Light-by-light-scattering contributions to the Lamb shift in light muonic atoms

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We consider one-loop light-by-light-scattering contributions to the Lamb shift of the $1s$, $2s$, $2p$ states in light muonic hydrogenlike atoms at $Z \leqslant 10$. The contributions are of the order $\alpha^5 m_\mu$ (with diverse dependence on the nuclear charge Z). Those include the contributions of the so-called Wichmann-Kroll potential $[\alpha(Z\alpha)^4 m_\mu]$, the virtual Delbrück scattering $[\alpha^2(Z\alpha)^3 m_\mu]$, etc. The results are obtained in a nonrelativistic approximation. For the calculation of the virtual-Delbrück-scattering contribution, we have constructed an effective potential in the coordinate space which may be applied to other calculations in muonic atoms.

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I. INTRODUCTION

Muonic atoms give an opportunity to develop and test a bound-state QED theory and probe a nuclear structure with a specific range of parameters not available with ordinary [electronic] atoms. Recently the accuracy of the measurement of the $2s - 2p$ Lamb shift in some light hydrogenlike muonic atoms has been dramatically improved [1,2]. The QED theory of the energy levels in muonic atoms is somewhat different from that in ordinary atoms. The Bohr radius in muonic atoms is comparable with the Compton wavelength of an electron. Because of that, an important role is played by the diagrams with the closed electron loops. Those contributions are specific for muonic atoms. The most important are those due to vacuum polarization. Their contribution to the energy is of the order $\alpha(Z\alpha)^2 m$.

Effects of the virtual light-by-light scattering contribute to higher orders. There are three types of such contributions, characteristic diagrams which are presented in Fig. 1. They are all of the order $\alpha^5 m$, but their dependence on the value of the nuclear Z charge is different.

The $\alpha(Z\alpha)^4 m$ contribution (see the graph 1:3 in Fig. 1) is the so-called Wichmann-Kroll (WK) contribution, which has been studied for a while (see, e.g., [3,4]). A number of the results have been achieved for muonic atoms using certain numerical approximations of the exact WK potential. In particular, the approximations, introduced in [3,5] on the basis of the results of numerical integration in [6], were numerously applied (e.g., in [3,7,8]). The result for the $2p - 2s$ Lamb shift with accuracy sufficient for applications in μH was found in [8] and confirmed in [9–11]. In [10,11] the result was also confirmed by direct calculations. The WK contributions to the $n = 2$ Lamb shift for some other light muonic atoms are obtained in, e.g., [12–14].

The $\alpha^2(Z\alpha)^3 m$ term is due to the virtual Delbrück scattering (see the 2:2 diagram in Fig. 1). It has also been studied for quite a long period (see, e.g., [3,4]). Still, some questions have been resolved only recently [10].

The initial calculations were based on a so-called scattering approximation [15] (where the Coulomb muon propagator is substituted for a free one). The substitution by itself is incorrect (see, e.g., discussion in [4,10]); however, the formulas which were eventually used in the numerical calculations were nevertheless correct (see below). Results on the contribution to the Lamb shift in some light atoms were published, e.g., in [3,9], but they were not very accurate.

The third type of contributions (see the 3:1 plot in Fig. 1) have not been calculated until recently. It was studied in [10,11], where also the virtual-Delbrück-scattering contribution was found with a sufficient accuracy for several light muonic atoms.

A kind of theorem on the 2:2 and 3:1 contributions was announced in [11] and proven in [10]. The papers considered an approximation of a static muon, where its nonrelativistic propagator is presented with a δ function over the energy.

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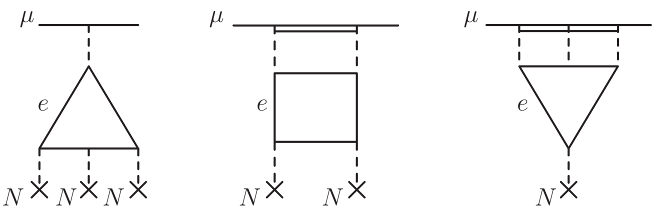

FIG. 1. Characteristic diagrams induced by the light-by-light scattering. The double horizontal line is for the nonrelativistic Coulomb Green's function of a muon.

