of the TB model as a open system, namely the Hall conductivity of a topological insulator (TI) [12–16], has attracted wide interest [17–21]. The absence of an established formalism for the corresponding open system current operator presents a great challenge. Many works on open system Hall response simply employ the same current operator given for a closed system [22–24]. Such research is therefore confined to momentum-independent noise, which is believed to have a trivial effect on the current operator.

In this work, we formalize the current operator for an open system TB model. Under the assumption that the number of electrons conserves, our general formulation is directly based on the continuity equation, with system dynamics given by the general form of the Markovian master equation. Subsequently, by assuming discrete translational invariance and under single-particle approximation, we recovered a formula that conforms to the expression of "flux" that has been derived and examined by Avron *et al.* [25]. The effect of the memoryless

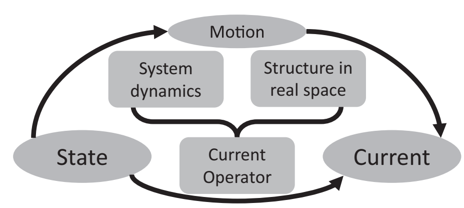

FIG. 1. Illustration of concept for current operator, which is a key characterization of the transport property of a quantum system. It is derived from the dynamics of the system, but it also relies on the structure of the system in real space.

environment on the current operator is formalized as additional terms that are each entirely given by a corresponding Lindblad operator. We show that such an open system term in the current operator is indeed trivial if the corresponding Lindblad operator is momentum independent.

As a demonstration, the open system current operator is then applied on a two-dimensional (2D) two-band TI undergoing dephasing and cooling. The robustness against noise in TIs has attracted much interest. And multiple theoretical predictions of such robustness has been made under certain noise, for observables [25,26] as well as the topology of the state [27]. Particularly, it has been shown that the Hall conductivity of a TI simulated by a photonic system is robust against Lindblad operators that are powers of the Landau-level lowering operators [26].

Here we show in an actual TI characterized by the TB model that such robustness stands against pure dephasing and cooling in the basis of energy at zero temperature as the current response is evaluated by the open system current operator, which also conforms with the result on "flux" (see Theorem 10 of Ref. [25]). Dephasing is known to affect the response of the system's density matrix to an electric field [22]. However, as we will demonstrate and analyze, the

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second term in the open system current operator cancels the effect of such an influence on the actual current response.

The remainder of this paper is organized as follows: In Sec. II, we give the general formalism of an operator the average value of which gives the mean probability current on a TB open system obeying the Markovian master equation. In Sec. III, we recover a simple expression of the current operator under translation invariance and single-particle approximation. In Sec. IV, by using quantum Hall response as an example, we demonstrate that the effect of dephasing on the current operator is not only nontrivial, but it also conspires to maintain the robustness of quantized Hall response. Finally, we conclude in Sec. V.

II. MARKOVIAN FORMALISM

We begin by generally considering a open TB system, where electrons are safely trapped within a collection of orbits. These orbits are localized in real space at discrete locations (sites), where multiple orbits are allowed to coexist. Such a structure in the real space thereby stated will be all that we need about the system Hamiltonian until Sec. III.

As for the environmental effect, in order to recover an open system current operator that produces current with the instantaneous state of the system, we restrict to the situation where the open system dynamics is entirely determined by the instantaneous state of the system. In other words, we require the open system in question to be undergoing Markovian dynamics, which is given generally by a Markovian master equation as follows:

$$\dot{\rho} = \mathcal{L}(\rho) \equiv -i[H, \rho] + \sum_j \gamma_j \left[V_j \rho V_j^\dagger - \frac{1}{2}\{V_j^\dagger V_j, \rho\} \right], \tag{1}$$

where H is the system Hamiltonian, V_j are the Lindblad operators defined on the subject system Hilbert space, and the real constant γ_j characterizes the rate of decoherence introduced by each Lindblad operator. This assumed Markovian dynamics is reasonable for a TB model under single-particle approximation and weak coupling with the environment.

Moreover, we also assume that the number of electrons conserve in the system. That is, although electrons are subjected to interaction with environmental degrees of freedom, there is no exchange of particles with the environment. Such restriction is physically possible since the environment in question could easily be photonic or phonon field. Therefore, the current and the electron number at each location must satisfy a continuity equation. Apparently, this continuity equation only relies on the electron number expectation at each site, which in turn only requires the system density matrix as follows:

$$\frac{\partial}{\partial t}\mathrm{Tr}[P_{\boldsymbol{r}}\rho(t)] + \nabla\cdot\boldsymbol{J}(\boldsymbol{r}) = 0, \tag{2}$$

where $\boldsymbol{r}$ denotes the location of a site in real space. For simplicity, this formalism above is given in the Schrodinger picture, with the instantaneous state of the open system given by density matrix $\rho(t)$. The total number of electrons at location $\boldsymbol{r}$ is produced by a time-independent operator $P_{\boldsymbol{r}} = \sum_\alpha c_\alpha^\dagger(\boldsymbol{r})c_\alpha(\boldsymbol{r})$, with the index denoting different orbits within the same site. Under a given structure in real space, the continuity equation above essentially defines the current from the motion of density matrix. Also note that $\boldsymbol{J}(\boldsymbol{r})$, the probability current of electrons at location $\boldsymbol{r}$, is not an operator here.

Current characterizes the transport of probability from one location to another, which is a natural result of state motion. The motion is in turn determined by both the nature of the system dynamics and the instantaneous state. Formalizing the current observable is essentially about simplifying the relations. To such end, we follow a procedure similar to that for the current operator in a closed system [9]. We consider the Fourier transformation of the continuity equation as follows:

$$\int d\boldsymbol{r} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\left[\frac{\partial}{\partial t}\langle P_{\boldsymbol{r}}\rangle + \nabla\cdot\boldsymbol{J}(\boldsymbol{r})\right] = \frac{\partial}{\partial t}\langle P_{\boldsymbol{q}}\rangle - i\boldsymbol{q}\cdot\boldsymbol{J}_{\boldsymbol{q}} = 0, \tag{3}$$

in which $\langle\bullet\rangle \equiv \mathrm{Tr}[\rho\,\bullet]$, $P_{\boldsymbol{q}} \equiv \int d\boldsymbol{r} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} P_{\boldsymbol{r}}$, and $\boldsymbol{J}_{\boldsymbol{q}} \equiv \int d\boldsymbol{r} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \boldsymbol{J}(\boldsymbol{r})$, where the arbitrary vector $\boldsymbol{q}$ is a dual vector of $\boldsymbol{r}$, hence also physically a wave vector.

