Nuclear-structure corrections to the hyperfine splitting in muonic deuterium

Marcin Kalinowski* and Krzysztof Pachucki†
Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland

Vladimir A. Yerokhin
Center for Advanced Studies, Peter the Great St. Petersburg Polytechnic University, 195251 St. Petersburg, Russia

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Nuclear structure corrections of orders $Z\alpha\, E_F$ and $(Z\alpha)^2 E_F$ are calculated for the hyperfine splitting of the muonic deuterium. The obtained results disagree with previous calculations and lead to a $5\,\sigma$ disagreement with the current experimental value of the $2S$ hyperfine splitting in muonic deuterium.

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I. INTRODUCTION

Nuclear structure effects represent the main limitation for precise theoretical description of atomic energy levels. These effects are particularly important for muonic atoms, where the Compton wavelength of the bound muon $\sim$2 fm is of the same order as the nuclear size. It is, therefore, not surprising that the uncertainty of modern theoretical predictions of energy levels of light muonic atoms is dominated by the uncertainty of nuclear-structure effects. Specifically, the current theoretical value of the hyperfine splitting (hfs) of the $2S$ state of muonic deuterium (μD) is [1]

$$E_{\rm hfs}(2S)_{\rm theo} = 6.2791\,(50)\,{\rm meV}, \tag{1}$$

where the uncertainty comes almost exclusively from the deuteron polarizability ($\pm$0.0049 meV). The theoretical value (1) was obtained in Ref. [1] by compiling two independent calculations, by Borie [2] and by Faustov *et al.* [3]. The theoretical result is in good agreement with the experimental value [4]:

$$E_{\rm hfs}(2S)_{\rm exp} = 6.2747(70)_{\rm stat}(20)_{\rm syst}\ {\rm meV}. \tag{2}$$

In this work we will demonstrate that the deuteron structure corrections to hfs in μD were previously treated incorrectly and that the good agreement with the experimental value was probably accidental. Specifically, the calculation by Borie [2] included only the elastic part of the nuclear structure, which does not reproduce even the correct sign of the total nuclear structure effect. Faustov and coworkers [3] included both the Zemach and the Low corrections at the same time, which is inconsistent, and, moreover, used an incorrect formula for the nuclear polarizability.

In the present work we derive the nuclear-structure corrections in μD induced by the two- and three-photon exchange between the bound muon and the nucleus and analyze discrepancies with the previous calculation [3] and the experimental result [4].

The leading-order hfs of atomic levels of light atoms is independent of nuclear structure and is given (for the nS states) by the Fermi contact term,

$$E_F = \frac{4 g_N m_r^3}{3 m_p\, m\, n^3} (Z\alpha)^4 \,\langle \vec{s}_N \cdot \vec{s}_\mu \rangle\,, \tag{3}$$

where n is the principal quantum number, Z is the nuclear charge number, α is the fine-structure constant, m and m_p are masses of the lepton and the proton, respectively, m_r is the reduced mass of the atom, $\vec{s}_\mu$ and $\vec{s}_N$ are the spin operators of the lepton and the nucleus, respectively, g_N is the "modified" g-factor of the nucleus defined by

$$\vec{\mu}_N = \frac{Ze\, g_N}{2\, m_p}\, \vec{s}_N, \tag{4}$$

$\vec{\mu}_N$ is the nuclear magnetic moment operator, and e is the elementary charge. Numerical values of E_F for the ground and the first excited state of muonic deuterium are

$$E_F = \begin{cases} 49.0875\,{\rm meV}, & \text{for the } 1S \text{ state}, \\ 6.13594\,{\rm meV}, & \text{for the } 2S \text{ state}. \end{cases} \tag{5}$$

If we assume the point nuclear model and account for all known QED corrections, the theoretical result for the $2S$ state is [1]

$$E_{\rm hfs}({\rm point}) = 6.17815(20)\,{\rm meV}, \tag{6}$$

which corresponds to the sum of entries $h_1 + h_2 + h_4 + h_5 + h_7 + h_8 + h_9 + h_{9b} + h_{12} + h_{13} + h_{14} + h_{18}$ from Table IV of Ref. [1]. The deviation of the experimental value (2) from the theoretical point-nucleus result (6) can be regarded as the "experimental value" of the nuclear-structure correction:

$$\delta E_{\rm nucl} = E_{\rm hfs}({\rm expt}) - E_{\rm hfs}({\rm point}) = 0.0966(73)\,{\rm meV}. \tag{7}$$

From the theoretical side the nuclear-structure correction for light atoms can be described within the $Z\,\alpha$ expansion,

$$\delta E_{\rm nucl} = \delta^{(1)} E_{\rm nucl} + \delta^{(2)} E_{\rm nucl} + \cdots\,, \tag{8}$$

where $\delta^{(1)} E_{\rm nucl}$ is the two-photon exchange correction of order $(Z\,\alpha)\,E_F$, $\delta^{(2)} E_{\rm nucl}$ is the three-photon exchange correction of order $(Z\,\alpha)^2\,E_F$, and $\cdots$ denotes smaller contributions

*mj.kalinowski@student.uw.edu.pl
†www.fuw.edu.pl/~krp

due to exchange of larger number of photons and radiative corrections. In the following discussion, we calculate the two-photon and the three-photon exchange nuclear-structure corrections for the hfs of nS states of muonic deuterium. Relativistic units ($\hbar = c = 1$) are employed throughout.

II. TWO-PHOTON EXCHANGE NUCLEAR STRUCTURE

The most straightforward way of including the nuclear effects is to assume that the nucleus is described by some elastic electric and magnetic form factors. This leads to the so-called elastic, or finite-nuclear-size (fns), corrections. The leading fns correction originates from the two-photon exchange. It was derived long ago by Zemach [5] and is given by

$$\delta E_{\rm Zem} = -2\, m_r\, Z\alpha\, r_Z\, E_F, \tag{9}$$

where r_Z is the Zemach radius defined by

$$r_Z = \int d^3r_1 \int d^3r_2\, \rho_M(r_1)\, \rho_E(r_2)\, |\vec{r}_1 - \vec{r}_2|. \tag{10}$$

(Note that the subscript Z in r_Z is not related to the nuclear charge.) In the above equation, ρ_E and ρ_M are the charge and the magnetic-moment distributions of the nucleus, respectively, i.e., the Fourier transform of the corresponding form factors. The numerical value of the Zemach correction for the $2S$ state of μD, in the nonrecoil limit, with $r_Z = 2.593(16)$ fm [6], is

$$\delta E_{\rm Zem} = -0.1177(33)\,{\rm meV}. \tag{11}$$

We note the opposite sign of the Zemach correction as compared to the experimental value of the total nuclear structure (7). This demonstrates that the description of the nucleus only through the elastic form factors is not adequate.

Various hfs corrections arise from excitations of the nucleus by the bound lepton, usually referred to as the *inelastic* nuclear-structure corrections. In the case of the proton, the inelastic contribution can be obtained from the experimentally accessible spin-dependent structure functions by using dispersion relations [7–10]. For other nuclei, including deuteron, the inelastic spin-dependent structure functions are unknown, and one has to rely on theoretical calculations.