It was proven that the approximation is a valid one. We discuss the accuracy of the approximation in this paper (see Sec. II). Using that approximation [10,11], the results on the 2:2 and 3:1 contributions to the Lamb shift in muonic hydrogen, deuterium, and helium ions have been found (see [14] for μT). It was also demonstrated that the related limit can be achieved both from the diagrams with the bound-muon Green's function (as shown in Fig. 1) and from those with the free Green's function (as were used in the scattering approximation in [3,9]). As far as the static-muon approximation is applicable, one may use both types of diagrams with the same result, which validates the working formulas used in [3,9].

In this paper we consider the effective potential for the virtual-Delbrück-scattering contribution to the Lamb shift in light muonic two-body atoms. We use the representation of the potential in momentum space in terms of an integral over Feynman parameters [10] and study the effective potential in the coordinate space by means of an analytic Fourier transform and subsequent numerical integrations over the Feynman parameters. For the effective potential in the coordinate space, we find both asymptotics (at $r \ll 1/m_e$ and $r \gg 1/m_e$). (Here and throughout the paper we apply the relativistic units in which $\hbar = c = 1$.) Eventually, we fit the numerical results and asymptotics, obtained here. The approximation is accurate at the level of 10^{-3} in the area where the muon wave function of low states is localized. Our main results are related to the virtual-Delbrück-scattering contribution to the Lamb shift; however, we present numerical results for all three light-by-light (LbL) contributions (see Fig. 1), because their comparison can be useful.

The $2p - 2s$ Lamb-shift interval cannot be successfully measured in all the two-body muonic atoms (because of the range of the interval); however, the theory of the Lyman-α transition is very similar. The data on such gross-structure transitions play an important role in determination of the rms charge radius of a large variety of elements (see, e.g., [16]). In this paper we tabulate the virtual light-by-light-scattering contribution to the Lamb shift of the $1s$, $2s$, $2p$ states, which is sufficient for the calculation of both the $2p - 2s$ interval and the energy of the $2p - 1s$ transition. The considered range of the nuclear charge is $Z = 1, ...10$.

II. THE EFFECTIVE POTENTIAL AND THE STATIC-MUON APPROXIMATION

As demonstrated in [10], once we can neglect various contributions to the muon propagator, such as the binding energy

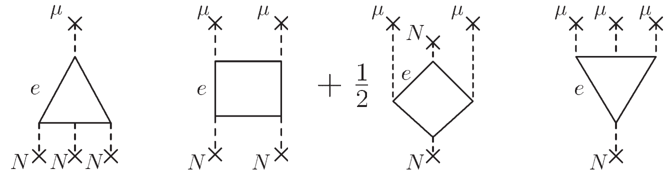

FIG. 2. "Double-external-field" approximation with a static nucleus and a static muon.

and those related to momentum transfer [between the muon and the electron loop] in comparison with its energy transfer q_0, we arrive at the nonrelativistic propagator reduced to $\delta(q_0)$. For $Z\alpha m_\mu/n \leqslant m_e$ (n is the principle quantum number), the energy transfer is determined by the m_e scale. In the opposite case, when $Z\alpha m_\mu/n \geqslant m_e$, the characteristic value of q_0 is determined by the value of the momentum (in the LbL loop), which in its turn is determined by the characteristic atomic momentum $Z\alpha m_\mu$. That means that once $Z\alpha \ll 1$, we can apply the static-muon approximation. (In [10] we considered a stronger condition $(Z\alpha)^2 m_\mu \ll m_e$.) All that is related, indeed, to only 2:2 and 3:1 contributions. The standard WK contribution does not require any conditions on the muon but only on the static regime of the nucleus. Those conditions are weaker and the validity of the WK potential is due to relativistic-recoil effects, i.e., due to corrections which are of higher order in both small parameters of the two-body Coulomb problem, $Z\alpha$ and m_μ/M, where M is the nuclear mass.

Once the static-muon approximation is applicable, we arrive at a "double-external-field" limit, the diagrams for which are presented in Fig. 2. In particular, that allows us to immediately set a relation between the 3:1 contribution and the 1:3 one (WK):

$$\Delta E_{3:1}(ns) = \frac{1}{Z^2}\, \Delta E_{1:3}(ns)\ , \tag{1}$$

since the related integrands differ by their normalization only. Note that Eq. (1) is correct only under the static-muon approximation. The corrections beyond the approximation are of different orders for $\Delta E_{3:1}$ and $\Delta E_{1:3}$.

