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High-order exceptional points in ultracold Bose gases

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We show that arbitrarily-high-order exceptional points (EPs) can be achieved in a repulsively interacting two-species Bose gas in one dimension. By exactly solving the non-Hermitian two-boson problem, we demonstrate the existence of third-order EPs when the system is driven across the parity-time symmetry-breaking transition. We further address the fourth-order EPs with three bosons and generalize the results to the N-body system, where the EP order can be as high as $N + 1$. Physically, such high order originates from the intrinsic ferromagnetic correlation in spinor bosons, which causes the entire system to collectively behave as a single huge spin. Moreover, we show how to create an ultrasensitive spectral response around EPs via an interaction anisotropy in different spin channels. Our work puts forward the possibility of atomic sensors made from highly controllable ultracold gases.

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I. INTRODUCTION

One of the most remarkable feature of non-Hermitian systems, as compared to Hermitian ones, is their extreme sensitivity to external perturbations around the spectral degeneracy, which is known as the exceptional point (EP) [1–3]. For conventional degeneracy in Hermitian systems, any perturbation will produce an energy shift that at most linearly depends on the perturbation strength $\sim\epsilon$, and the shift becomes negligibly small $\sim\epsilon^n$ for high perturbation order n. In contrast, around an EP of nth order, where n is the number of energy levels that simultaneously coalesce, it has been known that the perturbation can give rise to an energy shift $\sim\epsilon^{1/n}$, which grows with increasing n and becomes greatly magnified for large n. Such a sensitive response to tiny perturbations makes the non-Hermitian EP system an ideal candidate for sensors [4–9]. In the past few years, a second-order EP ($n = 2$) has been observed in various photonic, acoustic, and atomic systems [10–31]. While higher-order EPs have been studied by a number of theoretical works [32–39], their realizations in laboratories appear to be rather difficult. Very recently, two groundbreaking experiments have successfully achieved the third-order EPs and detected the enhanced sensitivity in coupled acoustic cavities [40] and optical microring system [41]. Given the power-law growing sensitivity of EP sensors in terms of the associated EP order, the search for non-Hermitian systems with high-order EPs is strongly demanded.

In this work we show how to achieve arbitrarily-high-order EPs in an ultracold gas of spinor bosons. Specifically, we consider a two-species Bose gas in one dimension across the parity-time-reversal ($\mathcal{PT}$) symmetry-breaking transition, which can be experimentally realized by using an rf field in combination with laser-induced dissipations [42].[1] We show that in the presence of spin-independent interactions, the EP order can be as high as $N + 1$, with N the total number of bosons. Such a high order originates from the intrinsic ferromagnetic correlation in spinor bosons, which makes the entire many-body system collectively behave as a single huge spin. At these high-order EPs, the large energy degeneracy can be lifted up by fine-tuning the few-body coupling strength to be anisotropic in spin channels, which can be utilized for atomic sensors. To demonstrate these results, we start with elaborating on the third-order EP by exactly solving the non-Hermitian two-boson problem and then address the fourth-order EP with three bosons, and finally approach the many-body system.

II. TWO-BODY PROBLEM

We consider two bosons in a trapped one-dimensional (1D) system with the Hamiltonian $H = \sum_{i=1,2} H_i^{(0)} + U_{2b}$ ($\hbar = 1$ throughout the paper),

$$H_i^{(0)} = \sum_\sigma \left(-\frac{1}{2m}\frac{\partial^2}{\partial x_{i\sigma}^2} + \frac{1}{2}m\omega^2 x_{i\sigma}^2 \right) + H_i^{\mathcal{PT}},$$
$$U_{2b} = \frac{1}{2}\sum_{i\neq j}\sum_{\sigma\sigma'} g_{\sigma\sigma'}\delta(x_{i\sigma} - x_{j\sigma'}). \quad (1)$$

Here m is the mass, $x_{i\sigma}$ is the coordinate of the ith particle with spin index $\sigma = \uparrow, \downarrow$, ω is the harmonic frequency, $g_{\sigma\sigma'}$

[1]As the $\mathcal{PT}$ potential acts on the single-particle state regardless of quantum statistics, the scheme implemented in the fermionic atom in Ref. [42] equally applies to bosonic atoms.