In our calculations we consider the elastic and inelastic contributions together and use a perturbation expansion over a small parameter, namely, the ratio of the average nucleon binding energy over the nucleon mass. Specifically, the two-photon exchange correction can be represented as

$$\delta^{(1)} E_{\rm nucl} = \delta E_{\rm Low} + \delta E_{\rm 1nucl} + \delta E_{\rm pol} + \cdots. \tag{12}$$

The leading-order term $\delta E_{\rm Low}$ was first derived by Low [11]. For the particular case of an nS state in μD, the numerical value of Low's correction in the point-nucleon model is

$$\delta E_{\rm Low} \approx -2\, m_r\, \alpha\, E_F\, \frac{g_n}{g_d} \langle R\rangle = \frac{2.640}{n^3}\,{\rm meV}. \tag{13}$$

Here R is the distance of the proton from the center of mass. Its expectation value was calculated using the AV18 potential [12] as $\langle R\rangle = 1.629$ fm.

A more detailed calculation of the leading-order term was performed by Friar and Payne [13,14], with inclusion of the finite nucleon size and meson exchange currents. Their result was reported for the $1S$ state of ordinary (electronic) deuterium, $\delta E_{\rm Low}(e{\rm D}) = 87.3$ kHz. Rescaling it to the nS states of μD, one gets

$$\delta E_{\rm Low} = \frac{m_r^4(\mu{\rm D})}{m_r^4(e{\rm D})}\, \frac{m(e)}{m(\mu)}\, \frac{1}{n^3}\, \delta E_{\rm Low}(e{\rm D}) = \frac{2.566}{n^3}\,{\rm meV}, \tag{14}$$

in good agreement with the approximate result of Eq. (13).

$\delta E_{\rm 1nucl}$ is the contribution induced by individual nucleons. It is given by the individual nucleon Zemach corrections:

$$\delta E_{\rm 1nucl} = -\frac{8\,\alpha^2}{3\,n^3}\, \frac{m_r^3}{m_p + m}\, \vec{s}_\mu \cdot \left\langle \sum_a g_a\, \vec{s}_a\, r_{aZ} \right\rangle, \tag{15}$$

where r_{aZ} is the effective radius of the nucleon a. If only the elastic part is included, the result for the proton is $r_{pZ} = 1.086(12)$ fm [6] and for the neutron, $r_{nZ} = -0.042$ fm [14]. The result for the proton effective radius that includes the recoil and polarizability contributions can be obtained from the $1S$ muonic hydrogen correction $\delta E_{\rm nucl}(\mu{\rm H}) = -1.131(24)$ meV obtained by Tomalak in Ref. [15], namely, $r_{pZ} = 0.883(19)$ fm, and for the neutron from Ref. [16], $r_{nZ} = 0.06(1)$ fm.

$\delta E_{\rm pol}$ is the contribution of the nuclear vector polarizability. It has been studied for ordinary atoms by Khriplovich and Milstein [17] and later by one of us (Pachucki) [18]. Results of Ref. [17] obtained in the logarithmic approximation were shown in Ref. [18] to be incorrect, because the coefficient of logarithm for an arbitrary chosen cutoff vanishes completely. Moreover, the derivation of Ref. [17] is applicable only for the "electronic" atoms but not for the muonic ones. Nevertheless, the result of Ref. [17] was later used by Faustov *et al.* in their calculations of the nuclear structure in μD [3]. In the present work we derive the nuclear vector polarizability correction $\delta E_{\rm pol}$ for muonic deuterium.

Following the method of Ref. [18], we obtain the vector polarizability corrections to hfs in μD in the form

$$\delta E_{\rm pol} = -e^2\, \psi^2(0) \int \frac{d\,\omega}{2\,\pi} \int \frac{d^3k}{(2\,\pi)^3} \times \frac{(\omega^2\, \epsilon^{klj} + k^i\, k^k\, \epsilon^{lij} - k^i\, k^l\, \epsilon^{kij})\, \sigma^j\, \alpha^{kl}}{\omega\,(\omega^2 + 2\,m\,\omega - k^2)\,(\omega^2 - 2\,m\,\omega - k^2)\,(\omega^2 - k^2)}, \tag{16}$$

where α^{kl} is the antisymmetric part of the scattering amplitude tensor T^{kl}. In the simplest case of the electric dipole coupling $-\vec{D}\cdot\vec{E}$, α^{kl} takes the form

$$\alpha^{kl} = \omega^2 \left\langle D^k \frac{1}{E_N - \tilde{H}_N - \omega}\, D^l + D^l \frac{1}{E_N - \tilde{H}_N + \omega}\, D^k \right\rangle, \tag{17}$$

where $\tilde{H}_N = H_N + k^2/(2\,M)$, H_N is the nuclear internal Hamiltonian, $E_N = \langle H_N\rangle$, $\vec{D} = \sum_a e_a \vec{R}_a$, and $\vec{R}_a$ is the nucleon position with respect to the mass center. After integration over ω and k, and expansion in the small parameter,

$$X = \sqrt{\frac{2(H_N - E_N)}{m_r}}, \tag{18}$$

we obtain the dipole polarizability correction as

$$\delta E_{\rm pol0} = -\frac{i\,e^2}{12\pi}\psi^2(0)\left(\frac{m_r}{m}\right)^2 \epsilon^{ijk}\sigma^k\langle D^i\, X\, D^j\rangle. \quad (19)$$

$\delta E_{\rm pol0}$ vanishes in the nonrelativistic limit, and its numerical contribution is expected to be small, because it requires the presence of both the spin-orbit and the quadrupole-spin interactions between nucleons.

There are, however, other polarizability corrections that yield significant numerical contributions. The first one is the correction due to the magnetic quadrupole interaction:

$$\delta H = -\sum_a \frac{e\,g_a}{2\,m_p}\, R_a^i\, s_a^j\, B^j_{,i}. \quad (20)$$

The corresponding contribution to the scattering tensor is

$$\delta\alpha^{kl} = i\,\omega\sum_a \frac{e\,g_a}{2\,m_p}\left\langle \vec{R}_a\vec{k}\,(\vec{s}_a\times\vec{k})^k\,\frac{1}{E_N-\tilde{H}_N-\omega}\,D^l + D^l\,\frac{1}{E_N-\tilde{H}_N+\omega}\,\vec{R}_a\vec{k}\,(\vec{s}_a\times\vec{k})^k - (k\leftrightarrow l,\omega\to-\omega)\right\rangle. \quad (21)$$

The first term in the small-X expansion gives the following correction to the hyperfine splitting of μD:

$$\delta E_{\rm pol1} = -\frac{2\alpha}{3}\frac{g_p-g_n}{g_d}E_F\, m_r^2\langle\vec{R}\,X\,\vec{R}\rangle. \quad (22)$$

The second correction of the same order in X comes from the magnetic dipole interaction, which is enhanced by the factor of $(g_p-g_n)^2$:

$$\delta H = -\frac{e}{2\,m_p}\left[\sum_a g_a\,\vec{s}_a - g_d\,\vec{s}_d\right]\vec{B} = -(\vec{\mu}-\langle\vec{\mu}\rangle)\,\vec{B}. \quad (23)$$