The potential for the 1:3 contribution was studied for a while and there are a number of efficient approximations, such as those mentioned above from [3,5]. (Still, we revisit the problem in Sec. IV.) An effective potential for the 2:2 contribution, an evaluation of which is the main purpose of this paper, is considered in detail in the next section.

III. THE EFFECTIVE POTENTIAL FOR THE VIRTUAL-DELBRÜCK-SCATTERING CONTRIBUTION

Following [10], the contribution of virtual Delbrück scattering to the Lamb shift in light muonic atoms can be presented in terms of a certain potential. In the momentum space the result reads [10]

$$\Delta E_{2:2} = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, V_{2:2}(\mathbf{q}^2)\, F(\mathbf{q}^2), \tag{2}$$

where the potential $V(\mathbf{q}^2)$ is discussed in detail in [11], and

$$F_{nl}(\mathbf{q}^2) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,[\Psi_{nl}(\mathbf{p}-\mathbf{q})]^*\Psi_{nl}(\mathbf{p}) = \int d^3\mathbf{r}\,[\Psi_{nl}(\mathbf{r})]^*\,e^{-i(\mathbf{q}\cdot\mathbf{r})}\Psi_{nl}(\mathbf{r}) \tag{3}$$

is the form factor of the atomic nl state, while $\Psi_{nl}(\mathbf{p})$ is its nonrelativistic Coulomb wave function (with the reduced mass m_r).

The potential $V_{2:2}(\mathbf{q}^2)$ is presented in momentum space as an integral over the Feynman parameters [10]

$$V_{2:2}(\mathbf{q}^2) = \frac{3}{4\pi}\,\alpha^2(Z\alpha)^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \int_0^1 dv \int_0^1 dw \int_0^1 dt \times \sum_{k=1,2}\left\{\frac{\mathcal{B}^{(k)}_{2:2}}{\left(s^{(k)}_{2:2}\,\mathbf{q}^2+m_e^2\right)} + \frac{\mathcal{C}^{(k)}_{2:2}\,\mathbf{q}^2}{\left(s^{(k)}_{2:2}\,\mathbf{q}^2+m_e^2\right)^2} + \frac{\mathcal{D}^{(k)}_{2:2}\,\mathbf{q}^4}{\left(s^{(k)}_{2:2}\,\mathbf{q}^2+m_e^2\right)^3}\right\}, \tag{4}$$

where $\mathcal{B}^{(k)}_{2:2}$, $\mathcal{C}^{(k)}_{2:2}$, $\mathcal{D}^{(k)}_{2:2}$, and $s^{(k)}_{2:2}$ are bulky dimensionless functions of those parameters considered in [10]. The parameter k is to distinguish two diagrams contributing to $V_{2:2}$: $k=1$ stands for the left 2:2 graph (see Fig. 2) and $k=2$ is for the right one.

The dependence on $\mathbf{q}^2$ is simple, which allows us to immediately perform the Fourier transformation

$$V_{2:2}(r) = \frac{4\pi}{r}\int_0^\infty \frac{dq}{(2\pi)^3}\,q\,\sin(qr)V_{2:2}(\mathbf{q}^2) \tag{5}$$

and to obtain a result in the coordinate space, which reads

$$V_{2:2}(r) = \frac{3}{4\pi}\,\alpha^2(Z\alpha)^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \int_0^1 dv \int_0^1 dw \int_0^1 dt \sum_{k=1,2}\exp\left(-\frac{m_e r}{\sqrt{s^{(k)}_{2:2}}}\right) \times\left\{\frac{\mathcal{B}^{(k)}_{2:2}}{4\pi s^{(k)}_{2:2}\,r} + \frac{\mathcal{C}^{(k)}_{2:2}}{\left(s^{(k)}_{2:2}\right)^3}\,\frac{2s^{(k)}_{2:2}-m_e r\sqrt{s^{(k)}_{2:2}}}{8\pi\,r} + \frac{\mathcal{D}^{(k)}_{2:2}}{\left(s^{(k)}_{2:2}\right)^4}\,\frac{8s^{(k)}_{2:2}-m_e r\left(7\sqrt{s^{(k)}_{2:2}}-m_e r\right)}{32\pi\,r}\right\}. \tag{6}$$

The explicit representation of the potential $V_{2:2}(r)$ is cumbersome, and for practical applications we further look for an efficient approximate formula. To derive it we first find the value of the potential in certain points in the coordinate space (see Fig. 3) and then fit them with a Padé approximation.