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is the coupling strength between spin σ and σ', and the $\mathcal{PT}$-symmetric potential is written as

$$H_i^{\mathcal{PT}} = \Omega(s_{x,i} + i\Gamma s_{z,i}), \tag{2}$$

with $s_{x,y,z}$ the spin-$\frac{1}{2}$ operators. Here Ω refers to the strength of the $\mathcal{PT}$ potential and Γ is a dimensionless parameter. In the single-particle sector, a second-order EP occurs at $\Gamma = 1$, where the two energy levels coalesce and the eigenstates undergo the $\mathcal{PT}$-symmetry-breaking transition [43].

According to the Lippmann-Schwinger equation, the two-body wave function $|\Psi\rangle$ satisfies

$$|\Psi\rangle = G_E U_{2b}|\Psi\rangle, \tag{3}$$

where $G_E = (E - H_1^{(0)} - H_2^{(0)})^{-1}$ is the noninteracting Green's function and E is the eigenenergy. We only focus on the relative motion of two particles as their center-of-mass motion can be factored out. By noting that U_{2b} only acts on spin triplets, we define the relevant spin states as $|1\rangle \equiv |\uparrow\uparrow\rangle$, $|0\rangle \equiv (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, and $|-1\rangle \equiv |\downarrow\downarrow\rangle$. Accordingly, $g_{\uparrow\uparrow}, g_{\uparrow\downarrow}, g_{\downarrow\downarrow}$ can be replaced by g_1, g_0, g_{-1}, denoting the coupling strengths in $m = 1, 0, -1$ spin-triplet channels. Now we introduce three variables $\{f_m\}$ in

$$\langle x|U_{2b}|\Psi\rangle = \sum_m f_m|m\rangle\delta(x), \tag{4}$$

with x the relative coordinate of two bosons. Combining (3) and (4), we arrive at three coupled equations in terms of $\{f_m\}$, which gives the E solution by solving

$$\mathrm{Det}\left(\frac{1}{g_m}\delta_{mm'} - \langle m|G_E(0,0)|m'\rangle\right) = 0. \tag{5}$$

Here the Green's function can be expanded as

$$G_E(x,x') = \sum_n \sum_j \frac{\psi_n(x)\psi_n^*(x')}{E_{rel} - E_n - \epsilon_j} \frac{|\mu_j^R\rangle\langle\mu_j^L|}{\langle\mu_j^L|\mu_j^R\rangle}, \tag{6}$$

where $E_{rel} = E - \omega/2$, $\psi_n(x)$ is the eigenfunction for the relative motion with eigenenergy $E_n = (n + 1/2)\omega$, and $|\mu_j^R\rangle$ and $|\mu_j^L\rangle$ are the left and right spin vectors defined through $H_{\mathcal{PT}}|\mu_j^R\rangle = \epsilon_j|\mu_j^R\rangle$ and $H_{\mathcal{PT}}^\dagger|\mu_j^L\rangle = \epsilon_j^*|\mu_j^L\rangle$, respectively;[2] here $H_{\mathcal{PT}} = \sum_i H_i^{\mathcal{PT}}$. Note that the spin expansion in (6) fails at the location of the EP ($\Gamma = 1$), where the single eigenvector is inadequate to expand the whole spin space. Because of this, we have further carried out the exact diagonalization to solve the spectrum at $\Gamma = 1$ and also confirmed that the two methods give consistent results in the regime $\Gamma \neq 1$.

