Solitary waves in optical fibers governed by higher-order dispersion

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An exact solitary wave solution is presented for the nonlinear Schrödinger equation governing the propagation of pulses in optical fibers including the effects of second-, third-, and fourth-order dispersions. The stability of this solitonlike solution with a sech^2 shape is proven. The main criteria governing the existence of such stable localized pulses propagating in optical fibers are also formulated. A unique feature of these solitonlike optical pulses propagating in a fiber with higher-order dispersion is that their parameters satisfy efficient scaling relations. The main term of the perturbation theory describing ultrashort localized pulses is also presented when absorption or gain is included in the nonlinear Schrödinger equation. We anticipate that this type of stable localized pulses could find practical applications in communications, slow-light devices, and ultrafast lasers.

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I. INTRODUCTION

The nonlinear Schrödinger equation has found wide application in describing phenomena in nonlinear dispersive media where it has been used, for example, to model the dynamics of a Bose-Einstein condensate [1–3], dispersive shock waves in hydrodynamic media [4], and EM propagation in optical waveguides [5]. The application of this equation to light propagation in optical fibers has led to an understanding of a rich variety of phenomena, including optical wave breaking, modulation instability, the generation of solitary waves and optical solitons [6,7], and parabolic pulses (similaritons) [8–10]. In the majority of these applications, the nonlinear Schrödinger equation (NLSE) is used in its simplest form including only a second-order term to describe the dispersion. Recently, however, with the advent of silicon photonics and other nanophotonic technologies, it has become possible to engineer a wide variety of optical waveguides with complex dispersion profiles, which are not adequately described by a simple second-order term in the expansion of the wave number with frequency.

Solitary waves governed by second- and fourth-order dispersions only, have been studied since the 1990s [11–15]. It has been found that for some conditions these quartic solitons can have decaying oscillating tails [13–16]. These studies have been based on the assumption that the third-order dispersion is zero which has limited the experimental observation of quartic solitons [17]. The recent advent of silicon photonics has provided a way to generate waveguide structures exhibiting a wide range of dispersion profiles wherein the propagation of pulses is described by the NLSE [18–22]. Experimental and numerical evidence for pure-quartic solitons and periodically modulated propagation for the higher-order quartic soliton has also been reported [23]. Recently, optical pulses in photonic crystal wavegides have been observed exhibiting slow-light propagation at particular frequencies [24].

In this paper we present an exact stationary solitonlike solution of the generalized nonlinear Schrödinger equation with second-, third-, and fourth-order dispersion terms. This stable solitonlike pulse has a velocity which depends on all orders of dispersion. The solitonlike solution has been derived by a regular method which will be published elsewhere. The stability of this solitonlike solution is also demonstrated. In addition, we have also found an approximate solution describing ultrashort localized pulses having the same form but with an exponentially varying amplitude in the case when an absorption or gain term is included in the nonlinear Schrödinger equation. Finally, we present the main criteria for the existence of stable solitary waves propagating in optical fibers.

II. SOLITARY WAVE SOLUTION

In this section, we present an exact solitary wave solution of the generalized NLSE with higher-order dispersion terms. It is also shown that this solution is stable and the parameters of such localized optical pulses satisfy efficient scaling relations. Moreover, the derived scaling relations show that the group velocity of the sech^2 solitonlike solution can be significantly reduced for appropriate parameters of the waveguide, which may find application in developing slow-light systems.

For the standard assumptions of the slowly varying envelope, instantaneous nonlinear response, and no higher-order nonlinearities, the generalized NLSE for the pulse envelope $\psi(z,\tau)$ has the form [25–27]

$$i\frac{\partial\psi}{\partial z}=\alpha\frac{\partial^2\psi}{\partial\tau^2}+i\sigma\frac{\partial^3\psi}{\partial\tau^3}-\epsilon\frac{\partial^4\psi}{\partial\tau^4}-\gamma|\psi|^2\psi-i\frac{\mu}{2}\psi, \quad (1)$$

where z is the longitudinal coordinate, $\tau=t-\beta_1 z$ is the retarded time, $\alpha=\beta_2/2$, $\sigma=\beta_3/6$, $\epsilon=\beta_4/24$, and γ is the nonlinear parameter. The parameter $\beta_k=(d^k\beta/d\omega^k)_{\omega=\omega_0}$

is the k-order dispersion of the optical fiber, and β is the propagation constant. The last term in the NLSE describes absorption or amplification depending on the sign of parameter μ.

We have found the following exact solitary wave solution of Eq. (1) for $\mu = 0$:

$$\psi(z,\tau) = u\,\mathrm{sech}^2[w(\tau - \eta - v^{-1}z)]\exp[i(\kappa z - \delta\tau + \phi)], \quad (2)$$

where η and ϕ represent the position and phase of the stable localized pulse at $z = 0$. The amplitude and inverse temporal width of the solitary wave are given by

$$u = \sqrt{\frac{-3}{10\gamma\epsilon}}\left(\frac{3\sigma^2}{8\epsilon} - \alpha\right), \quad w = \frac{1}{4}\sqrt{\frac{4\alpha}{5\epsilon} - \frac{3\sigma^2}{10\epsilon^2}}, \quad (3)$$

where $\alpha < 0$, $\epsilon < 0$, and $8\alpha\epsilon > 3\sigma^2$ with $\gamma > 0$. The velocity v of the solitary wave in the retarded frame and the parameters δ and κ are

$$v = \frac{8\epsilon^2}{\sigma(\sigma^2 - 4\alpha\epsilon)}, \quad \delta = -\frac{\sigma}{4\epsilon}, \quad (4)$$

$$\kappa = -\frac{4}{25\epsilon^3}\left(\frac{3\sigma^2}{8} - \alpha\epsilon\right)^2 - \frac{\sigma^2}{16\epsilon^3}\left(\frac{3\sigma^2}{16} - \alpha\epsilon\right). \quad (5)$$