To improve the accuracy of the fit, prior to fitting, we look for the asymptotics. The potential behaves as $\propto r^{-1}$ at short distances, as one should expect from (6), while at long distances it is $\propto r^{-4}$. The general situation is illustrated in the plot in Fig. 3. The range of characteristic values of x, which are of interest for light muonic atoms, is summarized in Table I.

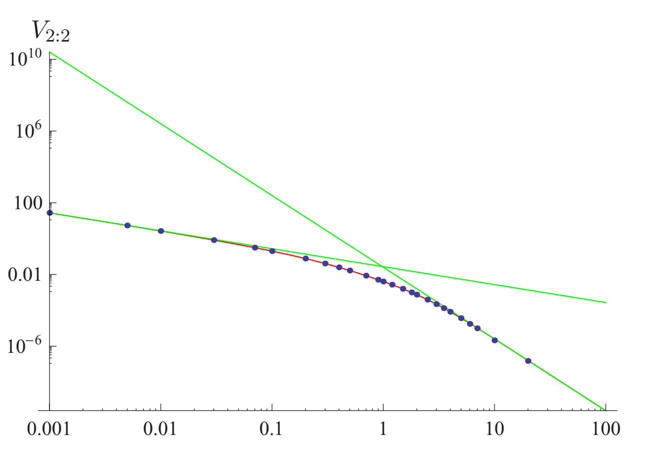

FIG. 3. The "data" (i.e., the results of our numerical calculation of (6) in coordinate space), their asymptotics, and the fit from (9) (see below). The potential $V_{2:2}$ is given in units of $-\alpha^2(Z\alpha)^2m_e$, and the distance is characterized with $x = rm_e$.

The short-distance asymptotic coefficient can be directly established from (6) in a rather straightforward way. The result of the numerical integration reads

$$V_{2:2}(r \ll 1/m_e) \simeq -0.027\,565(13)\,\frac{\alpha^2(Z\alpha)^2}{r}\,. \tag{7}$$

The large-distance asymptotic behavior is not that simple to establish from (6). Considering the LbL contributions (see Fig. 1) in the t channel, we note that some pure photonic intermediate states are possible there, which sets the branch point for $t = -\mathbf{q}^2$ to zero and eventually leads to a certain r^{-p} behavior at large distances for each of the LbL potentials (cf. [17]). In the case of $V_{2:2}(r)$ in the form of (6), that technically means a singularity of the effective dispersion-relation variable [cf. (4)] at $m_e^2/s^{(k)}_{2:2} = 0$, which should transform the exponential factor in (6) to r^{-p}.

Fortunately, the asymptotic behavior of the 2:2 potential can be successfully studied in a different way; namely, we find it from the virtual-Delbrück-scattering amplitude for soft photons [18–20] (cf. [21]) as

$$V_{2:2}(r \gg 1/m_e) \simeq -\frac{59}{2304}\,\frac{\alpha^2(Z\alpha)^2m_e}{(m_er)^4} \simeq -0.025\,61\,\frac{\alpha^2(Z\alpha)^2m_e}{(m_er)^4}\,. \tag{8}$$

With the asymptotic coefficients in hand, we fit the numerical results. The fit reads

$$V_{2:2}^{\rm approx}(r) = -\frac{\alpha^2(Z\alpha)^2}{r}\,\frac{7.236 + 0.3099x + 2.561x^2}{262.5 + 902.0x + 751.7x^2 + 458.6x^3 + 2.62x^4 + 100x^5}\,, \tag{9}$$

where $x = m_e r$. The fit has $\chi^2 = 9.5$ for 22 degrees of freedom. We estimate the accuracy of the fit as 1×10^{-3} for $x \leqslant 1$. In the interval of $1 < x < 10$ the uncertainty gradually increases to a few percent level. For higher x, thanks to the correct asymptotic behavior, the error does not exceed that level.

As an independent test of our fit, we compare the results obtained by using the fit for the $n = 2$ Lamb shift in the lightest two-body muonic atoms with the direct ones [10,11] (see Table II). The results are in perfect agreement within our estimation of the uncertainty of the fit as 10^{-3}.