In Fig. 1 we plot the four lowest energy levels for isotropic interactions, $g_1 = g_0 = g_{-1} \equiv g$, in both weak-coupling [Fig. 1(a)] and strong-coupling [Fig. 1(b)] regimes. We see that in both Figs. 1(a) and 1(b), the three lowest energy levels merge at $\Gamma = 1$, beyond which the upper and lower energies start to develop imaginary parts, while all three eigenvectors also coalesce at $\Gamma = 1$. These are all characteristic features of a third-order EP. Such a third order can be further checked through the spectral response to small perturbations, as shown below.

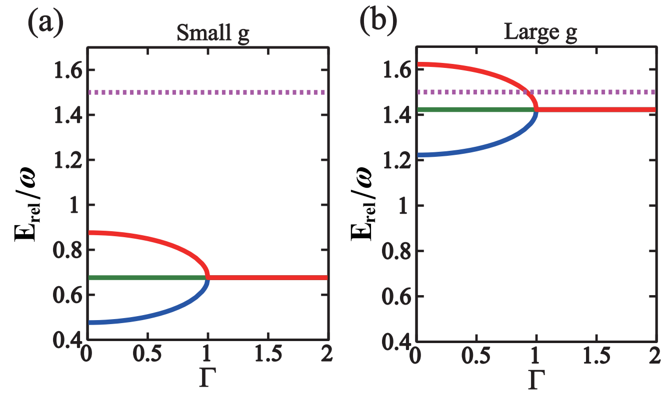

FIG. 1. Exact solution of the four lowest energy levels for two bosons with isotropic interactions $g_1 = g_0 = g_{-1} \equiv g$ for (a) weak coupling $g = 0.5\omega l$ and (b) strong coupling $g = 20\omega l$. The horizontal dashed lines in both (a) and (b) are for the spin-singlet state, which is immune to interactions. A third-order EP appears in both (a) and (b) at $\Gamma = 1$. Here $l = 1/\sqrt{m\omega}$ is the confinement length and we take $\Omega = 0.2\omega$.

We introduce external perturbations through the interaction anisotropy in spin channels, which is easy to implement in cold atoms by tuning the magnetic field. Here we take, for instance, a tiny interaction anisotropy in the $m = 0$ channel, i.e., $g_1 = g_{-1} = g$ and $g_0 = g + \epsilon$. The exact solution shows that the original degenerate energy levels at $\Gamma = 1$ split with the same amplitude $|\Delta E_{i=1,2,3}| \equiv \Delta E$. In Fig. 2(a) we plot ΔE as a function of ϵ, where a cube-root relation can be identified in the entire coupling regime:

$$\Delta E = C\epsilon^{1/3}. \tag{7}$$

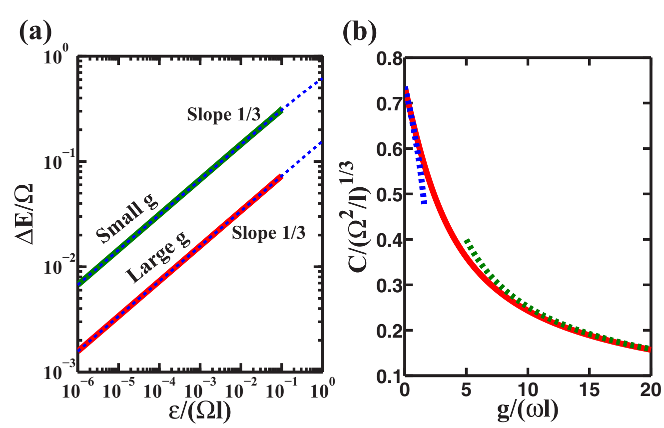

FIG. 2. (a) Energy splitting ΔE of two bosons at $\Gamma = 1$ as a function of interaction isotropy ϵ in the $m = 0$ channel. The upper green and lower red solid lines are for weak coupling $g = 0.5\omega l$ and strong coupling $g = 20\omega l$, respectively. Blue dashed lines are fitting curves according to the cube-root relation $\Delta E = C\epsilon^{1/3}$ [Eq. (7)]. (b) Coefficient C as a function of g. Blue and green dashed lines are obtained from the second-order perturbation theory in the small-g limit and the effective spin chain model in the large-g limit, respectively (see the text). Here $\Omega = 0.2\omega$.