The corresponding contribution to the scattering tensor is

$$\delta\alpha^{kl} = \left\langle[(\vec{\mu}-\langle\vec{\mu}\rangle)\times\vec{k}]^k\,\frac{1}{E_N-\tilde{H}_N-\omega}\,[(\vec{\mu}-\langle\vec{\mu}\rangle)\times\vec{k}]^l + (k\leftrightarrow l,\omega\to-\omega)\right\rangle, \quad (24)$$

which gives the following correction to the hyperfine splitting in μD:

$$\delta E_{\rm pol2} = -\frac{\alpha}{16}\frac{m_r^2}{m_p\,m}\,\frac{(g_p-g_n)^2}{g_d}\,E_F\langle(\vec{s}_p-\vec{s}_n)\,X\,(\vec{s}_p-\vec{s}_n)\rangle. \quad (25)$$

Further corrections are of higher order in X. The X^2 terms vanish, as shown in Ref. [18], and the next-order nonvanishing terms are proportional to X^3. Namely, the next-order (in X) term of the correction coming from $\delta\alpha^{kl}$ in Eq. (21) is

$$\delta E_{\rm pol3} = \frac{\alpha}{4}\frac{g_p-g_n}{g_d}E_F\frac{5m_r-2m}{3\,m^3}m_r^4\langle\vec{R}\,X^3\,\vec{R}\rangle. \quad (26)$$

Another X^3 correction, $\delta E_{\rm pol4}$, comes from the following spin dependent coupling to the electric field [18],

$$\delta H = -\vec{T}\cdot\frac{\partial\vec{E}}{\partial t}, \quad (27)$$

where $\vec{T}$ is defined as

$$\vec{T} = \sum_a\left[\frac{e_a}{2m_p}(g_a-1)-\frac{Z\,e}{2M}\right]\vec{s}_a\times\vec{x}_a. \quad (28)$$

The corresponding vector polarizability correction is

$$\delta\alpha^{kl} = i\,\omega^3\left\langle D^k\,\frac{1}{E_N-\tilde{H}_N-\omega}\,T^l + T^l\,\frac{1}{E_N-\tilde{H}_N+\omega}\,D^k - (k\leftrightarrow l,\omega\to-\omega)\right\rangle, \quad (29)$$

and the contribution to the hyperfine splitting of μD is

$$\delta E_{\rm pol4} = -\frac{\alpha}{3}\frac{g_p-g_n-1}{g_d}E_F\frac{m_r^4}{m^2}\langle\vec{R}\,X^3\,\vec{R}\rangle. \quad (30)$$

The last nuclear polarizability correction $\delta E_{\rm pol5}$ comes from the fourth-order terms of the expansion of the vector polarizability $\delta\alpha^{kl}$ in the small parameter $\vec{k}\cdot\vec{R}$. In order to derive this correction, we rewrite Eq. (16) by using $k^i\,T^{ik} = \omega\,T^{0k}$ and apply the nonrelativistic approximation. The result is

$$\delta E_{\rm hfs} = -e^2\,\psi^2(0)\int\frac{d\,\omega}{2\,\pi}\int\frac{d^3k}{(2\,\pi)^3}\,\frac{(\vec{k}\times\vec{\sigma})^i\,(T^{0i}-T^{i0})}{k^2\,(2\,m\,\omega-k^2)(2\,m\,\omega+k^2)}$$

$$= \frac{e^4}{m}\,\psi^2(0)\int\frac{d^3k}{(2\,\pi)^3}\,i\,\frac{(\vec{\sigma}\times\vec{k})^i}{k^4}\sum_{a,b}\left\langle j_a^0\,e^{i\,\vec{k}\cdot\vec{R}_a}\,\frac{1}{H_N-E_N+\frac{k^2}{2\,m_r}}\,j_b^i\,e^{-i\,\vec{k}\cdot\vec{R}_b}\right\rangle, \quad (31)$$

where, in the nonrelativistic approximation, $j_a^0 = 1$ and $j_b^i = -i\,\vec{\sigma}_b\times\vec{k}\,g_b/(4\,m_p)$. By neglecting H_N-E_N in the above expression one obtains the Low correction given by Eq. (13). The quadratic terms of the expansion of Eq. (31) in $\vec{k}\cdot\vec{R}_a$ gives $\delta E_{\rm pol1}$. We now consider the fourth power of $\vec{k}\cdot\vec{R}_a$ and obtain for μD the following correction:

$$\delta E_{\rm pol5} = \frac{\alpha}{15}\,E_F\,m_r^4\left\{\frac{5}{6}\frac{g_p+g_n}{g_d}\,\langle R^2\,X^3\,R^2\rangle - 2\,\frac{g_p-g_n}{g_d}\,\langle R^2\,\vec{R}\,X^3\,\vec{R}\rangle + \frac{g_p+g_n}{g_d}\,\langle(R^i\,R^j-R^2\,\delta^{ij}/3)\,X^3\,(R^i\,R^j-R^2\,\delta^{ij}/3)\rangle\right\}. \quad (32)$$

We are not aware of any further significant contributions, therefore we write the nuclear vector polarizability correction as

$$\delta E_{\rm pol} = \sum_{i=1}^{5} \delta E_{{\rm pol}\,i} \tag{33}$$

and assume a 5% uncertainty due to omitted $\delta E_{\rm pol0}$ and higher-order (in X) corrections. Our result disagrees with the result by Faustov and Martynenko [3], because they used an incorrect formula for the polarizability correction derived for electronic atoms and included in addition the Low correction.

The nuclear vector polarizability is presently the main source of the theoretical uncertainty of the total nuclear-structure correction. This means that in the future detailed investigations should reanalyze all possible contributions to the nuclear vector polarizability.

III. THREE-PHOTON-EXCHANGE ELASTIC CORRECTION

The $(Z\alpha)^2\,E_F$ elastic contribution to the hyperfine splitting $\delta^{(2)}E_{\rm fns}$ can be derived by following the approach developed earlier for the case of the Lamb shift in Ref. [19]. Instead of a direct use of the Dirac equation, which is possible but tedious, we shall split the total correction into low-energy δE_L and high-energy δE_H parts. Both these parts are separately divergent, so we employ dimensional regularization with $d = 3 - 2\epsilon$ (see Appendix for details) and cancel singularities $\sim 1/\epsilon$ in the sum $\delta^{(2)}E_{\rm fns} = \delta E_L + \delta E_H$. For convenience we assume lepton mass $m = 1$ from now on and restore it only in the final expression from dimensional analysis.

A. Low-energy part

In the low-energy part, where $p \sim Z\alpha$, the nonrelativistic approximation is valid. The nonrelativistic Hamiltonian H with the nuclear electric G_E and magnetic G_M form factors (with their respective Fourier transforms ρ_E, ρ_M), here normalized to unity, is given by

$$H = \frac{p^2}{2} + V(r) + \frac{4\pi\,Z\alpha}{d m_p}\, g_p (\vec{s}_p\cdot\vec{s}_\mu)_\epsilon\, \rho_M(r), \tag{34}$$

where the potential V is defined by its Fourier transform

$$V(p) = -G_E(p^2)\,\frac{4\pi\,Z\alpha}{p^2} \tag{35}$$

and where $(\vec{s}_p\cdot\vec{s}_\mu)_\epsilon$ is defined in (A3).