We focus on the average current on the sample rather than the current on a particular part of the sample. From $\boldsymbol{J}_{\boldsymbol{q}}$, we directly obtain the average current in a unit-size subject sample:

$$\bar{\boldsymbol{J}} \equiv \int d\boldsymbol{r}\,\boldsymbol{J}(\boldsymbol{r}) = \lim_{\boldsymbol{q}\to 0}\boldsymbol{J}_{\boldsymbol{q}}. \tag{4}$$

We note that the average current of an actual system must be finite, so the expression above is physically guaranteed as safe from singularity at $\boldsymbol{q}\to 0$.

Considering that trace is invariant under cyclic permutations, we have

$$i\boldsymbol{q}\cdot\boldsymbol{J}_{\boldsymbol{q}} = \frac{\partial}{\partial t}\langle P_{\boldsymbol{q}}\rangle = \mathrm{Tr}[P_{\boldsymbol{q}}\mathcal{L}(\rho)] = \langle\mathcal{L}'(P_{\boldsymbol{q}})\rangle,$$
$$\mathcal{L}'(P_{\boldsymbol{q}}) \equiv i[H, P_{\boldsymbol{q}}] + \sum_j \gamma_j\left[V_j^\dagger P_{\boldsymbol{q}} V_j - \frac{1}{2}\{V_j^\dagger V_j, P_{\boldsymbol{q}}\}\right]. \tag{5}$$

Note that, compared with $\mathcal{L}$, the right acting operator and the left acting operator in each term are exchanged due to the cyclic permutation in superoperator $\mathcal{L}'$.

As illustrated in Fig. 1, a current operator is essentially a direct linear relation between the instantaneous quantum state of the system and the current its instantaneous motion presents. Continuing from Eq. (4), we arrive at an expression for average current:

$$\bar{\boldsymbol{J}} = \nabla_{\boldsymbol{q}}(\boldsymbol{q}\cdot\boldsymbol{J}_{\boldsymbol{q}})|_{\boldsymbol{q}\to 0} = -i\nabla_{\boldsymbol{q}}\langle\mathcal{L}'(P_{\boldsymbol{q}})\rangle|_{\boldsymbol{q}\to 0}. \tag{6}$$

The general formalism of current operator is thereby given as follows:

$$\hat{\boldsymbol{J}} = -i\nabla_{\boldsymbol{q}}\mathcal{L}'(P_{\boldsymbol{q}})|_{\boldsymbol{q}\to 0}. \tag{7}$$

Admittedly, this result is not yet simple enough for numerical use. However, the only assumptions employed so far are the conservation of electron number, a real-space structure in continuous limit, and system dynamics characterized by a Markovian master equation.

III. CONDITIONAL RESULT

In the following, we shall recover a simple current operator under translation invariance and single-particle approximation. Apparently, one would need an infinitely large sample of solid-state matter to actually satisfy translation invariance. Our rationale is that the limit of infinite large is a good approximation to study the nature of any sufficiently large sample.

At continuous limit and without nonlinearity, the Hamiltonian of such a TB model is most generally denoted as follows:

$$H = \int d\boldsymbol{r}d\boldsymbol{r}' \sum_{\alpha,\beta} c_\alpha^\dagger(\boldsymbol{r}) h_{\boldsymbol{r}\boldsymbol{r}'}^{\alpha\beta} c_\beta(\boldsymbol{r}'). \tag{8}$$

Depending on indices, $h_{\boldsymbol{r}\boldsymbol{r}'}^{\alpha\beta}$ is either the transitional energy between two orbits (from either the same or different site), or the free energy of an orbit at a site.

For simplicity, we employ a Fourier transform,

$$c_\alpha(\boldsymbol{r}) = \int d\boldsymbol{k} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} c_{\boldsymbol{k},\alpha}, \tag{9}$$

where $c_{\boldsymbol{k},\alpha}$ is an annihilation operator in momentum space. Translation invariance is formally given as $h_{\boldsymbol{r}\boldsymbol{r}'}^{\alpha\beta} \equiv h_{\boldsymbol{R}}^{\alpha\beta}$, where $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}'$. The Hamiltonian can thus be rewritten as follows:

$$H = \int d\boldsymbol{k} \underbrace{\sum_{\alpha,\beta} c_{\boldsymbol{k},\alpha}^\dagger h_{\boldsymbol{k}}^{\alpha\beta} c_{\boldsymbol{k},\beta}}_{H(\boldsymbol{k})}, \tag{10}$$

in which $h_{\boldsymbol{k}}^{\alpha\beta} = \int d\boldsymbol{R} e^{i\boldsymbol{k}\cdot\boldsymbol{R}} h_{\boldsymbol{R}}^{\alpha\beta}$. It is obvious from this expression that momentum conserves.

Likewise, we give the general form of a Lindblad operator, and assume that it satisfies translation invariance:

$$V_j = \int d\boldsymbol{r}d\boldsymbol{r}' \sum_{\alpha,\beta} c_\alpha^\dagger(\boldsymbol{r}) v_{\boldsymbol{r}-\boldsymbol{r}',j}^{\alpha\beta} c_\beta(\boldsymbol{r}') = \int d\boldsymbol{k} V_{\boldsymbol{k},j}, \tag{11}$$

$$V_{\boldsymbol{k},j} \equiv \sum_{\alpha,\beta} c_{\boldsymbol{k},\alpha}^\dagger c_{\boldsymbol{k},\beta} \underbrace{\int dR e^{i\boldsymbol{k}\cdot\boldsymbol{R}} v_{\boldsymbol{R},j}^{\alpha\beta}}_{v_{\boldsymbol{k},j}^{\alpha\beta}}. \tag{12}$$

We note that operators of quadratic form in the creational and annihilation operators represent completely the linear operators on the single-electron Hilbert space, hence it can be shown that the thereby assumed form of master equation can cover all dynamical semigroups on this Hilbert space [28,29]. Moreover, translational invariant Lindblad operators comply with momentum conservation as well.