[2]By this definition, one can prove that $\sum_j |\mu_j^R\rangle\langle\mu_j^L|/\langle\mu_j^L|\mu_j^R\rangle$ is an identity matrix expanded in spin space. This is true except at the EP (when $\Gamma = 1$).

This relation ultimately confirms the existence of the third-order EP in the two-boson system. In Fig. 2(b) we further plot the coefficient C as a function of g. The asymptotic behaviors of C in weak- and strong-g limits will be discussed later. As a comparison, we note that the ground state of the Hermitian system is threefold degenerate when $\Omega = 0$ and $\epsilon = 0$, and the introduction of anisotropic interaction would split the triple state with induced energy splitting $\Delta E \propto \epsilon$. This suggests that the energy splitting around the third-order EP is much more sensitive to the tiny anisotropic interaction than the corresponding Hermitian system, which can be exploited for ultrasensitive sensing.

A remarkable result shown above is that, given the non-Hermitian potential (2), the order of the EP at $\Gamma = 1$ can be upgraded from 2 to 3 when the boson number increases from 1 to 2. Physically, this order upgrading can be traced back to the intrinsic ferromagnetic correlation in spin-$\frac{1}{2}$ bosons [44,45]. It can be seen easily in the strong-coupling regime, where the system can be described by an effective ferromagnetic spin chain $H = -J\sum_{\langle i,j\rangle}\mathbf{s}_i\cdot\mathbf{s}_j$ $(J > 0)$ [46,47], resulting in a ferromagnetic ground state. Since the $\mathcal{PT}$ potential $H_{\mathcal{PT}}$ commutes with the total spin, the ferromagnetic state is also the eigenstate of $H_{\mathcal{PT}}$. In the case of two bosons, the ferromagnetic state is a spin triplet $(S = 1)$ with three components, and in this subspace the operators $S_\alpha = \sum_i s_{\alpha,i}$ in $H_{\mathcal{PT}}$ just behave as spin-1 operators. Equivalently, the two bosons constitute a spin-1 object, and accordingly the EP order is upgraded to $2S + 1 = 3$.

Given above picture, the energy splitting under a small interaction anisotropy [see Eq. (7)] can be analyzed by expanding the two-body Hamiltonian only in spin-triplet space. In the weak-coupling limit, a second-order perturbation theory based on an unperturbed noninteracting system gives the cube-root relation (7) with $C = [\frac{\Omega^2}{\sqrt{2\pi l}}(1 - \frac{\sqrt{2}g\gamma}{\sqrt{\pi}\omega l})]^{1/3}$, where $\gamma \approx 0.577$ is the Euler constant. In the strong-coupling limit, we resort to the effective spin-chain model for spin-$\frac{1}{2}$ bosons [47]:

$$H_{\text{eff}} = J\left(-\frac{1}{g}\mathbf{s}_1\cdot\mathbf{s}_2 - \frac{2\epsilon}{g^2}s_{z,1}s_{z,2}\right) + \sum_i H_i^{\mathcal{PT}}. \tag{8}$$

Here J is the Heisenberg coupling due to the density overlap of two neighboring bosons and we have assumed $1/g_1 - 1/g \sim -\epsilon/g^2$. Expanding (8) in spin-triplet states, we obtain ΔE, following Eq. (7), with $C = [\frac{J\Omega^2}{g^2}]^{1/3}$. These asymptotic behaviors of C in weak- and strong-g limits can fit well the exact results [see Fig. 2(b)]. In addition, we have tried interaction anisotropies in other spin channels $(m = 1, -1)$ and found that the cube-root relation and the asymptotic behaviors of C are not qualitatively altered.