Because the characteristic momentum p is much smaller than the inverse of the nuclear size, the nuclear form factors can be expanded in p^2. We thus obtain

$$H = H_0 + H_{\rm hfs} + \delta V + \delta H_{\rm hfs}, \tag{36}$$

where

$$H_0 = \frac{p^2}{2} - Z\alpha\left[\frac{1}{r}\right]_\epsilon, \tag{37}$$

$$H_{\rm hfs} = \frac{4\pi\,Z\alpha}{d m_p}\, g_p (\vec{s}_p\cdot\vec{s}_\mu)_\epsilon \delta^{(d)}(\vec{r}), \tag{38}$$

$$\delta V = -4\pi\,Z\alpha\, G'_E(0)\,\delta^{(d)}(\vec{r}), \tag{39}$$

$$\delta H_{\rm hfs} = -\frac{4\pi\,Z\alpha}{d m_p}\, g_p (\vec{s}_p\cdot\vec{s}_\mu)_\epsilon\, G'_M(0)\nabla^2\delta^{(d)}(\vec{r}), \tag{40}$$

and where $[1/r]_\epsilon$ denotes a d-dimensional generalization of the $1/r$ potential. The corresponding correction to the hyperfine splitting of order $(Z\alpha)^6$ is

$$\delta E_L = 2\left\langle \delta V\, \frac{1}{(E_0 - H_0)'}\, H_{\rm hfs}\right\rangle + \langle \delta H_{\rm hfs}\rangle. \tag{41}$$

Calculating matrix elements in d dimensions and using $G'_E(0) = -r_p^2/6$, $G'_M(0) = -r_m^2/6$, with r_p and r_m being charge and magnetic radius of the nucleus, respectively, we obtain the following expression for the low-energy part:

$$\delta E_L = \frac{4}{d}(r_p Z\alpha)^2 E_F\left[-\frac{1}{4\epsilon} - \frac{1}{n} - \frac{1}{2} + \gamma - \ln\frac{n}{2} + \Psi(n) + \ln(Z\alpha) + \frac{r_m^2}{4 r_p^2 n^2}\right]. \tag{42}$$

The singularity $\sim 1/\epsilon$ in the above equation will cancel out with δE_H.

B. High-energy part

In the high-energy part δE_H, the lepton momentum is of the order of the inverse of the nuclear size, so one can employ the scattering approximation. Specifically, δE_H is given by the forward three-photon exchange amplitude, which can be represented by the three diagrams shown in Fig. 1. The resulting expression is

$$\delta E_H = \delta E_{H1} + \delta E_{H2} + \delta E_{H3}, \tag{43}$$

$$\delta E_{H1} = \psi^2(0)\int\frac{d^d p}{(2\pi)^d}\int\frac{d^d q}{(2\pi)^d}\,\langle \bar{t}|\, e\vec{\gamma}\cdot\vec{A}(-\vec{p}) \times \frac{1}{\not{p}-1}\gamma^0 V(\vec{p}-\vec{q})\frac{1}{\not{q}-1}\gamma^0 V(\vec{q})\,|t\rangle\,,$$

$$\delta E_{H2} = \psi^2(0)\int\frac{d^d p}{(2\pi)^d}\int\frac{d^d q}{(2\pi)^d}\,\langle \bar{t}|\, \gamma^0 V(-\vec{p}) \times \frac{1}{\not{p}-1}\gamma^0 V(\vec{p}-\vec{q})\frac{1}{\not{q}-1}e\vec{\gamma}\cdot\vec{A}(\vec{q})\,|t\rangle\,,$$

$$\delta E_{H3} = \psi^2(0)\int\frac{d^d p}{(2\pi)^d}\int\frac{d^d q}{(2\pi)^d}\,\langle \bar{t}|\, \gamma^0 V(-\vec{p}) \times \frac{1}{\not{p}-1}e\vec{\gamma}\cdot\vec{A}(\vec{p}-\vec{q})\frac{1}{\not{q}-1}\gamma^0 V(\vec{q})\,|t\rangle\,,$$

where $t = (1,\vec{0})$, $\not{p} = (1,\vec{p})^\nu\gamma_\nu$, and

$$A^i(q) = \frac{i\,Z\,e\,g_p}{4\,m_p}\sigma_p^{ik}\frac{q^k}{q^2}G_M(q^2). \tag{44}$$

After performing Dirac algebra, δE_H can be expressed in the coordinate representation as

$$\delta E_H = \frac{8\psi^2(0)}{d m_p}(Z\alpha)^3 g_p\,\langle \vec{s}_p\cdot\vec{s}_\mu\rangle_\epsilon \times\int d^d r\left\{2\pi\rho_M(r)\left[\mathcal{V}_E^{(2)}\right]^2 + \mathcal{V}_M\mathcal{V}_E\mathcal{V}_E^{(2)}\right\}, \tag{45}$$

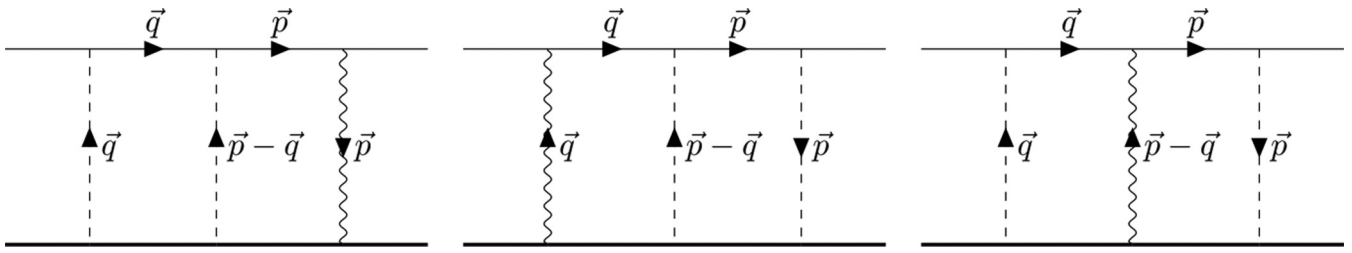

FIG. 1. Three diagrams representing contributions to the high-energy part δE_H. Wavy lines represent magnetic photons carrying potential (44), and dashed lines are electric photons carrying potential (35).

where d-dimensional potentials $\mathcal{V}_E$ and $\mathcal{V}_M$ are defined in Appendix. The first term under the integral sign is convergent due to the presence of ρ_M and thus can be evaluated in three dimensions. The second term, however, contains a singularity $\sim 1/\epsilon$, which has to be separated out. This is achieved by splitting the domain of integration into $(0,\Lambda)$ and (Λ,∞). The integral over (Λ,∞) is evaluated by using the asymptotic form of the potentials in d dimensions. The final result for the high-energy part is

$$\delta E_H = \frac{4}{d}(r_p Z\alpha)^2 E_F\left[\frac{1}{4\epsilon} + \ln r_{pp} + \frac{1}{2} + \gamma\right]. \tag{46}$$

Here r_{pp} is the effective radius defined by

$$\ln\frac{r_{pp}}{r_p} = \frac{1}{G'_E(0)}\int_0^\infty dr \ln\frac{r}{r_p}\frac{d}{dr}r^3\left\{2\pi\rho_M(r)\left[V_E^{(2)}\right]^2 + V_M V_E V_E^{(2)} + \frac{1}{r^2}\left[\frac{r}{2} - \frac{G'_E(0)}{r}\right]\right\}, \tag{47}$$

where $V_{E(M)}$ are three-dimensional versions of potentials $\mathcal{V}_{E(M)}$, which depend on electric and magnetic form factors, and are presented in Appendix.