Moreover, in the momentum space, using Eq. (9), we also have

$$P_{\boldsymbol{q}} = \sum_\alpha \int d\boldsymbol{k} c_{\boldsymbol{k}+\boldsymbol{q},\alpha}^\dagger c_{\boldsymbol{k},\alpha} = \sum_\alpha \int d\boldsymbol{k} c_{\boldsymbol{k}+\boldsymbol{q}/2,\alpha}^\dagger c_{\boldsymbol{k}-\boldsymbol{q}/2,\alpha}, \tag{13}$$

where arbitrary replacement $\boldsymbol{k} \to \boldsymbol{k} - \frac{\boldsymbol{q}}{2}$ is justified for being within an integration over $\boldsymbol{k}$.

We then evaluate $\mathcal{L}'(P_{\boldsymbol{q}})$. Consider that we're restricted to a single-electron Hilbert space, by using $c_{\boldsymbol{k},\alpha} c_{\boldsymbol{k}',\beta}^\dagger |\text{vac}\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\alpha\beta}|\text{vac}\rangle$ whenever it applies, we produce the following:

$$[H, P_{\boldsymbol{q}}] = c_{\boldsymbol{k}+\boldsymbol{q}/2,\alpha}^\dagger \big[h_{\boldsymbol{k}+\boldsymbol{q}/2}^{\alpha\beta} - h_{\boldsymbol{k}-\boldsymbol{q}/2}^{\alpha\beta}\big] c_{\boldsymbol{k}-\boldsymbol{q}/2,\beta}, \tag{14}$$

$$V_j^\dagger P_{\boldsymbol{q}} V_j = c_{\boldsymbol{k}+\boldsymbol{q}/2,\alpha}^\dagger \big[\big(v_{\boldsymbol{k}+\boldsymbol{q}/2,j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k}-\boldsymbol{q}/2,j}^{\gamma\beta}\big] c_{\boldsymbol{k}-\boldsymbol{q}/2,\beta}, \tag{15}$$

$$V_j^\dagger V_j P_{\boldsymbol{q}} = c_{\boldsymbol{k}+\boldsymbol{q}/2,\alpha}^\dagger \big[\big(v_{\boldsymbol{k}+\boldsymbol{q}/2,j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k}+\boldsymbol{q}/2,j}^{\gamma\beta}\big] c_{\boldsymbol{k}-\boldsymbol{q}/2,\beta}, \tag{16}$$

$$P_{\boldsymbol{q}} V_j^\dagger V_j = c_{\boldsymbol{k}+\boldsymbol{q}/2,\alpha}^\dagger \big[\big(v_{\boldsymbol{k}-\boldsymbol{q}/2,j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k}-\boldsymbol{q}/2,j}^{\gamma\beta}\big] c_{\boldsymbol{k}-\boldsymbol{q}/2,\beta}, \tag{17}$$

where index summation is assumed. We're only interested in the first-order differential around $\boldsymbol{q} \to 0$, hence it would suffice to only preserve the first order of $\boldsymbol{q}$. We give

$$h_{\boldsymbol{k}+\boldsymbol{q}/2}^{\alpha\beta} - h_{\boldsymbol{k}-\boldsymbol{q}/2}^{\alpha\beta} = \boldsymbol{q}\cdot\nabla_{\boldsymbol{k}} h_{\boldsymbol{k}}^{\alpha\beta} + O(\boldsymbol{q}^2), \tag{18}$$

$$\begin{aligned}\big(v_{\boldsymbol{k}+\boldsymbol{q}/2,j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k}-\boldsymbol{q}/2,j}^{\gamma\beta} &= \frac{\boldsymbol{q}}{2}\cdot\big[v_{\boldsymbol{k},j}^{\gamma\beta}\nabla_{\boldsymbol{k}}\big(v_{\boldsymbol{k},j}^{\gamma\alpha}\big)^* - \big(v_{\boldsymbol{k},j}^{\gamma\alpha}\big)^*\nabla_{\boldsymbol{k}} v_{\boldsymbol{k},j}^{\gamma\beta}\big]\\ &\quad + \big(v_{\boldsymbol{k},j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k},j}^{\gamma\beta} + O(\boldsymbol{q}^2),\end{aligned} \tag{19}$$

$$\begin{aligned}&\big(v_{\boldsymbol{k}+\boldsymbol{q}/2,j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k}+\boldsymbol{q}/2,j}^{\gamma\beta} + \big(v_{\boldsymbol{k}-\boldsymbol{q}/2,j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k}-\boldsymbol{q}/2,j}^{\gamma\beta}\\ &\quad = 2\big(v_{\boldsymbol{k},j}^{\gamma\alpha}\big)^* v_{\boldsymbol{k},j}^{\gamma\beta} + O(\boldsymbol{q}^2).\end{aligned} \tag{20}$$

Putting all of the above together, the expression of $\mathcal{L}'(P_{\boldsymbol{q}})$ to the first order of $\boldsymbol{q}$ reads

$$\begin{aligned}\mathcal{L}'(P_{\boldsymbol{q}}) = O(\boldsymbol{q}^2) + \int d\boldsymbol{k} \sum_{\alpha,\beta} c_{\boldsymbol{k}+\boldsymbol{q}/2,\alpha}^\dagger c_{\boldsymbol{k}-\boldsymbol{q}/2,\beta}\Big\{\big[i\boldsymbol{q}\cdot\nabla_{\boldsymbol{k}} h_{\boldsymbol{k}}^{\alpha\beta}\big]\\ + \frac{\gamma\boldsymbol{q}}{2}\cdot\sum_j\Big[v_{\boldsymbol{k},j}^{\gamma\beta}\nabla_{\boldsymbol{k}}\big(v_{\boldsymbol{k},j}^{\gamma\alpha}\big)^* - \Big(v_{\boldsymbol{k},j}^{\gamma\alpha}\Big)^*\nabla_{\boldsymbol{k}} v_{\boldsymbol{k},j}^{\gamma\beta}\Big]\Big\}.\end{aligned} \tag{21}$$

Finally, by using Eq. (7), we recover the current operator on a translational invariant TB model obeying the Markovian master equation. For more clarity, we decompose the current operator by momentum and direction. The average current in the l direction is then given by a simple summation of the current contribution expectation from each momentum components as follows:

$$\begin{aligned}\bar{J}_l &= \int d\boldsymbol{k}\,\mathrm{Tr}[\hat{J}_l(\boldsymbol{k})\rho_{\boldsymbol{k}}(t)],\\ \hat{J}_l(\boldsymbol{k}) &= \frac{\partial H(\boldsymbol{k})}{\partial k_l} - \sum_j \frac{i\gamma_j}{2}\left[\frac{\partial V_{\boldsymbol{k},j}^\dagger}{\partial k_l} V_{\boldsymbol{k},j} - V_{\boldsymbol{k},j}^\dagger \frac{\partial V_{\boldsymbol{k},j}}{\partial k_l}\right],\end{aligned} \tag{22}$$

where the density operator $\rho_{\boldsymbol{k}}(t)$ is on the subspace of Hilbert space of momentum $\boldsymbol{k}$. The first term of the current operator $\hat{J}_l(\boldsymbol{k})$ is consistent with that for a closed TB system, while the second term gives the transport of probability resulting from environment-induced dynamics.