The substitution of the retarded time $\tau = t - \beta_1 z$ into Eq. (2) shows that δ and $\kappa + \beta_1\delta$ are the frequency and wave-number shifts, respectively. This solitary wave solution we call a soliton below for simplicity. We emphasize that this soliton does not have a nontrivial free parameter. Moreover the velocity of such solitons is fixed because the generalized NLSE is not invariant with respect to Galilean transformations. Equation (3) with $\gamma > 0$ yields the next relations $\epsilon < 0$ and $\alpha < 3\sigma^2/8\epsilon$. Hence the velocity is positive $v > 0$ when $\beta_3 < 0$, and the velocity is negative $v < 0$ when $\beta_3 > 0$. In the case when $\beta_3 = 0$ the solution reduces to that given in Ref. [12]. Equation (1) for $\mu = 0$ can also be written as

$$i\frac{\partial\psi}{\partial z} = -\frac{\delta\mathcal{H}}{\delta\psi^*}, \quad (6)$$

where $\mathcal{H}$ is the Hamiltonian of the system. The stability of this soliton solution is proven by the method developed in Ref. [15]. This method yields the stability region which is the same as the region of existence of sech^2 solitons: $\beta_2 < 0$, $\beta_4 < 0$, and $2\beta_2\beta_4 > \beta_3^2$, where β_3 can be negative, positive, or zero. The proof is based on the boundedness of the Hamiltonian for a fixed value of soliton energy and an explicit soliton solution presented in Eq. (2).

The energy E of the solitons for $\mu = 0$ is given by

$$E = \int_{-\infty}^{+\infty}|\psi(z,\tau)|^2 d\tau = \frac{4}{\gamma\sqrt{5|\epsilon|}}\left(\frac{3\sigma^2}{8\epsilon} - \alpha\right)^{3/2}. \quad (7)$$

Note that the energy E and other parameters of the solitons satisfy simple scaling relations if the dispersion parameters are defined in the form: $\beta_k = \beta_k^{(0)} q$ where $k = 2$–4 and q is a positive dimensionless parameter. In this case, the scaling relations are

$$E = E_0 q, \quad u = u_0 q^{1/2}, \quad v = v_0 q^{-1}, \quad \kappa = \kappa_0 q, \quad (8)$$

and we have $w = w_0$ and $\delta = \delta_0$. Here E_0 is given by Eq. (7) with the changes $\alpha \mapsto \alpha_0$, $\sigma \mapsto \sigma_0$, and $\epsilon \mapsto \epsilon_0$ where $\alpha_0 = \beta_2^{(0)}/2$, $\sigma_0 = \beta_3^{(0)}/6$, and $\epsilon_0 = \beta_4^{(0)}/24$. The same change is assumed for all other relations in Eq. (8). Thus if the parameter q grows the energy E of the solitons and the absolute value of the inverse velocity $|v|^{-1}$ grow proportional to parameter q. It also follows from Eq. (8) that in this case the amplitude u of the solitons grows as $q^{1/2}$. However the width w^{-1} and the frequency shift δ do not change when the parameter q grows. Note that the velocity of the solitons is given by $v_s = v/(1 + \beta_1 v)$. Hence the velocity of the solitons tends to zero when $q \to \infty$ because Eq. (8) yields the scaling relation $v_s = v_0/(q + \beta_1 v_0)$. Nevertheless, the value of the parameter q is limited in optical fibers. Thus we have demonstrated that it is possible to create a new type of solitary wave propagating with reduced speed and high energy with suitable dispersion profiles.

We anticipate that the scaling feature of sech^2 solitons can find various practical applications. As an example, tunable all-optical delay systems that dynamically manipulate the group velocity of light have received a great deal of attention for optical information processing applications, such as data buffering and synchronization. Various slow-light devices have been explored as potential realizations of a practical delay system [28–32].

III. ENERGY AND MOMENTUM INTEGRALS OF MOTION

In this section, we consider the energy and momentum integrals of motion of sech^2 solitonlike solution in the dimensionless form. It is efficient to reduce the number of parameters of the NLSE using appropriate dimensionless variables. Without loss of generality we can define the next new variables,

$$\psi(z,\tau) = QU(\zeta,\xi), \quad Q = \frac{|\alpha|}{\sqrt{\gamma|\epsilon|}}, \quad (9)$$

where $\zeta = z/l$ and $\xi = \tau/\tau_0$. We also define here the length $l = |\epsilon|/\alpha^2$ and time $\tau_0 = \sqrt{\epsilon/\alpha}$ with $\alpha < 0$ and $\epsilon < 0$. In this case Eq. (1) has the dimensionless form

$$i\frac{\partial U}{\partial\zeta} = -\frac{\partial^2 U}{\partial\xi^2} + i\lambda\frac{\partial^3 U}{\partial\xi^3} + \frac{\partial^4 U}{\partial\xi^4} - |U|^2 U - i\frac{\Gamma}{2}U, \quad (10)$$

where $\lambda = \sigma/\sqrt{\alpha\epsilon}$ and $\Gamma = \mu l = \mu|\epsilon|/\alpha^2$ are two dimensionless parameters. We emphasize that λ does not depend on the parameter q when we consider the scaling relations given in Eq. (8). However the dimensionless parameter $\Gamma = \Gamma_0 q^{-1}$ tends to zero when $q \to \infty$.