III. THREE-BODY SYSTEM

We now turn to the three-boson problem. In the presence of a spin-independent interaction, it is easily drawn from previous analysis that the ground state is ferromagnetic with total spin $S = 3/2$ and the $\mathcal{PT}$ potential will result in an EP at $\Gamma = 1$ with order $2S + 1 = 4$. It is then promising to achieve an even sensitive spectral response as $\Delta E \sim \epsilon^{1/4}$, given that a proper perturbation is introduced. In the following we will show that such a fourth-root sensitivity can be induced by an anisotropy in three-body couplings.

We consider three trapped bosons experiencing small interaction anisotropy in, for instance, two-body $\uparrow\downarrow$ and/or three-body $\uparrow\downarrow\downarrow$ scattering channels. To simplify the analysis while keeping the essence of the physics, we concentrate on the strongly repulsive regime (with large two-body repulsion in all channels), where the system can be described by the following effective spin chain:

$$H_{\text{eff}} = \sum_{i=1}^{2}\left(-\frac{J}{g}\mathbf{s}_i\cdot\mathbf{s}_{i+1} + \epsilon_2 s_{z,i}s_{z,i+1}\right) + \epsilon_3 s_{z,1}s_{z,2}s_{z,3} + \sum_{i=1}^{3} H_i^{\mathcal{PT}}. \tag{9}$$

Here ϵ_2 and ϵ_3 respectively refer to the two-body and three-body interaction anisotropies.[3]

In Fig. 3 we show the spectral response for the four lowest energy levels to different types of interaction anisotropies. Depending on the anisotropy from the two-body ($\epsilon_2 \neq 0$ and $\epsilon_3 = 0$) or three-body ($\epsilon_3 \neq 0$ and $\epsilon_2 = 0$) sector, the spectral response shows distinct structures. In the case of only $\epsilon_2 \neq 0$, at $\Gamma = 1$ three different values are left for both the real and imaginary energies [see Figs. 3(a i) and 3(b i)]; accordingly, the original fourth-order EP splits into a third-order one with cube-root sensitivity and a trivial one [see Fig. 3(c i)]. In the case of $\epsilon_3 \neq 0$, the real and imaginary energies of four levels split simultaneously at $\Gamma = 1$ [see Figs. 3(a ii) and 3(b ii)] and the fourth-root scaling can be achieved [Fig. 3(c ii)]. That is to say, to maximize the spectrum sensitivity near the fourth-order EP, i.e., to realize $\Delta E \sim \epsilon^{1/4}$, a three-body interaction anisotropy is a crucial ingredient.

IV. MANY-BODY SYSTEM

Now we generalize the above discussion to a two-species boson system with arbitrary particle number N and under M-body interactions $(M \leqslant N)$. First, in the presence of spin-independent interaction which supports a ferromagnetic ground state, the system collectively behaves as a single huge spin with $S = \frac{N}{2}$ and a high EP order $2S + 1 = N + 1$ can be achieved. For the convenience of later discussion, we introduce another way to understand this result. Under a spin rotation around x, $H_{\mathcal{PT}}$ at $\Gamma = 1$ simply reproduces the spin raising operator $S_+ = S_x + iS_y$, with $S_\alpha = \sum_i s_{i,\alpha}$ the spin-$\frac{N}{2}$ operators. The S_+ operator can be expanded in $\{S_z\}$ space as an $(N+1)\times(N+1)$ matrix, which has one single eigenvalue (equal to 0) and one single eigenvector $(|S_z = \frac{N}{2}\rangle = |1, 0, \ldots, 0\rangle)$. This justifies the occurrence of the $(N+1)$th-order EP in $(N+1)$-dimensional spin space.

Second, when turning on a small M-body interaction isotropy, the original $(N+1)$th-order EP will generally split into a number of sub-EPs depending on the values of M and N. To see this, again we resort to the effective model

[3]In writing (9) we have omitted the term $\sim\epsilon_3\sum_i s_{z,i}$ as it does not contribute to the sensitive spectral response and can be eliminated by an additional tiny magnetic field.