The final result for the elastic three-photon exchange correction, $\delta^{(2)}E_{\rm fns} = \delta E_L + \delta E_H$, with m restored from dimensional analysis, is

$$\delta^{(2)}E_{\rm fns} = \frac{4}{3}E_F(m r_p Z\alpha)^2\left[-\frac{1}{n} + 2\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m r_{pp} Z\alpha) + \frac{r_m^2}{4r_p^2 n^2}\right]. \tag{48}$$

This expression is valid for any nucleus, both for muonic and electronic atoms.

For the dipole parametrization of the nuclear form factors

$$G_E(q) = G_M(q) = \frac{1}{(1+q^2/\Lambda^2)^2}, \tag{49}$$

one easily obtains the following results:

$$r_m = r_p, \quad r_Z/r_p = \frac{35}{16\sqrt{3}}, \quad r_{pp}/r_p = 5.274\,565\ldots, \tag{50}$$

which are independent of parameter Λ and thus valid for any nucleus. Numerical values of the elastic $(Z\alpha)^2 E_F$ correction for the $1S$ and $2S$ states of muonic hydrogen are

$$\delta^{(2)}E_{\rm fns}(\mu{\rm H}) = \begin{cases} -0.0093\,{\rm meV}, & \text{for the } 1S \text{ state}, \\ -0.00096\,{\rm meV}, & \text{for the } 2S \text{ state}. \end{cases} \tag{51}$$

The above result for the $2S$ state deviates from the value of -0.00065 meV calculated by Indelicato [20] and quoted by Antognini *et al.* [21] (entry h21 in Table III of that work). One of possible reasons could be the inclusion of the reduced mass in Indelicato's calculation, while our result is obtained in the nonrecoil limit. We claim that the recoil effect on this relativistic correction cannot be accounted for in terms of the reduced mass. Our analytic result is verified by a numerical calculation in the following subsection.

C. Numerical verification

The analytical expression (48) for the fns correction of order $(Z\alpha)^2 E_F$ has been verified by comparison with the numerical evaluation of the fns correction to all orders in $Z\alpha$. Specifically, the numerical all-order fns correction was obtained by evaluating the expectation value of the Fermi-Breit operator H_μ with solutions of the Dirac equation with an extended nucleus and subtracting the point-nucleus result. The (extended-nucleus) Fermi-Breit operator is

$$H_\mu = \frac{|e|}{4\pi}\frac{\boldsymbol{\alpha}\cdot[\boldsymbol{\mu}\times\boldsymbol{r}]}{r^3}F(r), \tag{52}$$

where $\boldsymbol{\mu}$ is the nuclear magnetic moment operator, $\boldsymbol{\alpha}$ is the vector of Dirac matrices, and $F(r)$ describes the radial distribution of the magnetic moment, $F(r)\approx 1$ outside of the nucleus. For the dipole parametrization (49), the distribution function is given by

$$F(r) = 1 - e^{-\lambda r}\left[1 + \lambda r + \tfrac{1}{2}(\lambda r)^2\right], \tag{53}$$

where $\lambda = 2\sqrt{3}/r_m$.

We define the relativistic fns correction that contains orders $(Z\alpha)^2 E_F$ and higher by subtracting the Zemach contribution from the numerical fns correction:

$$\delta^{(2+)}E_{\rm fns} = E_{\rm fns,num} - \delta E_{\rm Zem}. \tag{54}$$

Results for the relativistic fns correction are presented in Fig. 2, which demonstrates agreement between the numerical and analytical approaches for $Z = 0$. The difference between the numerical and analytical results scales linearly with Z, representing the contribution of orders $(Z\alpha)^3 E_F$ and higher.

IV. THREE-PHOTON EXCHANGE NUCLEAR-STRUCTURE CORRECTION

In this section we address the *inelastic three-photon exchange* nuclear-structure correction $\sim(Z\alpha)^2 E_F$ to the hyperfine splitting, which has not been studied in the literature so

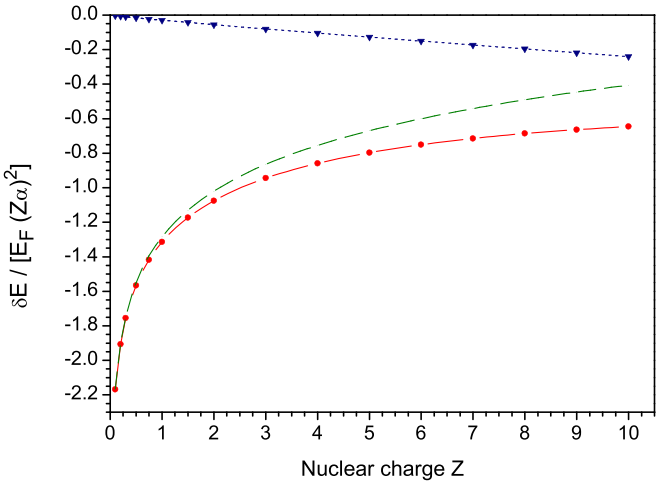

FIG. 2. Numerical and analytical results for the fns correction normalized by the factor of $E_F(Z\alpha)^2$, for the $1S$ state of muonic hydrogen-like ions with the nuclear root-mean-square radius 1 fm, as a function of nuclear charge Z. Solid line and dots (red) represent numerical results (54); dashed line (green) shows analytical results (48); dotted line and triangles (blue) show the difference.

far. We will perform an approximate treatment of this correction and evaluate the largest contribution, namely, that due to the electric dipole polarizability. Furthermore, we will demonstrate partial cancellations occurring between the elastic and inelastic parts for the case of muonic deuterium, μD.

The nuclear-structure correction $\delta^{(2)}E_{\rm nucl}$ can be represented as a sum of several contributions,

$$\delta^{(2)}E_{\rm nucl} = \delta E_{\rm DP} + \delta E_L(D) + \delta E_H(p) + \delta E_H(pn), \quad (55)$$

where $\delta E_{\rm DP}$ is the low-energy dipole polarizability contribution, $\delta E_L(D)$ is the elastic low-energy part, and $\delta E_H(p)$ and $\delta E_H(pn)$ are the high-energy corrections.

It is convenient to introduce the common factor

$$\mathcal{E} = \frac{4\pi\,\alpha}{3m_p}\,\psi^2(0), \quad (56)$$

which will frequently appear in formulas below.