In the case when $\Gamma = 0$ the soliton solution of Eq. (10) depends on a single fiber parameter λ and has the form

$$U(\zeta,\xi) = u_\lambda \mathrm{sech}^2\big[w_\lambda\big(\xi - \xi_0 - v_\lambda^{-1}\zeta\big)\big]\exp[i\Phi(\zeta,\xi)], \quad (11)$$

where $\lambda^2 < 8/3$ and ξ_0 is the position of the soliton at $\zeta = 0$. The dimensionless inverse velocity and the phase of the soliton are

$$v_\lambda^{-1} = \lambda^3/8 - \lambda/2, \quad \Phi(\zeta,\xi) = k_\lambda\zeta - d_\lambda\xi + \phi. \quad (12)$$

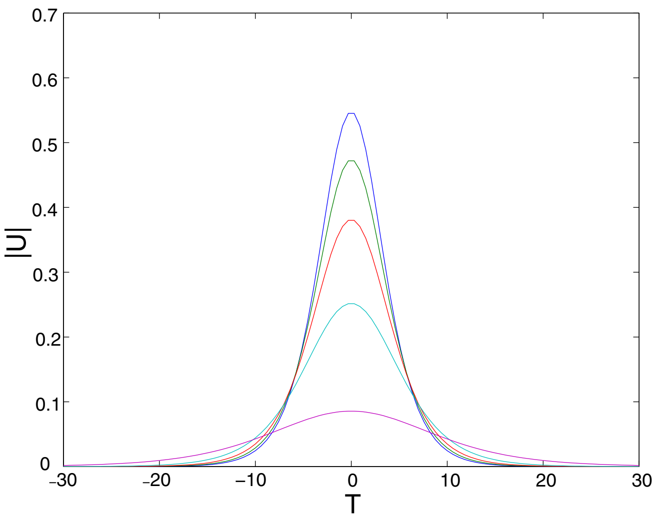

FIG. 1. Shape $|U| = u_\lambda \mathrm{sech}^2(w_\lambda T)$ of the soliton for $\lambda = \lambda_n$: $\lambda_0 = 0$, $\lambda_1 = \pm 0.6$, $\lambda_2 = \pm 0.9$, $\lambda_3 = \pm 1.2$, and $\lambda_4 = \pm 1.5$. Peak of amplitude monotonically decreases for increasing $|\lambda_n|$.

The amplitude u_λ and inverse width w_λ of the soliton are

$$u_\lambda = \sqrt{\frac{3}{10}}\left(1 - \frac{3\lambda^2}{8}\right), \quad w_\lambda = \sqrt{\frac{1}{20}\left(1 - \frac{3\lambda^2}{8}\right)}. \tag{13}$$

Thus the amplitude u_λ and width w_λ^{-1} of the soliton are related by $u_\lambda w_\lambda^{-2} = 2\sqrt{30}$. The functions k_λ and d_λ connected to the wave number and frequency shifts of the soliton are given by

$$k_\lambda = \frac{4}{25}\left(\frac{3\lambda^2}{8} - 1\right)^2 + \frac{\lambda^2}{16}\left(\frac{3\lambda^2}{16} - 1\right), \quad d_\lambda = \frac{\lambda}{4}. \tag{14}$$

In Fig. 1 we show the shape $|U|$ of solitons in Eq. (11) for different values of dimensionless parameter λ: $\lambda_0 = 0$, $\lambda_1 = \pm 0.6$, $\lambda_2 = \pm 0.9$, $\lambda_3 = \pm 1.2$, and $\lambda_4 = \pm 1.5$. We also plot in Fig. 2 the inverse velocity v_λ^{-1} and the inverse temporal width w_λ of the solitons for the region $|\lambda| < \sqrt{8/3}$. The

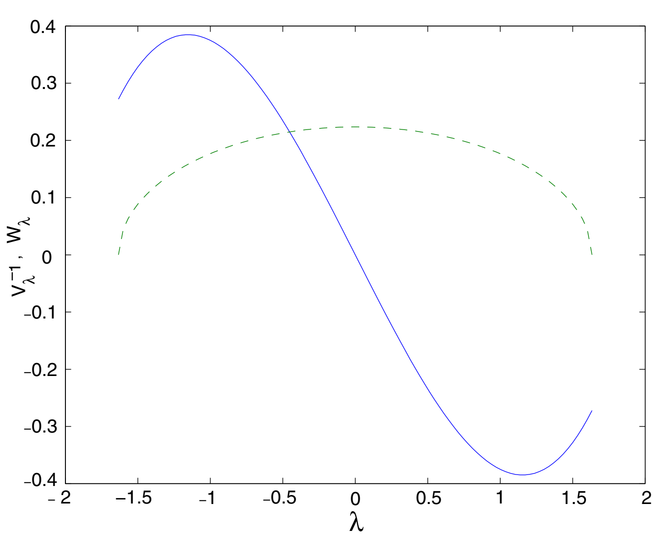

FIG. 2. Inverse velocity v_λ^{-1} (solid line) and inverse temporal width w_λ (dashed line) of the soliton for $-\sqrt{8/3} < \lambda < \sqrt{8/3}$.