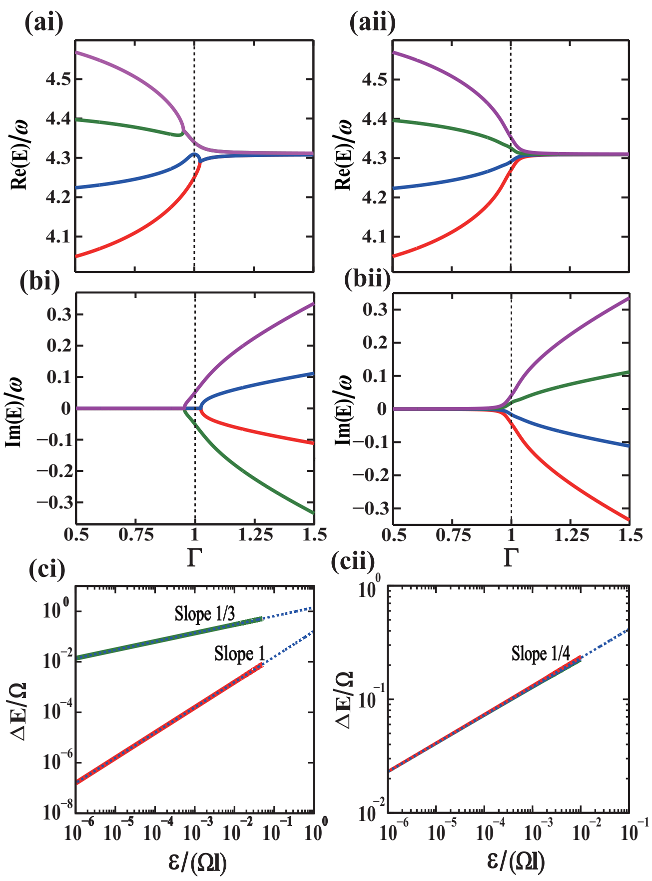

FIG. 3. Spectral response for the four lowest energy levels in the three-boson system to different types of interaction anisotropies. The real and imaginary parts of the energies are shown as a function of Γ only with two-body anisotropy (a i) $\epsilon_2/\Omega l = 0.01$ and (b i) $\epsilon_3/\Omega l = 0$, where we can find a pair of complex-conjugate eigenvalues with the same real parts and two purely real eigenvalues on the line $\Gamma = 1$, or only with three-body anisotropy (a ii) $\epsilon_3/\Omega l = 0.01$ and (b ii) $\epsilon_2/\Omega l = 0$, where two pairs of complex-conjugate eigenvalues appear simultaneously. Accordingly, the energy shift at $\Gamma = 1$ is plotted as a function of (c i) ϵ_2 (c1) or (c ii) ϵ_3. Here $\Omega = 0.2\omega$.

in the strong-coupling regime and work only within $S = \frac{N}{2}$ subspace, where the spin-dependent Hamiltonian at $\gamma = 1$ can be generally written as

$$H_{\rm SD} = \Omega(S_x + iS_z) + \epsilon \sum_i c_i \prod_{j=0}^{M-1} s_{z,i+j}. \tag{10}$$

Here c_i is the position-dependent coupling constant due to the trapping potential and we have omitted other less important terms $\sim \prod_{j=0}^{n} s_{z,i+j}$ with $n < M-1$, which produces a less sensitive spectral response. Again under a spin rotation around x, the $\mathcal{PT}$ term becomes an S_+ operator and the perturbation terms become M-rank polynomials in terms of $s_{y,i}$, which in the ferromagnetic subspace will generate terms like $S_\pm^m$ (with $m \leqslant M$). In Figs. 4(a) and 4(b) we show typical structures of the Hamiltonian matrix for $M = 2$ and $M = 3$, where the nonzero elements can at most extend to the second and the third super- or subdiagonals. According to a mathematic study in Ref. [48], this is the structure of Jordan blocks