The virtual-Delbrück-scattering situation is very different from the WK one. As mentioned, the WK potential $V_{1:3}(r)$ [17] is valid when one can neglect the recoil effects, i.e., it is a result of an expansion not only in $Z\alpha$, but also in m/M. Because of the recoil nature of the corrections, the WK potential is applicable in both ordinary and muonic atoms. In the former we are interested in a large range of distances at $x \gg 1$, while the latter deals only with $x \sim 1$ or $x \ll 1$. The 2:2 potential is applicable only for muonic atoms [10,11] and therefore the area with $x \gg 1$ and even with $x \geqslant 1$ is of low interest. It still may appear in evaluation of the energy for the highly excited states with $n^2/Z \gg 1$, but most of the applications rely on a study of the lower states with $n = 1, 2$. For such states the accuracy of the Padé approximation (9) is at the level of 10^{-3}.

TABLE I. Characteristic differences of the wave functions of the low states ($1s, 2s, 2p$) in light two-body muonic atoms. Here, $\kappa = Z\alpha m_r/m_e$ is the characteristic momentum of the muonic states in the units of m_e, while $x_n = n^2/\kappa$ is the characteristic radius of the nl state in units of $\lambda\!\!\!^{-}_e = \hbar/m_e c$.

Ion	Z	κ	x_1	x_2
^{1}H	1	1.356	0.737	2.950
^{2}H	1	1.428	0.700	2.800
^{3}H	1	1.454	0.688	2.751
^{3}He	2	2.908	0.344	1.375
^{4}He	2	2.935	0.341	1.363
^{6}Li	3	4.443	0.225	0.900
^{7}Li	3	4.455	0.224	0.898
^{9}Be	4	5.960	0.1678	0.671
^{10}B	5	7.460	0.1341	0.536
^{11}B	5	7.467	0.1339	0.536
^{12}C	6	8.968	0.1115	0.446
^{13}C	6	8.975	0.1114	0.446
^{14}N	7	10.48	0.0954	0.382
^{15}N	7	10.48	0.0954	0.382
^{16}O	8	11.99	0.0834	0.334
^{17}O	8	11.99	0.0834	0.334
^{18}O	8	12.00	0.0834	0.333
^{19}F	9	13.50	0.0741	0.296
^{20}Ne	10	15.00	0.0667	0.267
^{21}Ne	10	15.01	0.0666	0.267
^{22}Ne	10	15.01	0.0666	0.266

Note that this is the accuracy of the approximation of $V_{2:2}(r)$ potential. Meanwhile, the very applicability of that potential due to the static-muon approximation has lower accuracy (see above).

As an example of applicability of the $x \gg 1$ area to practical cases, we mention neutral antiprotonic helium, where the characteristic size of the antiproton orbit is comparable with the $1s$ orbit of an electron in a hydrogen atom (see, e.g., [22]).

IV. NUMERICAL RESULTS

The purpose of the paper is a derivation of an effective potential for the 2:2 contribution to the muonic-atom Lamb shift at medium Z, which has been done in the previous section. It is interesting to compare the numerical results with those from other LbL terms, and in particular, with the WK ones.

There are two fits for the WK potential for the muonic atoms, which are available in literature. [The potential is valid by itself for ordinary and muonic atoms; however, the purpose of the fit determines the range of the distances of interest (see above).] One of them is [5]

$$V_{1:3}(r) = 0.3617\,\frac{\alpha(Z\alpha)^2}{\pi}\,\frac{Z\alpha}{r} \times \exp[0.3728\,x - \sqrt{2.906 + 11.4\,x + 4.417\,x^2}]. \tag{10}$$

Another fit applied in numerical calculations in muonic atoms is [3]

$$V_{1:3} = \frac{\alpha(Z\alpha)^3}{\pi^3 r}\begin{cases}\frac{-0.1755+0.1559x+0.0880x^2}{x^6} & \text{for } x \geqslant 1\\ \frac{0.649-0.208x}{1.374x^3+1.41x^2+2.672x+1} & \text{for } x \leqslant 1\end{cases}. \tag{11}$$

Both fits are based on numerical calculations by Vogel [6] for the interval of $0.1 < x \leqslant 1$, and in that area the fits

TABLE II. The 2:2 contributions to the $2s$ and $2p$ Lamb shift in light muonic atoms. The results of direct calculations are taken from [10,11]. The uncertainty of the integration over the fit in (9) is the statistical one. The error due to the static-muon approximation is the same for the direct calculations and for those from the fit. The characteristic value of x is $x = x_2$.