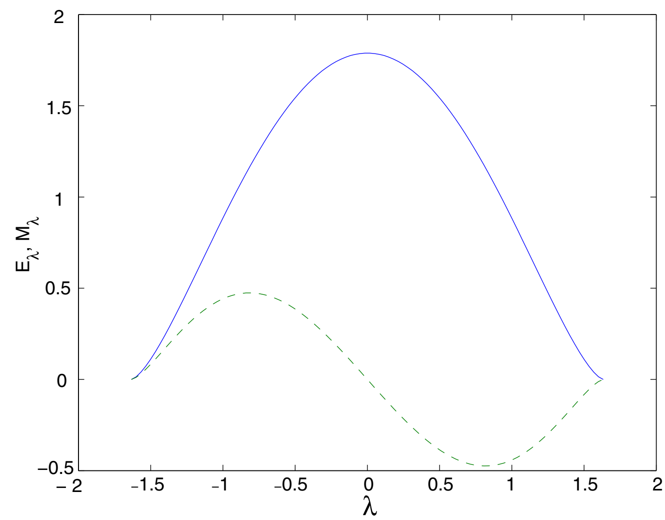

FIG. 3. Energy E_λ (solid line) and momentum M_λ (dashed line) of the soliton for interval $-\sqrt{8/3} < \lambda < \sqrt{8/3}$.

dimensionless energy of the soliton is the integral of motion when $\Gamma = 0$. In this case we have

$$E_\lambda = \int_{-\infty}^{+\infty} |U|^2 d\xi = \frac{4}{\sqrt{5}}\left(1 - \frac{3\lambda^2}{8}\right)^{3/2}. \tag{15}$$

Another integral of motion when $\Gamma = 0$ is the momentum. The dimensionless momentum of the soliton is

$$M_\lambda = \int_{-\infty}^{+\infty} i\left(U\frac{\partial U^*}{\partial \xi} - U^*\frac{\partial U}{\partial \xi}\right)d\xi$$
$$= -\frac{2\lambda}{\sqrt{5}}\left(1 - \frac{3\lambda^2}{8}\right)^{3/2}. \tag{16}$$

Hence we have the relation $M_\lambda = -(\lambda/2)E_\lambda$. In Fig. 3, the energy E_λ and momentum M_λ of the solitons are plotted over the region $|\lambda| < \sqrt{8/3}$.

IV. ABSORPTION AND AMPLIFICATION OF OPTICAL PULSES

In this section, the perturbation theory based on scaling transformation is applied to Eq. (10). Thus we consider below Eq. (10) where the last term on the right side describes absorption or amplification of the pulses in the optical fiber for $\Gamma > 0$ and $\Gamma < 0$, respectively. It is shown in Appendices A and B that in the case when $|\Gamma| \ll 1$ the perturbation method based on scaling transformation leads to efficient procedure. In this approach the scaling transformation is given by

$$U(\zeta, \xi) = F(\zeta)V(Z, T), \tag{17}$$

where the function $V(Z, T)$ has the next expansion,

$$V(Z, T) = V_0(Z, T) + \Gamma V_1(Z, T) + \Gamma^2 V_2(Z, T) + \cdots. \tag{18}$$

It is shown in Eqs. (A12) and (A13) (see also Appendix B) that scaling function has the form $F(\zeta) = e^{-\Gamma\zeta}$ and new variables Z and T are

$$Z = \frac{1}{2\Gamma}(1 - e^{-2\Gamma\zeta}), \quad T = T_0 + e^{-\Gamma\zeta}(\xi - \xi_0). \tag{19}$$

The transformation of Eq. (10) by Eqs. (17) and (19) yields the equation for the function $V(Z, T)$ as

$$i\frac{\partial V}{\partial Z} + \frac{\partial^2 V}{\partial T^2} - i\lambda\sqrt{1-2\Gamma Z}\frac{\partial^3 V}{\partial T^3} - (1-2\Gamma Z)\frac{\partial^4 V}{\partial T^4} + |V|^2 V = \frac{i\Gamma}{1-2\Gamma Z}\left[\frac{V}{2} + (T - T_0)\frac{\partial V}{\partial T}\right]. \tag{20}$$

We assume that two conditions are satisfied: $|\Gamma| \ll 1$ and $2|\Gamma|Z \ll 1$. The second inequality is $|1 - e^{-2\Gamma\zeta}| \ll 1$, and hence it yields the next relation $2|\Gamma|\zeta \ll 1$.