A. Elastic low-energy contribution

The elastic low-energy part $\delta E_L(D)$ can be obtained from Eq. (42) by replacing the proton charge r_p and magnetic r_m radii with their deuteron counterparts r_d and r_{md}, because at low energies muon sees the nucleon as a whole. The resulting expression is

$$\delta E_L(D) = \frac{4}{d}(r_d\alpha)^2\mathcal{E}\left[-\frac{1}{4\epsilon} - \frac{1}{n} - \frac{1}{2} + \gamma - \ln\frac{n}{2} + \Psi(n) + \ln\alpha + \frac{r_{md}^2}{4r_d^2n^2}\right] g_d\,\langle\vec{s}_d\cdot\vec{s}_\mu\rangle_\epsilon, \quad (57)$$

where $\langle\vec{s}_d\cdot\vec{s}_\mu\rangle_\epsilon$ is defined in Eq. (A3).

B. Polarizability contribution

We derive here the leading dipole polarizability correction $\delta E_{\rm DP}$ and represent it as a sum of two terms:

$$\delta E_{\rm DP} = \delta E_{\rm DP1} + \delta E_{\rm DP2}. \quad (58)$$

The first one $\delta E_{\rm DP1}$ is obtained by taking the second-order matrix element with dipole interaction in the nonrelativistic approximation and perturbing it with the magnetic dipole-dipole interaction $H_{\rm hfs}$. The result is

$$\delta E_{\rm DP1} = \alpha^2\delta_{\rm hfs}\langle\psi\phi_N|\vec{R}\cdot\vec{\nabla}\left[\frac{1}{r}\right]_\epsilon \times \frac{1}{E_N+E_0-H_N-H_0}\vec{R}\cdot\vec{\nabla}\left[\frac{1}{r}\right]_\epsilon|\psi\phi_N\rangle. \quad (59)$$

Here $\vec{R}$ is the proton position with respect to the deuteron mass center, and $\delta_{\rm hfs}$ denotes the first-order perturbation due to the hyperfine interaction $H_{\rm hfs}$ which is an analog of $H_{\rm hfs}$ in Eq. (38) for the deuterium,

$$H_{\rm hfs} = \frac{4\pi\alpha}{dm_p}\delta^{(d)}(\vec{r})\,g_d(\vec{s}_d\cdot\vec{s}_\mu)_\epsilon, \quad (60)$$

where H_N is the nuclear Hamiltonian and H_0 is the nonrelativistic muon Hamiltonian defined in Eq. (37). The perturbative treatment of $H_{\rm hfs}$ means that the polarizability correction is expressed as a sum of two terms, originating from perturbations of the denominator and the wave function. However, the first term vanishes, and $\delta E_{\rm DP1}$ becomes

$$\delta E_{\rm DP1} = 2\alpha^2\langle\psi\phi_N|\vec{R}\cdot\vec{\nabla}\left[\frac{1}{r}\right]_\epsilon\frac{1}{E_0+E_N-H_0-H_N}\vec{R}\cdot\vec{\nabla} \times\left[\frac{1}{r}\right]_\epsilon\frac{1}{(E_0-H_0)'}H_{\rm hfs}|\psi\phi_N\rangle. \quad (61)$$

Because we are interested in the leading correction only, we neglect the D-wave in the ground deuteron state and neglect the Coulomb corrections, so

$$\delta E_{\rm DP1} = \frac{12}{d^2}\alpha^2\mathcal{E}\sum_\Lambda\langle\phi_N|\,\vec{R}\,|\Lambda\rangle\,\langle\Lambda|\,\vec{R}\,|\phi_N\rangle\,g_d\,\langle\vec{s}_d\cdot\vec{s}_\mu\rangle_\epsilon \times\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{4\pi}{p^2}\frac{4\pi}{q^2}\frac{\vec{p}\cdot\vec{q}}{(\vec{p}-\vec{q})^2}\frac{2}{p^2+2\Lambda}. \quad (62)$$

After integration we obtain the following expression for the polarizability correction:

$$\delta E_{\rm DP1} = \frac{4}{d}(r_s\alpha)^2\mathcal{E}\left(\frac{1}{4\epsilon}+\frac{2}{3}-\frac{1}{2}\ln 2\bar{E}\right)\langle g_d\vec{s}_d\cdot\vec{s}_\mu\rangle_\epsilon, \quad (63)$$

where $\bar{E}$ is the mean excitation energy defined by

$$\ln\frac{\bar{E}}{m} = \frac{1}{r_s^2}\langle\phi_N\,|\,\vec{R}\ln\left[\frac{(H_N-E_N)}{m}\right]\vec{R}\,|\,\phi_N\rangle, \quad (64)$$

and r_s is the deuteron structure radius

$$r_s = \sqrt{\langle R^2\rangle}. \quad (65)$$

The mean excitation energy (64) was calculated in Ref. [19] using the AV18 potential [12], with the result

$$\bar{E} = 7.37(7)\,\text{MeV}. \quad (66)$$

The pionless EFT in the next-to-leading order [22] reproduces this result with 2% accuracy.

The second dipole polarizability contribution $\delta E_{\rm DP2}$ is the Coulomb distortion correction to the leading two-photon exchange contribution $\delta E_{\rm pol1}$ in Eq. (22). We derive it by considering the nonrelativistic formula for the second-order correction to energy that comes from the electric dipole interaction and the magnetic quadrupole term in Eq. (20):

$$\delta E_{\rm DP2} = 4\pi\alpha^2\,\frac{g_p-g_n}{dm_p}\langle\vec{s}_d\cdot\vec{s}_\mu\rangle_\epsilon\delta_C\langle\psi\phi_N|\vec{R}\cdot\vec{\nabla}\left[\frac{1}{r}\right]_\epsilon\frac{1}{H_0+H_N-E_0-E_N}\vec{R}\cdot\vec{\nabla}\,\delta(\vec{r})|\psi\phi_N\rangle, \tag{67}$$

where δ_C denotes the Coulomb correction, namely, that beyond $E_{\rm pol1}$ in Eq. (22). This Coulomb correction is the forward-scattering three-photon exchange amplitude, which takes the following form in the momentum representation:

$$\delta E_{\rm DP2} = \frac{3\alpha^2}{d}\frac{g_p-g_n}{g_d}E_F\sum_{\Lambda'}\langle\phi_N|R^k|\Lambda\rangle\,\langle\Lambda|R^i|\phi_N\rangle\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{4\pi\,p^k}{p^2}\frac{4\pi\,q^i}{\frac{q^2}{2}+\Lambda}\left[\frac{1}{\frac{p^2}{2}+\Lambda}\frac{1}{|\vec{p}-\vec{q}|^2}+\frac{2}{|\vec{p}-\vec{q}\,|^4}\right]. \tag{68}$$

After integration we obtain for $\delta E_{\rm DP2}$

$$\delta E_{\rm DP2} = \frac{4}{3}(\alpha\,r_s)^2\frac{g_p-g_n}{g_d}E_F\left(\frac{3}{4}-\ln 2\right). \tag{69}$$

Its value is heavily suppressed by the numerical factor in the parentheses.