From Eq. (12) we observe that $v_{\boldsymbol{r}\boldsymbol{r}',j}^{\alpha\beta} \equiv \delta_{\boldsymbol{r}\boldsymbol{r}'} v_{\boldsymbol{r}\boldsymbol{r}',j}^{\alpha\beta}$ would give $\nabla_{\boldsymbol{k}} V_{\boldsymbol{k},j} \equiv 0$, which is trivial in the above formalism. In other words, site-local noise, which produces momentum-independent noise, is trivial to the current operator. We note that a microscopic formalism mirroring this result is also given (Appendix A). For both sufficient conditions guaranteeing trivial open system influence on the current operator, the physical explanations are the same: The dynamics of site-local noise do not transport electrons from one site to another.

Apparently, Eq. (22), though given by a different principle, is the same as a gauge invariant formulation of the principle of virtual work with momentum as the control [25]. Moreover, we note that this expression also conforms with the mean velocity of an electron, of which we give a much simpler derivation in Appendix B, where it also leads to another validation of our formula.

The derivation here is limited to the mean current density over the bulk of a sample, but as long as the current and the charge distribution on the sample were stable, it has to reflect the current passing through a barrier on the sample, which in turn has to be consistent with the current measured through external circuitry. Also, moving electrons can also generate a magnetic field that reflects their mean velocity, which presents another way to measure the electric current. For greater access to the real-time dynamics, one can also use simulation by cold atom in optical lattice [30]. A wide range of dissipation can be implemented [31] in such a system, and the instantaneous location of a cold atom (substitution of an electron) can be more easily measured [32].

IV. DEMONSTRATION

In this section, as an example, we apply the formalism above to the evaluation of Hall conductivity.

A. Model

Consider a simple translational invariant 2D two-band TI subject to both dephasing and cooling. Without electron-electron interaction, each momentum component of this system is simply a two-level system, the master equation of which reads

$$\dot{\rho}_{\boldsymbol{k}} = -i[H_{\boldsymbol{k}}, \rho_{\boldsymbol{k}}] + \gamma_{\boldsymbol{k}}^{D}[\tilde{\sigma}_{z,\boldsymbol{k}}\rho_{\boldsymbol{k}}\tilde{\sigma}_{z,\boldsymbol{k}} - \rho_{\boldsymbol{k}}] + \gamma_{\boldsymbol{k}}^{C}[2\tilde{\sigma}_{-,\boldsymbol{k}}\rho_{\boldsymbol{k}}\tilde{\sigma}_{+,\boldsymbol{k}} - \{\tilde{\sigma}_{+,\boldsymbol{k}}\tilde{\sigma}_{-,\boldsymbol{k}}, \rho_{\boldsymbol{k}}\}], \tag{23}$$

where $\boldsymbol{k}$ denotes the momentum, $\tilde{\sigma}_{l,\boldsymbol{k}}$ is a Pauli matrix in the eigenbasis of $H(\boldsymbol{k})$, $\tilde{\sigma}_{\pm,\boldsymbol{k}} = \frac{1}{2}(\tilde{\sigma}_{x,\boldsymbol{k}} \pm i\tilde{\sigma}_{y,\boldsymbol{k}})$, and $\gamma_{\boldsymbol{k}}^{D}$ and $\gamma_{\boldsymbol{k}}^{C}$ are, respectively, rates of pure dephasing and cooling in the basis of energy at momentum $\boldsymbol{k}$.

For simplicity, we map the density matrix of each two-level system to a basis of Pauli matrix. By denoting $H_{\boldsymbol{k}} = \vec{d}_{\boldsymbol{k}} \cdot \vec{\sigma}$ and $\rho_{\boldsymbol{k}} = \frac{1}{2}(\vec{n}_{\boldsymbol{k}} \cdot \vec{\sigma} + I)$, the master equation is rewritten exactly in the following form:

$$\frac{\partial}{\partial t}\vec{n}_{\boldsymbol{k}} = 2\vec{d}_{\boldsymbol{k}} \times \vec{n}_{\boldsymbol{k}} + 2\gamma_{\boldsymbol{k}}^{DC}[(\vec{n}_{\boldsymbol{k}} \times \vec{z}_{\boldsymbol{k}}) \times \vec{z}_{\boldsymbol{k}}] - 2\gamma_{\boldsymbol{k}}^{C}[(\vec{n}_{\boldsymbol{k}} \cdot \vec{z}_{\boldsymbol{k}}) + 1]\vec{z}_{\boldsymbol{k}}, \tag{24}$$

where $\vec{z}_{\boldsymbol{k}} = \vec{d}_{\boldsymbol{k}}/|\vec{d}_{\boldsymbol{k}}|$ and $\gamma_{\boldsymbol{k}}^{DC} \equiv \gamma_{\boldsymbol{k}}^{D} + \frac{1}{2}\gamma_{\boldsymbol{k}}^{C}$. In terms of the density matrix in the eigenbasis of free energy (the first term), the second term corresponds with the loss of its nondiagonal values, and only the third term affects the population. Hence $\gamma_{\boldsymbol{k}}^{DC}$ characterizes the total rate of decoherence.

To preserve translation invariance in space, in the evaluation of Hall response, the electric field is introduced via vector potential, making the Hamiltonian time dependent instead. Formally we use the Pierels substitution [9]:

$$\boldsymbol{k} \to \boldsymbol{k} + \frac{(-e)}{\hbar}\boldsymbol{E}\, t. \tag{25}$$

The overall system must remain physically unchanged under the static electric field $\boldsymbol{E}$, so the treatment above is applied to both the Hamiltonian and the Lindblad operators. Since transport arises from instantaneous motion, the current observable at every moment is given by the instantaneousbrk dynamics.