Equations (18) and (20) and the above two conditions lead to the next equation,

$$i\frac{\partial V_0}{\partial Z} = -\frac{\partial^2 V_0}{\partial T^2} + i\lambda\frac{\partial^3 V_0}{\partial T^3} + \frac{\partial^4 V_0}{\partial T^4} - |V_0|^2 V_0. \tag{21}$$

Equation (17) and new variables $Z(\zeta)$ and $T(\zeta, \xi)$ defined in Eq. (19) yield the approximate solution of Eq. (10) to the first order in small parameter $|\Gamma|$ as

$$U(\zeta, \xi) = \exp(-\Gamma\zeta)V_0[Z(\zeta), T(\zeta, \xi)], \tag{22}$$

where the function $V_0(Z, T)$ is defined in Eq. (21). We emphasize that the scaling function $F(\zeta) = e^{-\Gamma\zeta}$ and new variables Z and T in Eq. (19) are found under condition that Eq. (21) for the function $V_0(Z, T)$ has the same form as Eq. (10) with $\Gamma = 0$. Hence the solitary wave solution of Eq. (21) is given by Eq. (11) with the changes $\zeta \mapsto Z$ and $\xi - \xi_0 \mapsto T - T_0$. Using this exact solution of Eq. (21) we can write by Eq. (22), the approximate solution of Eq. (10), to the first order in small parameter $|\Gamma| \ll 1$ as

$$U(\zeta, \xi) = u_\lambda e^{-\Gamma\zeta}\text{sech}^2\{w_\lambda[e^{-\Gamma\zeta}(\xi - \xi_0) - v_\lambda^{-1}f(\zeta)]\} \times \exp[ik_\lambda f(\zeta) - id_\lambda e^{-\Gamma\zeta}(\xi - \xi_0) + i(\phi - d_\lambda\xi_0)], \tag{23}$$

where $f(\zeta) = (1 - e^{-2\Gamma\zeta})/2\Gamma$. It is also assumed here that the condition $2|\Gamma|\zeta \ll 1$ is satisfied. The transformation of the solution in Eq. (23) to the function $\psi(z, \tau)$ by Eq. (9) yields the approximate solution of Eq. (1) as

$$\psi(z, \tau) = ue^{-\mu z}\text{sech}^2\{we^{-\mu z}[\tau - \eta - v^{-1}\mu^{-1}\text{sh}(\mu z)]\} \times \exp[i\kappa g(z) - i\delta e^{-\mu z}(\tau - \eta) + i(\phi - \delta\eta)], \tag{24}$$

where $g(z) = (1 - e^{-2\mu z})/2\mu$. It is assumed that two conditions are satisfied: $|\mu\epsilon|/\alpha^2 \ll 1$ and $2|\mu|z \ll 1$. Equation (24) for different signs of the parameter μ describes decay ($\mu > 0$) or amplification ($\mu < 0$) of the localized pulses. It follows from Eq. (24) that the initial pulse at $z = 0$ is given by the solitary wave solution in Eq. (2). In the limit when $\mu \to 0$ the solution in Eq. (24) tends to the exact solution given in Eq. (2). It also follows from Eq. (24) that the velocity of the peak amplitude $|\psi(z, \tau)|$ of the pulses in the retarded frame is $v(z) = v\,\text{sech}(\mu z)$.

Equation (1) leads to the differential equation for the energy $E(z)$ of localized pulses as

$$\frac{dE(z)}{dz} = -\mu E(z), \quad E(z) = \int_{-\infty}^{+\infty} |\psi(z, \tau)|^2 d\tau. \tag{25}$$

This exact equation has the solution $E(z) = Ee^{-\mu z}$ where the energy E is given in Eq. (7). It is worth noting that the approximate solution in Eq. (24) leads to the same energy $E(z)$ as the exact Eq. (25). It has been observed by numerical simulations [33] that perturbation of the NLSE in such systems does not destroy the form of the ultrashort pulses for enough long propagating distances. However the energy of the pulses is changing as in Eq. (25). These numerical results confirm that Eq. (1) leads to the production of stable short pulses with changing energy which is consistent with our approximate analytical solution.

In the case when the velocity $v_s = (v^{-1} + \beta_1)^{-1}$ of the localized pulses is negative, it is useful to change the coordinate system to the inverse direction. It can be shown that such a transformation in the solution $\psi(z, \tau)$ of Eq. (1) is given by $\psi \mapsto \psi^*$, $z \mapsto -z$, $\beta_1 \mapsto -\beta_1$, $\beta_3 \mapsto -\beta_3$, and $\mu \mapsto -\mu$. The solutions in Eqs. (2) and (24) are invariant to this transformation because the phase ϕ is an arbitrary constant. However, the velocity v in the retarded frame and the parameter β_1 change sign after this transformation, and then the velocity of the localized pulses v_s becomes positive.

Note that we have neglected in the generalized NLSE the Raman and higher-order nonlinear effects which lead to the next necessary condition $w^{-1} > \tau_c$ for the pulse width. Hence the width of pulses is restricted by some characteristic time τ_c depending on the fiber parameters. Moreover Eq. (13) leads to the relation $\lambda^2 < 8/3$ which is a necessary condition for the existence of the soliton solution. These two criteria for the existence of sech^2 solitons can be written as

$$2\beta_2\beta_4 > \beta_3^2, \quad |\beta_4|(2\beta_2\beta_4 - \beta_3^2)^{-1/2} > \sqrt{0.3}\tau_c, \tag{26}$$

where $\beta_2 < 0$ and $\beta_4 < 0$. Note that the dispersion parameters of silicon-based structures (for example, those in Ref. [17]) satisfy the criteria in Eq. (26) for appropriate geometry and materials of the structures. We also emphasize that in the case when $\beta_3 < 0$ the velocity v_s of the soliton is positive and the inequality $v_s < \beta_1^{-1}$ is satisfied. In the limiting case when $\beta_3 = 0$, we have the relation $v_s = \beta_1^{-1}$ where β_1^{-1} is the group velocity.