C. High-energy contribution $\delta E_H(pn)$

When muon momentum is of the order of the inverse of internucleon distance, the muon sees different positions of the proton and the neutron inside the nucleus, and effectively one can discern which photon interacts with which nucleon in the three-photon exchange. Because one can neglect the nuclear excitation energy in comparison to the muon kinetic energy, the high-energy contribution can be represented as an expectation value of the effective interaction potential,

$$\delta E_H(pn) = \langle\phi_N|\delta V_H|\phi_N\rangle, \tag{70}$$

where

$$\delta V_H = \psi^2(0)\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}[\delta V_{H1}+\delta V_{H2}+\delta V_{H2}] \tag{71}$$

and

$$\begin{aligned}
\delta V_{H1} &= \langle\bar{t}|\,e\vec{\gamma}\cdot\vec{A}_a(-\vec{p})e^{i\vec{p}\cdot\vec{R}_a}\frac{1}{\not{p}-1}\gamma^0V_b(\vec{p}-\vec{q})e^{i(\vec{q}-\vec{p})\cdot\vec{R}_b}\frac{1}{\not{q}-1}\gamma^0e^{-i\vec{q}\cdot\vec{R}_c}V_c(\vec{q})\,|t\rangle\,,\\
\delta V_{H2} &= \langle\bar{t}|\,\gamma^0V_c(-\vec{p})e^{i\vec{p}\cdot\vec{R}_c}\frac{1}{\not{p}-1}\gamma^0V_b(\vec{p}-\vec{q})e^{i(\vec{q}-\vec{p})\cdot\vec{R}_b}\frac{1}{\not{q}-1}e^{-i\vec{q}\cdot\vec{R}_a}e\vec{\gamma}\cdot\vec{A}_a(\vec{q})\,|t\rangle\,,\\
\delta V_{H3} &= \langle\bar{t}|\,\gamma^0V_b(-\vec{p})e^{i\vec{p}\cdot\vec{R}_b}\frac{1}{\not{p}-1}e\vec{\gamma}\cdot\vec{A}_a(\vec{p}-\vec{q})e^{i(\vec{q}-\vec{p})\cdot\vec{R}_a}\frac{1}{\not{q}-1}\gamma^0e^{-i\vec{q}\cdot\vec{R}_c}V_c(\vec{q})\,|t\rangle\,.
\end{aligned}$$

Indices $(a,b,c)\in\{p,n\}$ discern whether the interacting nucleon is the proton or the neutron. We neglect all the contributions where the electric photon interacts with the neutron ($b=c=p$) and thus are left with two cases. We first consider the case when the magnetic photon hits the neutron ($a=n$). This gives the following correction:

$$\delta E_H(pn) = \frac{8\psi^2(0)}{dm_p}\alpha^3g_n\,\langle\vec{s}_n\cdot\vec{s}_\mu\rangle_\epsilon\,\langle\phi_N|\int d^dr\left\{2\pi\rho_M(\vec{r}+2\,\vec{R})\left[\mathcal{V}_E^{(2)}\right]^2+\mathcal{V}_M\left(\vec{r}+2\,\vec{R}\right)\mathcal{V}_E\mathcal{V}_E^{(2)}\right\}|\phi_N\rangle, \tag{72}$$

which is almost the same as Eq. (45), but has magnetic terms shifted by the proton-neutron distance. All the potentials in Eq. (72) are functions of r, unless explicitly written and magnetic potentials refer to the neutron form factor, while electric ones to the proton one, both through definitions (A5). The treatment of Eq. (72) follows the same pattern as that of Eq. (45). The only difference is that magnetic factors are shifted by $2\,\vec{R}$, which leads to additional terms. The result is

$$\delta E_H(pn) = \frac{4}{d}(r_p\alpha)^2\mathcal{E}g_n\langle\vec{s}_n\cdot\vec{s}_\mu\rangle_\epsilon\left(\frac{1}{4\epsilon}+\gamma-\frac{1}{2}+\frac{3r_{pn}^2+2r_s^2}{r_p^2}+\ln 2r_l\right), \tag{73}$$

TABLE I. Effective radii and g-factors for the three-photon nuclear-structure correction for μD, given by Eq. (77). Proton-neutron radius r_{pn} is taken in the point-nucleon limit ($r_{pn} = r_s$), and r_l is obtained with the AV18 potential [12].

Variable	Value	Units	Source
r_p	0.84087(39)	fm	Ref. [23]
r_d	2.1256(8)	fm	Ref. [24]
r_s	1.954661(79)	fm	Ref. [19]
r_{md}	2.312(10)	fm	Ref. [25]
r_{pp}	4.435	fm	Eq. (47)
r_{pn}	1.955	fm	Eq. (75)
r_{pZ}	0.883(19)	fm	Ref. [15]
r_{nZ}	0.06(1)	fm	Ref. [16]
r_l	1.339	fm	Eq. (74)
g_p	5.585 694 702(17)		Ref. [26]
g_d	0.8574382311(48)		Ref. [26]
g_n	−3.82608545(90)		Ref. [26]

where r_l is defined as

$$\ln(2mr_l) = \langle\phi_N|\ln(2mR)|\phi_N\rangle\,, \tag{74}$$

and its value can be found in Table I. The effective proton-neutron radius r_{pn} in Eq. (73) is defined as

$$\begin{aligned} 2\pi r_{pn}^2 = \langle\phi_N| \int d^3r \Big\{ & 2\pi\rho_M(\vec r + 2\,\vec R)\big[V_E^{(2)}\big]^2 \\ & + V_M(\vec r + 2\,\vec R)\,V_E\,V_E^{(2)} \\ & + \frac{1}{|\vec r + 2\,\vec R|}\left[\frac{1}{2} - \frac{G_E'(0)}{r^2}\right]\Big\}|\phi_N\rangle, \end{aligned} \tag{75}$$

where ρ_M and V_M correspond to the neutron, while G_E and V_E correspond to the proton. It is worth noting that in the point-nucleon limit the proton-neutron effective radius r_{pn}, given by Eq. (75), is equal to the deuteron structure radius ($r_{pn} \to r_s$).

TABLE II. Nuclear structure corrections for hyperfine splitting of the $1S$ and $2S$ states of muonic deuterium, in meV. Numerical results are obtained with the AV18 potential [12].

Correction	$1S$	$2S$	Source
$\delta E_{\rm pol1}$	−1.1007	−0.1376	Eq. (22)
$\delta E_{\rm pol2}$	−0.0823	−0.0103	Eq. (25)
$\delta E_{\rm pol3}$	0.1513	0.0189	Eq. (26)
$\delta E_{\rm pol4}$	−0.1979	−0.0283	Eq. (30)
$\delta E_{\rm pol5}$	−0.0327	−0.0041	Eq. (32)
$\delta E_{\rm pol}$	−1.2623(631)	−0.1578(79)	Eq. (33)
$\delta E_{\rm 1nucl}$	−0.9450(224)	−0.1181(28)	Eq. (15)
$\delta E_{\rm Low}$	2.566	0.3208	Eq. (14)
$\delta^{(1)}E_{\rm nucl}$	0.3587(670)	0.0448(84)	Eq. (12)
$\delta^{(2)}E_{\rm nucl}$	−0.0547(137)	−0.0065(16)	Eq. (77)
$\delta E_{\rm nucl,theo}$	0.304(68)	0.383(86)	Eq. (8)
$\delta E_{\rm nucl,exp}$		0.0966(73)	Eq. (7)
Difference		0.0583(113)	