B. Robustness

By using Lindblad operators from both cooling and pure dephasing, extensive operator and vector gymnastics produces the open system current operator as follows:

$$\hat{J}_y(\boldsymbol{k}) = \left\{\frac{\partial \vec{d}_{\boldsymbol{k}}}{\partial k_y} - \gamma_{\boldsymbol{k}}^{DC}\left[\vec{z}_{\boldsymbol{k}} \times \frac{\partial \vec{z}_{\boldsymbol{k}}}{\partial k_y}\right]\right\} \cdot \vec{\sigma} + \gamma_{\boldsymbol{k}}^{C}\mathrm{Tr}\left[\tilde{\sigma}_{x,\boldsymbol{k}}\frac{\partial \tilde{\sigma}_{y,\boldsymbol{k}}}{\partial k_y}\right]|e_{\boldsymbol{k}}\rangle\langle e_{\boldsymbol{k}}|, \tag{26}$$

where $|e_{\boldsymbol{k}}\rangle$ is the instantaneous upper energy eigenstate of the momentum $\boldsymbol{k}$ component. Note that it returns to the closed system current operator $\hat{J}_{0y}$ as $\gamma_{\boldsymbol{k}}^{D}, \gamma_{\boldsymbol{k}}^{C} \to 0$.

For an isolated two-level system, redefining each level by adding an arbitrary phase wouldn't alter the physics of the system. However, for a translational invariant two-band TB model, such a phase factor within each momentum component is acquired in Fourier transformation from the full Hamiltonian in real space, and hence is not physically trivial. Under an arbitrary cooling rate, the second line in Eq. (26) can be engineered to be either trivial or nontrivial by adjusting the phase of the basis on which the Pauli matrix $\tilde{\sigma}_x$ is defined. And for simplicity, we hereby assume the former and ignore this term.

At zero temperature and under energy-based dephasing and cooling, we follow Berry's approach [33] by solving Eq. (24) in a noninertial frame moving with $\vec{z}_{\boldsymbol{k}}$ as it became time dependent under Eq. (25). The response of density matrix to an electric field in the x direction thus reads

$$\frac{\partial \rho_{\boldsymbol{k}}^{S}}{\partial E_x} = -\left\{\left[d_{\boldsymbol{k}} + \gamma_{\boldsymbol{k}}^{DC}\vec{z}_{\boldsymbol{k}} \times \bullet\right]^{-1}\left[\vec{z}_{\boldsymbol{k}} \times \frac{\partial \vec{z}_{\boldsymbol{k}}}{\partial k_x}\right]\right\} \cdot \vec{\sigma}, \tag{27}$$

where $\rho_{\boldsymbol{k}}^{S}$ denotes the steady state of momentum $\boldsymbol{k}$, $d_{\boldsymbol{k}} = |\vec{d}_{\boldsymbol{k}}|$, and E_x denotes the electric field in the x direction. Through tedious derivations, we observe that

$$\mathrm{Tr}\left[\hat{J}_y(\boldsymbol{k})\frac{\partial \rho_{\boldsymbol{k}}^{S}}{\partial E_x}\right] = \mathrm{Tr}\left[\hat{J}_{0y}(\boldsymbol{k})\left(\frac{\partial \rho_{\boldsymbol{k}}^{S}}{\partial E_x}\right)_{\gamma_{\boldsymbol{k}}^{DC} \to 0}\right]. \tag{28}$$

Strikingly, though both the response of density matrix and the current operator are affected by decoherence, the current response to electric field remains unaffected for each momentum component. This result is consistent with that by Avron *et al.* [25].

As a comparison, we denote the analytical formalism of current response evaluated by closed system current operator

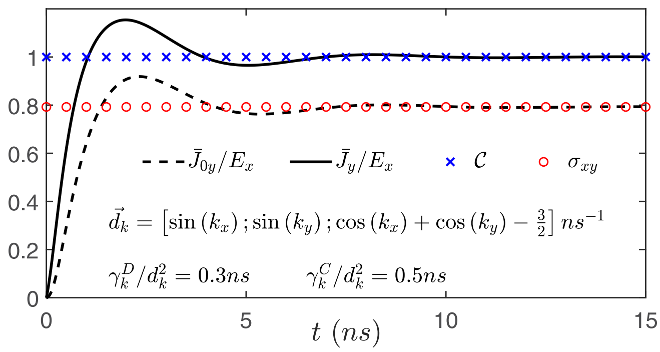

FIG. 2. Numerical simulation of open system Hall response. Units of $e = h = 1$ are employed. J_y is the current given by open system current operator, it is stabilized at Chern number $\mathcal{C}$. In comparison, J_{0y} and its analytical steady value σ_{xy} are produced by closed system current operator.

as follows:

$$\sigma_{xy} \equiv \int_{BZ} d\boldsymbol{k}\mathrm{Tr}\bigg[\hat{J}_{0y}(\boldsymbol{k})\frac{\partial \rho_{\boldsymbol{k}}^S}{\partial E_x}\bigg]. \tag{29}$$

A simple Chern insulator model is used in our simulation [13], of which the topological invariant $\mathcal{C}$ is called the Chern number. As shown in Fig. 2, when the electric field is applied, the average current $\bar{J}_{0y}$, which is evaluated by $\hat{J}_{0y}(\boldsymbol{k})$, is affected by the dephasing and cooling. It eventually is stabilized at σ_{xy}, which varies with dephasing rate and deviates from the topological invariant. However, the average current $\bar{J}_y$, as given by $\hat{J}_y$, is stabilized at the topological invariant.