V. DISCUSSION AND CONCLUSION

The nonlinear Schrödinger equation is a powerful mathematical model which has found wide application in the nonlinear description of many different physical systems, including water waves, Bose-Einstein condensates, and plasmas. One of the most important applications of the equation, however, has been in the description of pulse propagation in single-mode optical waveguides. In this application, by far the most widespread use of the equation has been in the simplest form where the dispersion of the waveguide is characterized only up to the second order. In this form the equation predicts many well-characterized phenomena including soliton propagation, modulation instability, self-steepening, and parabolic pulse propagation for various appropriate combinations of the nonlinearity and second-order dispersion parameters.

The dispersion of conventional single-mode optical fiber waveguides can be well approximated by an expansion up to second order around a central wavelength, and consequently, pulse propagation is appropriately modeled by the NLSE with only a second-order dispersion term. It is now possible, however, to generate optical waveguides with a wide range of dispersion profiles either by drawing optical fibers containing an array of holes or more recently by using silicon photonics and slot-based waveguides. For waveguides with complex dispersion profiles, it is becoming increasingly important to consider higher-order terms in the expansion describing the dispersion of the waveguide, particularly, in the region close to the zero group-velocity dispersion (GVD) wavelength, and consequently, to look for solutions of the NLSE containing higher-order dispersion terms.

Twenty-five years ago, Höök and Karlsson [11] considered the formation of pulses in a waveguide characterized by second- and fourth-order dispersion terms only. This equation is appropriate when the third-order dispersion term vanishes, that is, at a local minimum or maximum of the GVD. They noted that an exact solitary solution to this equation existed having a sech^2 shape. These quartic solitons as they became known have mainly been regarded as having mathematical interest only as it was immediately recognized that the solution cannot apply as soon as any perturbation (for example, Raman-induced intrapulse self-frequency shifting) introduced a frequency offset from the point where the third-order dispersion vanishes. Consequently, this solution has been largely ignored in the past two decades, although it has recently been the subject of renewed interest with the development of silicon-photonics-enabled waveguides.

We present here an exact solitary wave solution that applies for all orders of dispersion up to the fourth order and is not subject to the problem discussed above. Indeed, although this general solution reduces to that found earlier when the third-order dispersion vanishes, more importantly, the solution exhibits interesting behavior when the third-order dispersion is nonzero. In this case the velocity of the pulse can be significantly reduced, enabling slow-light pulse propagation by appropriate manipulation of the dispersion parameters. The energy of the exact solution also grows as the group velocity decreases, leading to other interesting potential applications. We also show by using a perturbation technique that, in the presence of gain or loss, the optical pulse remains stable and propagates with an exponentially growing or decaying amplitude. It is important to note that this solution contains no free parameters, unlike the familiar sech-shaped soliton family, which has a well-known amplitude-width relationship. As with the normal soliton solution, the sech^2 soliton exists only in the anomalous dispersion regime subject to relationships among the three dispersion parameters. These constraints, however, have already been shown to be met by previously characterized silicon-photonics-enabled waveguides, which can therefore support these solutions.

In addition to demonstrating the existence of this sech^2 solution, we have shown that the pulses are stable and by using a perturbation approach we have shown that the solution also applies in the more realistic situation when absorption or gain terms are included in the NLSE. We anticipate that these solitary waves could find significant application in optical systems, whereas the solution may also find application in the other areas of application of the nonlinear Schrödinger equation.

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APPENDIX A: SCALING TRANSFORMATION

The perturbation theory for the generalized nonlinear Schrödinger equation can be developed using a scaling transformation (ST). The ST depends on the form of the NLSE and the perturbation term. First, we demonstrate this perturbation method using the NLSE in its simplest form including only the second-order term to describe the dispersion. This allows us to compare this approach with well-known results. The more complicated form of the NLSE with different perturbation terms can be treated using the same method with an appropriate ST. Thus we consider in this appendix the normalized NLSE with the perturbation term as

$$i\frac{\partial U}{\partial \zeta} + \frac{1}{2}\frac{\partial^2 U}{\partial \xi^2} + |U|^2 U = -i\frac{\Gamma}{2}U, \tag{A1}$$

where the dimensional parameter $\Gamma > 0$ or $\Gamma < 0$ describes absorption or amplification of the pulses in an optical fiber. This NLSE with the perturbation term on the right side has been studied experimentally [34] and theoretically [35,36]. In the general case, the perturbed solution of the NLSE can be written in the form

$$U(\zeta, \xi) = F(\zeta)V(Z, T), \tag{A2}$$

where $Z = Z(\zeta)$, $T = T(\zeta, \xi)$, and $F(\zeta)$ is a real function of dimensionless propagation distance ζ. Equations (A1) and (A2) lead to the next equation for the function $V(Z, T)$,

$$i\frac{\partial V}{\partial Z}\frac{dZ}{d\zeta} + \frac{1}{2}\frac{\partial^2 V}{\partial T^2}\left(\frac{\partial T}{\partial \xi}\right)^2 + F^2|V|^2 V = -\frac{i}{2}\Gamma V - \frac{i}{F}\frac{dF}{d\zeta}V - i\frac{\partial V}{\partial T}\frac{\partial T}{\partial \zeta} - \frac{1}{2}\frac{\partial V}{\partial T}\frac{\partial^2 T}{\partial \xi^2}. \tag{A3}$$