D. High-energy contribution $\delta E_H(p)$

In the case when all three photons interact with the proton, one should use the complete four-vector current, as in the two-photon case. We are not able to perform such a calculation at present, and thus we assume that the dominating contribution comes from the elastic part in the nonrecoil limit:

$$\delta E_H(p) = \frac{4}{d}(r_p\alpha)^2\mathcal{E}\left[\frac{1}{4\epsilon} + \ln r_{pp} + \frac{1}{2} + \gamma\right] g_p\langle\vec s_p\cdot\vec s_\mu\rangle_\epsilon. \tag{76}$$

E. Total three-photon exchange correction

Summing all contributions in Eq. (55) and restoring m's from dimensional analysis, we obtain the three-photon nuclear-structure correction of order $(Z\,\alpha)^2\,E_F$ for μD:

$$\begin{aligned} \delta^{(2)}E_{\rm nucl} = \frac{4}{3}(m\alpha)^2E_F\Big\{ & r_{pn}^2\frac{3g_n}{2g_d} + r_{md}^2\frac{1}{4n^2} + r_s^2\left[\frac{g_n}{g_d} - \gamma\right. \\ & \left. + \frac{g_p - g_n}{g_d}\left(\frac{3}{4} - \ln 2\right) + \frac{1}{6} - \frac{1}{2}\ln\frac{2\bar E}{m}\right] \\ & + r_d^2\left[2\gamma - \frac{1}{n} - \ln\frac{n}{2} + \Psi(n) + \ln\alpha\right] \\ & + r_p^2\left[g_p\ln(mr_{pp}) + g_n\ln(2mr_l) - g_n\right]\frac{1}{2g_d}\Big\}. \end{aligned} \tag{77}$$

In deriving this final formula we made use of an approximate identity $\langle g_d\vec s_d\rangle \approx \langle g_p\vec s_p + g_n\vec s_n\rangle\,/$ to cancel out the ϵ singularity and $\langle\vec s_p\rangle = \langle\vec s_n\rangle = \langle\vec s_d\rangle\,/2$ for further simplifications. Numerical values for all parameters in the final formula are listed in Table I.

Our final results for the three-photon exchange deuteron structure correction $\delta^{(2)}E_{\rm nucl}$ are presented in Table II. We assume a 25% uncertainty of these results, due to neglect of the polarizability corrections beyond the electric dipole contribution and of the unknown inelastic proton contribution, although we must admit that we can not well justify this uncertainty estimate.

Our results for the three-photon exchange structure correction disagree with corresponding formulas obtained by Faustov *et al.* [3] [given by Eqs. (55), (58), and (59) of that work]. The reason for this disagreement is twofold. First, Faustov *et al.* considered only the low-energy part of the nuclear-structure correction, omitting completely the high-energy part. Second, the term proportional to the deuteron magnetic radius r_{md} in their calculation contains an additional factor of $1/n^3$, due to a mistake in evaluation of the matrix element presented in Eq. (A8).

V. SUMMARY

A summary of all known nuclear-structure contributions to the hyperfine splitting of the $1S$ and $2S$ states in μD is presented in Table II. We find that the total theoretical result for the deuteron structure correction for the $2S$ state differs from the experimental value by about $5\,\sigma$. A possible reason for this discrepancy might be our insufficient knowledge of the spin-dependent coupling of nucleus to the electromagnetic field,

in particular the unknown corrections to the nonrelativistic current in Eq. (31). Another possible reason could be a mistake in calculations of QED effects for the point nucleus, although it looks much less probable since these calculations were performed independently by two groups [2,3].

Summarizing, in the present work we have calculated the two- and three-photon exchange nuclear structure corrections to the hyperfine splitting of the nS states in μD. The obtained results disagree with the previous theoretical calculation [3] and with the experimental result [4].

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APPENDIX: DIMENSIONAL REGULARIZATION

In order to extend spin $\frac{1}{2}$ into d dimensions, we define antisymmetric tensor

$$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j], \tag{A1}$$

that in the three-dimensional limit simplifies to

$$\sigma^{ij} \overset{d\to 3}{=} 2\varepsilon^{ijk} s^k. \tag{A2}$$

Additionally, for the deuteron we define $\sigma_d^{ij} = \sigma_p^{ij} + \sigma_n^{ij}$ and use the following convenient notation:

$$(\vec{s}_a \cdot \vec{s}_b)_\epsilon = \tfrac{1}{8}\sigma_a^{ij}\sigma_b^{ij}. \tag{A3}$$

Throughout our calculations we extensively used the following result for the general d-dimensional integral:

$$\int \frac{d^d p}{(2\pi)^d} \frac{4\pi}{p^n} e^{i\vec{p}\cdot\vec{r}} = 2^{2-n}\pi^{1-d/2} \frac{\Gamma\left(\frac{d-n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} r^{n-d}. \tag{A4}$$

Two special cases are of particular importance (with $d = 3 - 2\epsilon$):

$$\mathcal{V}(r) = \int \frac{d^d p}{(2\pi)^d} \frac{4\pi}{p^2} e^{i\vec{p}\cdot\vec{r}} = \pi^{\epsilon-1/2}\Gamma(1/2-\epsilon)\frac{1}{r^{1-2\epsilon}},$$

$$\mathcal{V}^{(2)}(r) = \int \frac{d^d p}{(2\pi)^d} \frac{4\pi}{p^4} e^{i\vec{p}\cdot\vec{r}} = \frac{1}{4}\pi^{\epsilon-1/2}\Gamma(-1/2-\epsilon) r^{1+2\epsilon}.$$

We define also the associated potentials:

$$\mathcal{V}_X = 4\pi \int \frac{d^d p}{(2\pi)^d} \frac{G_X(p^2)}{p^2} e^{i\vec{p}\cdot\vec{r}},$$

$$\mathcal{V}_X^{(2)} = 4\pi \int \frac{d^d p}{(2\pi)^d} \frac{G_X(p^2)}{p^4} e^{i\vec{p}\cdot\vec{r}}, \tag{A5}$$

where $X \in \{E, M\}$. Asymptotic forms of these potentials at $r \to \infty$ are

$$\mathcal{V}_X \to \mathcal{V} + \text{local terms},$$
$$\mathcal{V}_X^{(2)} \to \mathcal{V}^{(2)} + G_X'(0)\mathcal{V} + \text{local terms}.$$

The corresponding three-dimensional potentials are

$$V_X = 4\pi \int \frac{d^3 p}{(2\pi)^3} \frac{G_X(p^2)}{p^2} e^{i\vec{p}\cdot\vec{r}},$$

$$V_X^{(2)} = 4\pi \int \frac{d^3 p}{(2\pi)^3} \frac{[G_X(p^2)-1]}{p^4} e^{i\vec{p}\cdot\vec{r}}, \tag{A6}$$

with the asymptotic form

$$V_X \to \frac{1}{r} + \text{local terms},$$

$$V_X^{(2)} \to -\frac{r}{2} + \frac{G_X'(0)}{r} + \text{local terms}. \tag{A7}$$

Furthermore, we point out that in the framework of dimensional regularization the following matrix element vanishes:

$$\langle \vec{p}\, \delta^{(d)}(\vec{r})\, \vec{p} \rangle = 0. \tag{A8}$$

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