		Contribution [meV]	
Atom, state	x	Eq. (9)	Direct
μH ($2s$)	2.95	−0.001 791(4)	−0.001 793(3)
μH ($2p$)		−0.000 642(1)	−0.000 642(2)
μD ($2s$)	2.80	−0.001 966(4)	−0.001 968(3)
μD ($2p$)		−0.000 733(1)	−0.000 734(2)
μ^4He$^+$ ($2s$)	1.36	−0.027 28(3)	−0.027 31(4)
μ^4He$^+$ ($2p$)		−0.015 88(2)	−0.015 88(3)

TABLE III. The WK contributions to the $2s$ and $2p$ Lamb shift in light muonic atoms. The results of direct calculations are taken from [10,11]. The uncertainty of the fits for $x > 1$ is *a priori* unclear and not shown.

		Contribution [meV]			
Atom, state	x	Eq. (10)	Eq. (11)	Eq. (12)	Direct
μH $(2s)$	2.95	0.001 240	0.001 238	0.001 243	0.001 2472(7)
μH $(2p)$		0.000 2196	0.000 2196	0.000 2270	0.000 228 87(4)
μD $(2s)$	2.80	0.001 362	0.001 358	0.001 364	0.001 3693(7)
μD $(2p)$		0.000 2609	0.000 2609	0.000 2691	0.000 271 23(4)
μ^4He$^+$ $(2s)$	1.36	0.037 67	0.037 30	0.037 69	0.037 833(22)
μ^4He$^+$ $(2p)$		0.017 68	0.017 68	0.017 82	0.017 8676(15)

well agree with the numerical results (at the level of 10^{-3}). They both utilize the known leading asymptotic term at low x. They are different in area $x > 1$. The advantage of (10) is more smooth behavior around $x = 1$ and therefore a better extrapolation to the low end of the $x > 1$ interval, while the fit in (11) accommodates the asymptotic term at $x \gg 1$ and is better at high end of the interval.

We use our own fit of Vogel's data [6]

$$V_{\rm app.\,WK}(x) = \frac{\alpha(Z\alpha)^3}{r}\frac{5.026 + 0.02676x + 0.2829x^2}{240.0 + 725.4x + 542.2x^2 + 649.8x^3 + 150.2x^4 + 9.457x^5 + 100x^6}, \qquad (12)$$

which fits the data for $0.1 < x \leqslant 1$ with a fractional uncertainty better than 10^{-3} and correctly reproduces the asymptotics at low r [17] (see also [23,24]) and at high r [17] (see also [5,25]). In contrast to the fit (11) from [3], our fit in (12) has smooth behavior at around $x = 1$.

The application of the fits to the $n = 2$ Lamb shift in muonic hydrogen is rather questionable (see Table I), since we essentially need to integrate over an interval outside of the data area of [6], which was used to derive the fit. The smooth behavior at around $x = 1$ and a correct $x \gg 1$ asymptotics (mentioned above) should deliver a reasonable result, but its accuracy is unclear.

Previously, while calculating the results for muonic hydrogen, deuterium, and helium [10,11,14] we have used a direct

TABLE IV. The LbL contributions to the Lamb shift of the $1s$ state in a light two-body muonic atom. The contributions are given in units of $\alpha^3(Z\alpha)^2m_r$ and meV. The results are given for the total LbL contribution and for its components (see Fig. 1). We present in the table the central values, while the accuracy of the calculation is discussed in the text.