In the perturbation method, we use the next expansion,

$$V(Z, T) = V_0(Z, T) + \Gamma V_1(Z, T) + \Gamma^2 V_2(Z, T) + \cdots, \tag{A4}$$

where $|\Gamma| \ll 1$. We also require that the equation for the function $V_0(Z, T)$ would have the same form as Eq. (A1) with $\Gamma = 0$. This requirement yields the next relations for new variables Z and T and scaling function $F(\zeta)$,

$$\frac{dZ}{d\zeta} = \left(\frac{\partial T}{\partial \xi}\right)^2, \quad \frac{dZ}{d\zeta} = F^2, \tag{A5}$$

where $F \to 1$ and $Z \to \zeta$ at $\Gamma \to 0$. Equation (A5) leads to the equation $F = \pm\partial T/\partial\xi$. The solution of this equation with positive sign ($F = \partial T/\partial\xi$) and the boundary condition $T = T_0$ at $\xi = \xi_0$ is

$$T = T_0 + F(\zeta)(\xi - \xi_0). \tag{A6}$$

We have used the positive sign in the above equation because we require that $F \to 1$ and $T - T_0 \to \xi - \xi_0$ at $\zeta \to 0$. It is shown below that $dF(\zeta)/d\zeta = -\Gamma F(\zeta)$ which yields the relation $dF/d\zeta = O(\Gamma)$. This relation and Eq. (A6) lead to the next relation $\partial T/\partial\zeta = O(\Gamma)$. Moreover Eq. (A6) leads to the equation $\partial^2 T/\partial\xi^2 = 0$, and hence the right side of Eq. (A3) has the first order to small parameter $|\Gamma|$. However this is correct only for restricted distances as $2|\Gamma|\zeta \ll 1$ which we show below.

Thus Eqs. (A3)–(A5) lead to the equation for the function $V_0(Z, T)$ as

$$i\frac{\partial V_0}{\partial Z} + \frac{1}{2}\frac{\partial^2 V_0}{\partial T^2} + |V_0|^2 V_0 = 0. \tag{A7}$$

Equation (A7) has the integral of motion,

$$\mathcal{E}_0 = \int_{-\infty}^{+\infty} |V_0(Z,T)|^2 dT, \quad \frac{d\mathcal{E}_0}{dZ} = 0. \tag{A8}$$

This is correct for any solution of Eq. (A7) decreasing to zero at $T \to \pm\infty$ and integrable on L^2. The normalized energy of the optical pulses is

$$E(\zeta) = \int_{-\infty}^{+\infty} |U(\zeta,\xi)|^2 d\xi = F(\zeta)\int_{-\infty}^{+\infty} |V(Z,T)|^2 dT. \tag{A9}$$

We have used here the relation $d\xi = dT/F(\zeta)$ (for the fixed variable ζ) which follows from Eq. (A6). Equation (A1) leads to the well-known equation for normalized energy,

$$\frac{dE(\zeta)}{d\zeta} = -\Gamma E(\zeta). \tag{A10}$$

Equations (A9) and (A10) in the first order to small parameter $|\Gamma|$ are

$$E_0(\zeta) = F(\zeta)\int_{-\infty}^{+\infty} |V_0(Z,T)|^2 dT,$$
$$\frac{dE_0(\zeta)}{d\zeta} = -\Gamma E_0(\zeta). \tag{A11}$$

Thus Eqs. (A8) and (A11) yield the next equations: $E_0(\zeta) = \mathcal{E}_0 F(\zeta)$ and $dF(\zeta)/d\zeta = -\Gamma F(\zeta)$. The last equation with the condition $F \to 1$ at $\Gamma \to 0$ leads to the function $F(\zeta)$,

$$F(\zeta) = \exp(-\Gamma\zeta). \tag{A12}$$

We assume that the next condition is satisfied: $Z \to \zeta$ at $\Gamma \to 0$. In this case Eqs. (A5), (A6), and (A12) yield the functions $Z(\zeta)$ and $T(\zeta,\xi)$ as

$$Z(\zeta) = \frac{1}{2\Gamma}(1 - e^{-2\Gamma\zeta}), \quad T(\zeta,\xi) = T_0 + e^{-\Gamma\zeta}(\xi - \xi_0). \tag{A13}$$

We emphasize that the scaling function $F(\zeta)$ and new variables Z and T are defined under the condition that the equation for the function $V_0(Z, T)$ would have the same form as Eq. (A1) at $\Gamma = 0$. This is a basic principle of our approach based on scaling transformation. Thus the solution of Eq. (A7) is the same as solution of Eq. (A1) for $\Gamma = 0$ with the appropriate change in variables: $\zeta \mapsto Z$ and $\xi - \xi_0 \mapsto T - T_0$.