(a)

$$\begin{pmatrix} * & * & * & 0 & \cdots & 0 & 0 \\ * & * & * & * & \ddots & \ddots & 0 \\ * & * & * & \ddots & \ddots & \ddots & \vdots \\ 0 & * & \ddots & \ddots & \ddots & * & 0 \\ \vdots & \ddots & \ddots & \ddots & * & * & * \\ 0 & \ddots & \ddots & * & * & * & * \\ 0 & 0 & \cdots & 0 & * & * & * \end{pmatrix}$$

(b)

$$\begin{pmatrix} * & * & * & * & 0 & \cdots & 0 \\ * & * & * & * & \ddots & \ddots & \vdots \\ * & * & * & \ddots & \ddots & \ddots & 0 \\ * & * & \ddots & \ddots & \ddots & * & * \\ 0 & \ddots & \ddots & \ddots & * & * & * \\ \vdots & \ddots & \ddots & * & * & * & * \\ 0 & \cdots & 0 & * & * & * & * \end{pmatrix}$$

FIG. 4. General $(N+1) \times (N+1)$ Hamiltonian matrix for N bosons in the (rotated) ferromagnetic spin space with (a) two-body and (b) three-body interaction anisotropy. Here $*$ denotes a nonzero element.

J_{N+1} with perturbations constituting the $(M+1)$-Hessenberg matrix, under which the $(N+1)$th-order EP splits into $[\frac{N+1}{M+1}]$ groups of sub-EP and each with order $M+1$. That is to say, a tiny perturbation ϵ in the M-body couplings can generate an energy splitting as $\epsilon^{1/(M+1)}$ in the eigenspectrum. This covers our previous analyses on the spectral response to the two- and three-body interaction anisotropies.

We emphasize that the present scheme of generating high-order EPs only requires the involvement of two spin states, rather than multimode (with the mode number equal to the EP order) as in previous proposals [33–38], and thus should be even practical to implement. A similar high-order EP was proposed previously in a two-site Bose-Hubbard model [32,39], where the spectral response under a specific two-body interaction was discovered up to cube-root sensitivity [32].

V. EXPERIMENTAL RELEVANCE

Experimentally, a two-species Bose gas with a nearly-spin-independent interaction can be achieved by using the lowest two hyperfine states of ^{87}Rb atoms, i.e., $|\uparrow\rangle = |F=1, m_F=0\rangle$ and $|\downarrow\rangle = |F=1, m_F=-1\rangle$, where the bare scattering lengths in different spin channels are rather close [49]. The two-body interaction anisotropy can be further fine-tuned through the magnetic field. By applying an rf field to couple these two states and tune the rf frequency to match their Zeeman splitting, the s_x term (with strength Ω) can be realized, while the third hyperfine state $|F=1, m_F=-1\rangle$ can be adiabatically eliminated in the presence of a large quadratic Zeeman shift (much greater than Ω). The non-Hermitian term is_z can be implemented by laser-induced dissipations as in a previous experiment [42].[1] To generate the three-body interaction, one can tune the magnetic field near an Efimov resonance in a particular collision channel [50,51] or directly utilize the transverse confinement to create visible three-body strengths in quasi-1D geometry [52–55]. The spectral response discussed in this work can be easily measured in a cold-atom experiment using radio-frequency spectroscopy and the magnitude of energy splitting in the presence of a tiny perturbation can be used to judge the sensitivity of the corresponding sensor system.

VI. SUMMARY

We have demonstrated the existence of arbitrarily-high-order EPs in the non-Hermitian 1D two-species Bose gas. This is facilitated by the ferromagnetic correlation in interacting spinor bosons such that the EP order directly scales as the number of bosonic atoms. Moreover, we have pointed out that a small interaction anisotropy in spin channels can be used to generate an ultrasensitive spectral response. Specifically, a two-body (three-body) interaction anisotropy is responsible for a cube-root (fourth-root) spectral response. Our work thus can serve as a guideline for making sensors based on ultracold atoms. Stimulated by this work, in the future it would be interesting to explore more intriguing physics due to the interplay of a non-Hermitian interaction and high spin.

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