Nucleus	Z	$\Delta E_{1:3}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{2:2}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{3:1}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{\rm LbL}(1s)$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{\rm LbL}(1s)$ [meV]
^{1}H	1	0.005 804	−0.008 095	0.005 804	0.003 513	0.006 903
^{2}H	1	0.006 073	−0.008 410	0.006 073	0.003 736	0.007 734
^{3}H	1	0.006 167	−0.008 520	0.006 167	0.003 814	0.008 038
^{3}He	2	0.040 28	−0.026 23	0.010 07	0.024 13	0.2034
^{4}He	2	0.040 49	−0.026 35	0.010 12	0.024 26	0.2063
^{6}Li	3	0.1118	−0.047 97	0.012 43	0.076 30	1.474
^{7}Li	3	0.1120	−0.048 02	0.012 44	0.076 39	1.479
^{9}Be	4	0.2227	−0.071 50	0.013 92	0.1651	5.704
^{10}B	5	0.3737	−0.096 03	0.014 95	0.2926	15.81
^{11}B	5	0.3738	−0.096 06	0.014 95	0.2927	15.83
^{12}C	6	0.5656	−0.1213	0.015 71	0.4600	35.87
^{13}C	6	0.5657	−0.1213	0.015 72	0.4601	35.90
^{14}N	7	0.7988	−0.1471	0.016 30	0.6681	71.00
^{15}N	7	0.7989	−0.1471	0.016 30	0.6682	71.04
^{16}O	8	1.073	−0.1731	0.016 77	0.9170	127.4
^{17}O	8	1.073	−0.1732	0.016 77	0.9171	127.5
^{18}O	8	1.074	−0.1732	0.016 77	0.9172	127.5
^{19}F	9	1.390	−0.1995	0.017 15	1.207	212.5
^{20}Ne	10	1.747	−0.2260	0.017 47	1.539	334.5
^{21}Ne	10	1.747	−0.2260	0.017 47	1.539	334.6
^{22}Ne	10	1.747	−0.2260	0.017 47	1.539	334.7

TABLE V. The LbL contributions to the Lamb shift of the $2s$ state in a light two-body muonic atom. The contributions are given in units of $\alpha^3(Z\alpha)^2m_r$ and meV. The results are given for the total LbL contribution and for its components (see Fig. 1). We present in the table the central values, while the accuracy of the calculation is discussed in the text.

Nucleus	Z	$\Delta E_{1:3}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{2:2}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{3:1}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{\rm LbL}(2s)$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{\rm LbL}(2s)$ [meV]
^{1}H	1	0.000 6323	−0.000 9114	0.000 6323	0.000 3532	0.000 6941
^{2}H	1	0.000 6592	−0.000 9498	0.000 6592	0.000 3687	0.000 7631
^{3}H	1	0.000 6686	−0.000 9632	0.000 6686	0.000 3740	0.000 7880
^{3}He	2	0.004 404	−0.003 188	0.001 101	0.002 317	0.019 53
^{4}He	2	0.004 431	−0.003 207	0.001 108	0.002 332	0.019 83
^{6}Li	3	0.013 19	−0.006 236	0.001 465	0.008 416	0.1625
^{7}Li	3	0.013 21	−0.006 246	0.001 468	0.008 432	0.1633
^{9}Be	4	0.028 39	−0.009 825	0.001 774	0.020 34	0.7027
^{10}B	5	0.050 96	−0.013 83	0.002 039	0.039 17	2.117
^{11}B	5	0.050 99	−0.013 84	0.002 040	0.039 19	2.121
^{12}C	6	0.081 68	−0.018 21	0.002 269	0.065 75	5.127
^{13}C	6	0.081 72	−0.018 21	0.002 270	0.065 78	5.132
^{14}N	7	0.1210	−0.022 87	0.002 470	0.1006	10.69
^{15}N	7	0.1210	−0.022 88	0.002 470	0.1006	10.70
^{16}O	8	0.1693	−0.027 78	0.002 645	0.1442	20.03
^{17}O	8	0.1693	−0.027 78	0.002 646	0.1442	20.04
^{18}O	8	0.1694	−0.027 79	0.002 646	0.1442	20.06
^{19}F	9	0.2269	−0.032 90	0.002 801	0.1968	34.64
^{20}Ne	10	0.2938	−0.038 19	0.002 938	0.2585	56.20
^{21}Ne	10	0.2938	−0.038 19	0.002 937	0.2586	56.23
^{22}Ne	10	0.2938	−0.038 20	0.002 938	0.2586	56.24

calculation instead of the fits. To verify the accuracy of the previous fits and our fit, we compare our results of a direct calculation and the results from the fits for $2s$, $2p$ for a few light atoms where the characteristic values of x are the largest (see Table III). The error of our fit is about 1%, while for the others it is at a few-percent level. Eventually we estimate

TABLE VI. The LbL contributions to the Lamb shift of the $2p$ state in a light two-body muonic atom. The contributions are given in units of $\alpha^3(Z\alpha)^2m_r$ and meV. The results are given for the total LbL contribution and for its components (see Fig. 1). We present in the table the central values, while the accuracy of the calculation is discussed in the text. That is a nonrelativistic calculation and therefore the results for $2p_{1/2}$ and $2p_{3/2}$ are the same.