APPENDIX B: FIRST-ORDER SOLUTIONS

In this appendix we present the general form of the solutions of the NLSE to the first order in the small parameter $|\Gamma|$. This general form given by Eq. (B3) can also be applied to the generalized NLSE with higher-order dispersion. Equations (A1) and (A3) with the function $F(\zeta) = \exp(-\Gamma\zeta)$ and variables Z and T in Eq. (A13) lead to the next equation,

$$i\frac{\partial V}{\partial Z} + \frac{1}{2}\frac{\partial^2 V}{\partial T^2} + |V|^2 V = \frac{i\Gamma}{1 - 2\Gamma Z}\left(\frac{V}{2} + (T - T_0)\frac{\partial V}{\partial T}\right). \tag{B1}$$

We assume that the function $V_0(Z, T)$ is decreasing to zero at $T \to \pm\infty$ and integrable on L^2. Moreover we also assume that two conditions are satisfied: $|\Gamma| \ll 1$ and $2|\Gamma|Z \ll 1$. The last condition can also be written as $2|\Gamma|\zeta \ll 1$. In this case Eq. (B1) and the expansion in Eq. (A4) lead to the system of equations for the functions $V_n(Z, T)$ with $n = 0, n = 1, n = 2, \ldots$. As an example, the functions $V_0(Z, T)$ and $V_1(Z, T)$ are given by Eq. (A7) and the next equation,

$$i\frac{\partial V_1}{\partial Z} + \frac{1}{2}\frac{\partial^2 V_1}{\partial T^2} + 2|V_0|^2 V_1 + V_0^2 V_1^*$$
$$= i\left(\frac{V_0}{2} + (T - T_0)\frac{\partial V_0}{\partial T}\right). \tag{B2}$$

Thus, if the conditions $|\Gamma| \ll 1$ and $2|\Gamma|\zeta \ll 1$ are satisfied, the approximate solution of Eq. (A1) to the first order in the small parameter $|\Gamma|$ is

$$U(\zeta,\xi) = F(\zeta)V_0[Z(\zeta), T(\zeta,\xi)], \tag{B3}$$

where the function $V_0(Z, T)$ is a solution of Eq. (A7). Moreover, we require that this function is integrable on L^2. For an example, the soliton solution of Eq. (A7) has the form

$$V_0(Z,T) = A\,\mathrm{sech}[A(T - T_0) + BZ]\exp[i(A^2 - B^2)Z/2$$
$$- iB(T - T_0) + i\Phi_0]. \tag{B4}$$

In this case Eqs. (A13), (B3), and (B4) yield the approximate solution of Eq. (A1) to the first order in small parameter $|\Gamma|$ as

$$U(\zeta,\xi) = Ae^{-\Gamma\zeta}\mathrm{sech}[Ae^{-\Gamma\zeta}(\xi - \xi_0) + D(1 - e^{-2\Gamma\zeta})]$$
$$\times \exp[iC(1 - e^{-2\Gamma\zeta}) - iBe^{-\Gamma\zeta}(\xi - \xi_0) + i\Phi_0], \tag{B5}$$

where $D = B/2\Gamma$ and $C = (A^2 - B^2)/4\Gamma$. In the case when $A = 1$, $B = 0$, and $\xi_0 = \Phi_0 = 0$, this solution reduces to the expression in Refs. [35,36].

Note that our approach leads to an infinite variety of approximate solutions of Eq. (A1). In the general case the solutions of Eq. (A7) can be found by the inverse scattering method which leads to the higher-order solitons and an infinite variety of soliton forms. Hence the solutions of Eq. (A7) found by the inverse scattering method and Eqs. (A13) and (B3) yield the appropriate approximate solutions of Eq. (A1) to the first order in the small parameter $|\Gamma|$.

Generalized NLSE with higher-order dispersion. The perturbation method based on ST leads to an approximate solution of the normalized NLSE given by Eq. (10). In this case, Eqs. (A2), (A4)–(A6), and (A8)–(A12) are valid as well. Thus the function $F(\zeta)$ and new variables Z and T have the same form as in Eqs. (A12) and (A13). The NLSE given in Eq. (10) can be written with new variables as

$$i\frac{\partial V}{\partial Z}+\frac{\partial^2 V}{\partial T^2}-i\lambda\sqrt{1-2\Gamma Z}\frac{\partial^3 V}{\partial T^3}-(1-2\Gamma Z)\frac{\partial^4 V}{\partial T^4}+|V|^2V = \frac{i\Gamma}{1-2\Gamma Z}\left[\frac{V}{2}+(T-T_0)\frac{\partial V}{\partial T}\right], \tag{B6}$$

where the function $V(Z,T)$ is defined in Eq. (A2). We assume that the function $V_0(Z,T)$ is decreasing to zero at $T\to\pm\infty$ and integrable on L^2 and two conditions are satisfied: $|\Gamma|\ll 1$ and $2|\Gamma|\zeta\ll 1$. In this case, Eq. (B6) yields the equation,

$$i\frac{\partial V_0}{\partial Z}=-\frac{\partial^2 V_0}{\partial T^2}+i\lambda\frac{\partial^3 V_0}{\partial T^3}+\frac{\partial^4 V_0}{\partial T^4}-|V_0|^2V_0. \tag{B7}$$

This equation has the same form as Eq. (10) with $\Gamma=0$. Hence Eqs. (A13) and (B3) and the exact solution in Eq. (11) lead to the approximate solution of Eq. (10) to the first order in small parameter $|\Gamma|$. Note that this method can also be used when the perturbation term has a more complicated form than we consider above. In the more general case when the last term in Eq. (10) has a different form, the function $F(\zeta)$ and hence new variables $Z(\zeta)$ and $T(\zeta,\xi)$ also have different forms.

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