C. Site-local noise

We note that site-local noise, although trivial to the expression of current operator, is nonetheless nontrivial to the Hall conductivity. As a simple example, we consider the translational invariant but site-local cooling and dephasing terms:

$$\gamma_{SL}^D[\sigma_z\rho_{\boldsymbol{k}}\sigma_z - \rho_{\boldsymbol{k}}] + \gamma_{SL}^C[2\sigma_-\rho_{\boldsymbol{k}}\sigma_+ - \{\sigma_+\sigma_-, \rho_{\boldsymbol{k}}\}], \tag{30}$$

where σ_l are the Pauli matrix in the original basis, and momentum-independent $\gamma_{SL}^D, \gamma_{SL}^C$ are rates of site-local pure dephasing and cooling, respectively. Correspondingly, in Bloch vector space, the contributions of these terms are as follows:

$$2\gamma_{SL}^{DC}[(\vec{n}_{\boldsymbol{k}} \times \vec{Z}) \times \vec{Z}] - 2\gamma_{SL}^C[(\vec{n}_{\boldsymbol{k}} \cdot \vec{Z}) + 1]\vec{Z}, \tag{31}$$

where $\vec{Z} = [0, 0, 1]$ and $\gamma_{SL}^{DC} \equiv \gamma_{SL}^D + \frac{1}{2}\gamma_{SL}^C$. Simulation with site-local noise can then be realized by appending these terms to the Bloch vector equations of motion Eq. (24), while evaluation of the current is unaffected due to the trivial effect of site-local noise on the current operator.

As shown in Fig. 3, site-local noise clearly affects the Hall conductivity. Moreover, although pure dephasing and cooling in the energy basis are trivial to the Hall conductivity at zero temperature as they are imposed alone, the same proposition does not extend to the situation where they are combined with other noise. In the presence of other forms of noise, adding

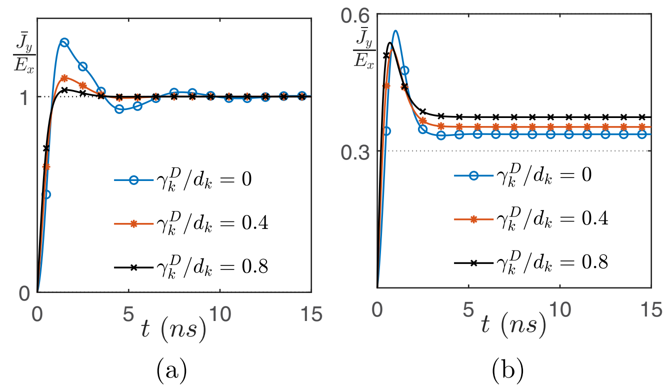

FIG. 3. Demonstration of the effect of energy-based pure dephasing in the presence of other noise. The same model and units as in Fig. 2 are used. $\gamma_{\boldsymbol{k}}^C = 0.4d_{\boldsymbol{k}}$ in both figures, whereas (a) $\gamma_{SL}^D = \gamma_{SL}^C = 0$ and (b) $\gamma_{SL}^D = \gamma_{SL}^C = 0.3\ \mathrm{ns}^{-1}$.

or subtracting the energy-basis pure dephasing rate clearly affects the Hall conductivity.

V. CONCLUSION

In conclusion, we formalized the current operator for a TB model subject to general memoryless noise under a conserved number of electrons. A general formalism for average current is produced, and a simple expression, Eq. (22), is recovered under translation invariance and single-particle approximation from two different perspectives. Though this expression has been given by other principles before, our derivation based on local charge conservation shows that it is indeed the open system current operator, and the other derivation in Appendix B shows that its mean value is consistent with the closed system current operator given by the full system Hamiltonian. As far as theoretical analysis goes, this result can serve as a solid foundation to the research on the transport property of the open system TB model beyond momentum-independent noise.

Moreover, we have also shown, from two distinct perspectives, that site-local noise is trivial to the expression of the current operator. The effect of momentum-independent noise is shown as indeed trivial to the formalism of current operator, in support of numerous previous research on open systems where the closed system current operator is directly employed. Admittedly, the definition of site-local noise in the general microscopic model is different from that in a translational invariant Markovian system without electron interactions, whereas the relation of the two remains opaque. However, these two definitions are never both applicable to the same system characterization.

Finally, upon examining Hall conductivity, we find that quantized Hall response is robust against energy-based pure dephasing, even as both the density matrix and current operator are nontrivially affected. This conforms with previous findings. However, we note that the effect of energy-based pure dephasing is not simply trivial to Hall conductivity, since we have also shown that in the context of other forms of noise, the influence on Hall current becomes nontrivial.

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APPENDIX A: MICROSCOPIC SUFFICIENT CONDITION FOR TRIVIAL OPEN SYSTEM EFFECT ON CURRENT OPERATOR

We generally consider a full system Hamiltonian $\mathcal{H} = H \otimes I_E + \sum_l A_{Sl} \otimes A_{El} + I_S \otimes H_E$, where I_S, I_E are the identity operators on the system and the environment, respectively, A_{Sl}, A_{El} are operators on the system and the environment, respectively, and H and H_E are the Hamiltonian of the system and the environment.

We then use Liouville equation with the full Hamiltonian before tracing out the environment.

The lack of particle exchange with the environment allows a straightforward extension to the full system as follows:

$$P_{\boldsymbol{r}} \rightarrow P_{\boldsymbol{r}} \otimes I_E. \tag{A1}$$

With some operator gymnastics, the full system continuity equation can eventually be unfolded as follows:

$$\mathrm{Tr}\{P_{\boldsymbol{r}}[H, \rho]\} + \sum_l \mathrm{Tr}\{[P_{\boldsymbol{r}}, A_{Sl}]F_l\} = i\nabla \cdot \boldsymbol{J}(\boldsymbol{r}). \tag{A2}$$

where $F_l \equiv \mathrm{Tr}_E\{A_{El}\rho_{SE}(t)\}$, in which ρ_{SE} is the full system density matrix. We observe that the effect of the environment is clearly included in the second term, the presence of which is clearly able to have a nontrivial influence to the value of $\boldsymbol{J}(\boldsymbol{r})$ even with the same open system state ρ. Moreover, we also observe that site-local noise, which here we define by $[P_{\boldsymbol{r}}, A_{Sl}] = 0$, is trivial to the current operator.

APPENDIX B: MEAN VELOCITY OF A SINGLE ELECTRON

The mean velocity of a single electron in the Heisenberg picture is generally given by the mean value of $\dot{X}_{SE} = i[\mathcal{H}, X_{SE}]$, where X_{SE} is the coordinate operator on the full system. Similar to Eq. (A1), the lack of particle exchange with environment gives $X_{SE} = X \otimes I_E$, with X being the coordinate operator on the system. Under Markovian dynamics, $\dot{X}$ is given as follows [22]:

$$\dot{X} = \mathcal{L}'(X) = i[H, X] + D^*(X), \tag{B1}$$

where X is the location operator. The superoperator $\mathcal{L}'$ is the same as defined in Eq. (5) due to the same permutation, hence $D^*(X) \equiv \sum_j \gamma_j [V_j^\dagger X V_j - \frac{1}{2}\{V_j^\dagger V_j, X\}]$.