Nucleus	Z	$\Delta E_{1:3}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{2:2}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{3:1}$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{\rm LbL}(2p)$ Units of $[\alpha^3(Z\alpha)^2m_r]$	$\Delta E_{\rm LbL}(2p)$ [meV]
^{1}H	1	0.000 1116	−0.000 3265	0.000 1155	−0.000 095 43	−0.000 1875
^{2}H	1	0.000 1300	−0.000 3543	0.000 1300	−0.000 094 24	−0.000 1951
^{3}H	1	0.000 1353	−0.000 3642	0.000 1353	−0.000 093 55	−0.000 1971
^{3}He	2	0.002 065	−0.001 848	0.000 5161	0.000 7332	0.006 180
^{4}He	2	0.002 095	−0.001 867	0.000 5237	0.000 7518	0.006 394
^{6}Li	3	0.008 568	−0.004 338	0.000 9520	0.005 182	0.1001
^{7}Li	3	0.008 597	−0.004 349	0.000 9552	0.005 203	0.1007
^{9}Be	4	0.021 43	−0.007 548	0.001 339	0.015 22	0.5258
^{10}B	5	0.041 72	−0.011 30	0.001 669	0.032 09	1.734
^{11}B	5	0.041 76	−0.011 31	0.001 670	0.032 12	1.738
^{12}C	6	0.070 29	−0.015 51	0.001 952	0.056 73	4.423
^{13}C	6	0.070 33	−0.015 52	0.001 954	0.056 76	4.429
^{14}N	7	0.1076	−0.020 08	0.002 196	0.089 72	9.535
^{15}N	7	0.1076	−0.020 08	0.002 197	0.089 75	9.543
^{16}O	8	0.1540	−0.024 93	0.002 406	0.1315	18.27
^{17}O	8	0.1540	−0.024 94	0.002 407	0.1315	18.28
^{18}O	8	0.1541	−0.024 94	0.002 408	0.1315	18.29
^{19}F	9	0.2098	−0.030 03	0.002 590	0.1824	32.10
^{20}Ne	10	0.2751	−0.035 31	0.002 751	0.2425	52.72
^{21}Ne	10	0.2751	−0.035 32	0.002 751	0.2425	52.74
^{22}Ne	10	0.2751	−0.035 32	0.002 751	0.2426	52.76

the accuracy of our fit as follows: at $0.1 < x \leqslant 1$ it is below 1×10^{-3}, and it gradually reduces for $x < 0.1$ and $x > 1$ down to a 1% level.

The results for $n = 1, 2$ states in a two-body muonic atom are summarized in Tables IV, V, and VI for all three LbL contributions (the 1:3, 2:2, 3:1 ones). The uncertainty of the fits is discussed above, as well as the uncertainty of the static-muon approximation.

V. CONCLUSIONS

In conclusion, we have derived a representation for an effective potential induced by the virtual Delbrück scattering in the leading nonrelativistic approximation. We have obtained its numerical values in a number of points in the coordinate space and found an efficient Padé approximation. The accuracy of the Padé approximation is the highest for $m_e r < 1$, which allowed us to find the contributions to the Lamb shift of the low states in light two-body muonic atoms. We estimate the accuracy of the numerical evaluation as at the level of one part in a thousand, which is higher than the accuracy of the leading nonrelativistic approximation by itself.

The uncertainty of the Padé approximation for the potential is the best for $m_e r < 1$ (at the level of 10^{-3}), and it gradually increases to the few-percent level for $m_e r \simeq 10$. The data of the numerical evaluation of the potential itself at higher $m_e r$ are not accurate enough; however, the Padé approximation is constrained by the long-distance asymptotic behavior, which we have established by an independent evaluation.

In particular, we have tabulated the related contributions to the Lamb shift of the $1s, 2s, 2p$ states in muonic atoms with $Z \leqslant 10$. Those states are sufficient for two important problems, namely, for a theory of the $n = 2$ Lamb shift and of the Lyman-α interval.

We have also compared the results for the virtual-Delbrück-scattering contribution and the Wichmann-Kroll one. At $Z = 1$ they are comparable (being of opposite signs). They increase with the value of Z, but the Wichmann-Kroll one increases faster. At $Z = 10$ the virtual-Delbrück-scattering contribution is between 10% and 20% of the Wichmann-Kroll contribution, depending on the state.

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