Within translation invariance, in momentum representation we use $X \rightarrow i\nabla_{\boldsymbol{k}}$ and $V_j \rightarrow V_{\boldsymbol{k},j}$. Focusing on D^* and omitting the indices, to an arbitrary quantum state denoted Ψ, we have

$$\begin{aligned} 2D^*(X)\Psi &= (2V^\dagger \nabla_{\boldsymbol{k}} V - \nabla_{\boldsymbol{k}} V^\dagger V - V^\dagger V \nabla_{\boldsymbol{k}})\Psi \\ &= 2V\nabla_{\boldsymbol{k}}[V\Psi] - \nabla_{\boldsymbol{k}}[V^\dagger V \Psi] - V^\dagger V \nabla_{\boldsymbol{k}}[\Psi] \\ &= -i[(\nabla_{\boldsymbol{k}} V^\dagger)V - V^\dagger(\nabla_{\boldsymbol{k}} V)]\Psi. \end{aligned} \tag{B2}$$

The result above is obviously consistent with Eq. (22). Moreover, we can give another validation as follows: By treating the environmental degree of freedom as essentially the same as orbital degrees of freedom, we can use the established result of closed system current operator on the full system. And this derivation above can connect to this full system current operator as follows:

$$\langle \mathcal{L}'(X) \rangle = \langle \dot{X} \rangle = \langle \dot{X}_{SE} \rangle = \mathrm{Tr}\left[\rho_{SE} \int d\boldsymbol{k} \nabla_{\boldsymbol{k}} H_{SE}(\boldsymbol{k})\right], \tag{B3}$$

where from left to right the key assumptions are (i) Markovian approximation is appropriate for the system in question; (ii) lack of the exchange of particles with the environment; (iii) the full system satisfies momentum conservation $\mathcal{H} = \int d\boldsymbol{k} H_{SE}(\boldsymbol{k})$.

APPENDIX C: REPRESENTATION OF CURRENT OPERATOR UNDER COOLING IN BLOCH SPHERE

Here we outline how to obtain the Bloch representation of open system current operator for cooling. With $V(\boldsymbol{k}) = \sqrt{2}\tilde{\sigma}_-(\boldsymbol{k})$, The key derivations are as follows:

$$\begin{aligned} &\left[\frac{\partial V^\dagger(\boldsymbol{k})}{\partial k_s} V(\boldsymbol{k}) - V^\dagger(\boldsymbol{k}) \frac{\partial V(\boldsymbol{k})}{\partial k_s}\right] \\ &= i\left(\frac{\partial \vec{x}}{\partial k_s} \times \vec{x} + \frac{\partial \vec{y}}{\partial k_s} \times \vec{y}\right) \cdot \vec{\sigma} + i\left(\vec{x} \cdot \frac{\partial \vec{y}}{\partial k_s} - \frac{\partial \vec{x}}{\partial k_s} \cdot \vec{y}\right) I \\ &= i(\vec{\omega} \times \vec{x} \times \vec{x} + \vec{\omega} \times \vec{y} \times \vec{y}) \cdot \vec{\sigma} - 2i(\vec{\omega} \cdot \vec{z}) I \\ &= -i[\vec{z} \times (\vec{\omega} \times \vec{z})] \cdot \vec{\sigma} - 4i(\vec{\omega} \cdot \vec{z}) \frac{1}{2}[\vec{z} \cdot \vec{\sigma} + I], \end{aligned} \tag{C1}$$

where we denote $\vec{\omega}$, which satisfies $\frac{\partial}{\partial k_s}\vec{l} = \vec{\omega} \times \vec{l}$ where $l = x, y, z$, the subscript s denotes an arbitrary direction, and $\sigma_l = \vec{l} \cdot \vec{\sigma}$. We then have $\vec{y} \cdot \frac{\partial}{\partial k_s}\vec{x} = -\vec{x} \cdot \frac{\partial}{\partial k_s}\vec{y} = \vec{\omega} \cdot \vec{z}$, which gives the third line. Eventually, for the fouth line, where $\frac{1}{2}[\vec{z} \cdot \vec{\sigma} + I] \equiv |e\rangle\langle e|$, we depend on the following:

$$\vec{a} = \frac{1}{b^2}[\vec{b} \times (\vec{a} \times \vec{b}) + (\vec{a} \cdot \vec{b})\vec{b}], \tag{C2}$$

$$\vec{\omega} = (\vec{\omega} \cdot \vec{x})\vec{x} + (\vec{\omega} \cdot \vec{y})\vec{y} + (\vec{\omega} \cdot \vec{z})\vec{z}. \tag{C3}$$

We note that the derivation for pure dephasing is trivial. Equation (26) is thereby obtained.

APPENDIX D: ROBUSTNESS OF HALL CONDUCTIVITY UNDER PURE DEPHASING AND COOLING IN THE BASIS OF ENERGY

Here we show the robustness of Hall conductivity against pure dephasing. From Eqs. (26) and (27), the key derivations are as follows:

$$\begin{aligned} &\left[\frac{\partial \vec{d}}{\partial k_y} - \gamma\left(\vec{z} \times \frac{\partial \vec{z}}{\partial k_y}\right)\right] \cdot [d + \gamma \vec{z} \times \bullet\,]^{-1}\left[\vec{z} \times \frac{\partial \vec{z}}{dk_x}\right] \\ &= [-\gamma + \vec{d} \times \bullet\,]^{-1}[-\gamma + \vec{d} \times \bullet\,]\left[\frac{\partial \vec{d}}{\partial k_y} - \gamma\left(\vec{z} \times \frac{\partial \vec{z}}{\partial k_y}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&\cdot[d+\gamma\vec{z}\times\bullet]^{-1}\left[\vec{z}\times\frac{\partial}{dk_x}\vec{z}\right]\\
&=\frac{\vec{z}\times\left[\vec{z}\times\frac{\partial}{dk_x}\vec{z}\right]}{d^2+\gamma^2}\cdot[-\gamma+\vec{d}\times\bullet]\left[\frac{\partial\vec{d}}{\partial k_y}-\gamma\left(\vec{z}\times\frac{\partial\vec{z}}{\partial k_y}\right)\right]\\
&=\left(-\frac{\partial}{\partial k_x}\vec{z}\right)\cdot\left(\vec{z}\times\frac{\partial}{\partial k_y}\vec{z}\right),
\end{aligned}
\tag{D1}
$$

where in the fourth line we rely on a formula as follows:

$$
\begin{aligned}
&\{[\beta+\alpha\vec{z}\times\bullet]^{-1}\vec{a}\}\cdot\{[-\alpha+\beta\vec{z}\times\bullet]^{-1}\vec{b}\}\\
&\quad=\frac{1}{\beta\alpha}(\vec{z}\cdot\vec{a})(\vec{z}\cdot\vec{b})+\frac{1}{\beta^2+\alpha^2}[(\vec{z}\times\vec{a})\cdot\vec{b}],
\end{aligned}
\tag{D2}
$$

in which $\vec{a}$, $\vec{b}$ are arbitrary vectors, α, β are arbitrary parameters, and $\vec{z}$ is a unit vector. The proof of this formula is in turn tedious but straightforward.

[1] A. Bohm, Generalization of the electromagnetic current operator, Phys. Rev. D **17**, 3127 (1978).

[2] J. Shi, P. Zhang, D. Xiao, and Q. Niu, Proper Definition of Spin Current in Spin-Orbit Coupled Systems, Phys. Rev. Lett. **96**, 076604 (2006).

[3] F. Bottegoni, H.-J. Drouhin, G. Fishman, and J.-E. Wegrowe, Probability- and spin-current operators for effective Hamiltonians, Phys. Rev. B **85**, 235313 (2012).

[4] R. Gebauer and R. Car, Current in Open Quantum Systems, Phys. Rev. Lett. **93**, 160404 (2004).

[5] A. Bodor and L. Diósi, Conserved current in Markovian open-quantum systems, Phys. Rev. A **73**, 064101 (2006).

[6] J. Salmilehto, P. Solinas, and M. Möttönen, Conservation law of operator current in open quantum systems, Phys. Rev. A **85**, 032110 (2012).

[7] O. K. Andersen and O. Jepsen, Explicit, First-Principles Tight-Binding Theory, Phys. Rev. Lett. **53**, 2571 (1984).

[8] S. Banerjee, R. R. P. Singh, V. Pardo, and W. E. Pickett, Tight-Binding Modeling and Low-Energy Behavior of the Semi-Dirac Point, Phys. Rev. Lett. **103**, 016402 (2009).

[9] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).

[10] M.-H. Liu, P. Rickhaus, P. Makk, E. Tóvári, R. Maurand, F. Tkatschenko, M. Weiss, C. Schönenberger, and K. Richter, Scalable Tight-Binding Model for Graphene, Phys. Rev. Lett. **114**, 036601 (2015).

[11] J. K. Asbóth, L. Oroszlány, and A. Pályi, *A Short Course on Topological Insulators* (Springer, Cham, 2016).

[12] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu, and J. Zwanziger, *The Geometric Phase in Quantum Systems* (Springer, Berlin, 2003).

[13] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Topological field theory of time-reversal invariant insulators, Phys. Rev. B **78**, 195424 (2008).

[14] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. **82**, 3045 (2010).

[15] J. E. Moore, The birth of topological insulators, Nature (London) **464**, 194 (2010).

[16] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. **83**, 1057 (2011).

[17] A. Rivas, O. Viyuela, and M. A. Martin-Delgado, Density-matrix Chern insulators: Finite-temperature generalization of topological insulators, Phys. Rev. B **88**, 155141 (2013).

[18] H. Z. Shen, M. Qin, X. Q. Shao, and X. X. Yi, General response formula and application to topological insulator in quantum open system, Phys. Rev. E **92**, 052122 (2015).

[19] S. Vajna, B. Horovitz, B. Dóra, and G. Zaránd, Floquet topological phases coupled to environments and the induced photocurrent, Phys. Rev. B **94**, 115145 (2016).

[20] Y. Hu, P. Zoller, and J. C. Budich, Dynamical Buildup of a Quantized Hall Response from Nontopological States, Phys. Rev. Lett. **117**, 126803 (2016).

[21] W. Q. Zhang, H. Z. Shen, and X. X. Yi, Hall conductance for open two-band system beyond rotating-wave approximation, Sci. Rep. **7**, 16243 (2017).

[22] J. E. Avron, M. Fraas, G. M. Graf, and O. Kenneth, Quantum response of dephasing open systems, New J. Phys. **13**, 053042 (2011).

[23] H. Z. Shen, W. Wang, and X. X. Yi, Hall conductance and topological invariant for open systems, Sci. Rep. **4**, 6455 (2014).

[24] L. Zhou, D. Y. Tan, and J. Gong, Effects of dephasing on quantum adiabatic pumping with nonequilibrium initial states, Phys. Rev. B **92**, 245409 (2015).

[25] J. E. Avron, M. Fraas, and G. M. Graf, Adiabatic Response for Lindblad Dynamics, J. Stat. Phys. **148**, 800 (2012).

[26] V. V. Albert, B. Bradlyn, M. Fraas, and L. Jiang, Geometry and Response of Lindbladians, Phys. Rev. X **6**, 041031 (2016).

[27] J. C. Budich, P. Zoller, and S. Diehl, Dissipative preparation of Chern insulators, Phys. Rev. A **91**, 042117 (2015).

[28] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).

[29] T. F. Havel, Robust procedures for converting among Lindblad, Kraus and matrix representations of quantum dynamical semigroups, J. Math. Phys. **44**, 534 (2003).

[30] L. M. Sieberer, M. Buchhold, and S. Diehl, Keldysh field theory for driven open quantum systems, Rep. Prog. Phys. **79**, 096001 (2016).

[31] Z. Gong, S. Higashikawa, and M. Ueda, Zeno Hall Effect, Phys. Rev. Lett. **118**, 200401 (2017).

[32] I. Bloch, J. Dalibard, and S. Nascimbène, Quantum simulations with ultracold quantum gases, Nat. Phys. **8**, 267 (2012).

[33] M. V. Berry, Quantum Phase Corrections from Adiabatic Iteration, Proc. R. Soc. London, Ser. A **414**, 31 (1987).