

Detecting nondecomposability of time evolution via extreme gain of correlationsTanjung Krisnanda,^{1,*} Ray Ganardi,¹ Su-Yong Lee,² Jaewan Kim,² and Tomasz Paterek^{1,3}¹*School of Physical and Mathematical Sciences, Nanyang Technological University, 637371 Singapore, Singapore*²*School of Computational Sciences, Korea Institute for Advanced Study, Hoegi-ro 85, Dongdaemun-gu, Seoul 02455, Korea*³*MajuLab, CNRS-UCA-SU-NUS-NTU International Joint Research Unit, UMI 3654 Singapore, Singapore*

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Noncommutativity is one of the most elementary nonclassical features of quantum observables. Here we propose a method to detect noncommutativity of interaction Hamiltonians of two probe objects coupled via a mediator. If these objects are open to their local environments, our method reveals nondecomposability of temporal evolution into a sequence of interactions between each probe and the mediator. The Hamiltonians or Lindblad operators can remain unknown throughout the assessment, we only require knowledge of the dimension of the mediator. Furthermore, no operations on the mediator are necessary. Technically, under the assumption of decomposable evolution, we derive upper bounds on correlations between the probes and then demonstrate that these bounds can be violated with correlation dynamics generated by non-commuting Hamiltonians, e.g., Jaynes-Cummings coupling. An intuitive explanation is provided in terms of multiple exchanges of a virtual particle which lead to the excessive accumulation of correlations. A plethora of correlation quantifiers are helpful in our method, e.g., quantum entanglement, discord, mutual information, and even classical correlation. Finally, we discuss exemplary applications of the method in quantum information: the distribution of correlations and witnessing dimension of an object.

DOI: [10.1103/PhysRevA.98.052321](https://doi.org/10.1103/PhysRevA.98.052321)**I. INTRODUCTION**

All classical observables are functions of positions and momenta. Since there is no fundamental limit on the precision of position and momentum measurement in classical physics, all classical observables are, in principle, measurable simultaneously. Quite differently, the Heisenberg uncertainty principle forbids simultaneous exact knowledge of quantum observables corresponding to position and momentum. The underlying nonclassical feature is their noncommutativity: Any pair of noncommuting observables cannot be simultaneously measured to arbitrary precision, as first demonstrated by Robertson in his famous uncertainty relation [1]. Other examples of nonclassical phenomena with underlying noncommutativity of observables include violations of Bell inequalities [2,3] or, more generally, noncontextual inequalities; e.g., see [4]. Here we describe a method to detect noncommutativity of interaction Hamiltonians, and generally nondecomposability of temporal evolution, from the dynamics of correlations.

Consider the situation depicted in Fig. 1, where the probe systems A and B do not interact directly but only via the mediator C ; i.e., there is no Hamiltonian term H_{AB} . In general, we allow all objects to be open systems and study whether the evolution operator cannot be represented by a sequence of operations between each probe and the mediator, i.e., $\Lambda_{BC}\Lambda_{AC}$, or vice versa. For the special case in which all systems are closed, nondecomposability implies noncommutativity of interaction Hamiltonians, i.e., $[H_{AC}, H_{BC}] \neq 0$.

Indeed, for commuting Hamiltonians, the unitary evolution operator is decomposable into $U_{BC}U_{AC}$, where, for example, $U_{AC} = \exp(-itH_{AC})$ and we set $\hbar = 1$. We show that for decomposable evolution, correlations between A and B are bounded. We also show with concrete dynamics generated by noncommuting Hamiltonians that these bounds can be violated. The bounds derived depend solely on the dimensionality of C and not on the actual form of the evolution operators. Hence, these operators can remain unknown throughout the assessment. This is a desired feature, as experimenters usually do not reconstruct the evolution operators via process tomography. It also allows applications of the method to situations where the physics is not understood to the extent that reasonable Hamiltonians or Lindblad operators can be written down. Furthermore, the assessment does not depend on the initial state of the tripartite system and does not require any operations on the mediator. It is therefore applicable to a variety of experimental situations; Refs. [5–8] provide concrete examples.

We begin by presenting the general bounds on the amount of correlations one can establish if the evolution is decomposable. It is shown that these bounds are generic and hold for a plethora of correlation quantifiers. We then calculate concrete bounds on exemplary quantifiers and show how they can be violated in a system of two fields coupled by a two-level atom. We discuss the origin of the violation in terms of “Trotterized” evolution, where a virtual particle is exchanged between A and B multiple times if the Hamiltonians do not commute but only once if they do commute. Finally, we focus on immediate applications in quantum information and discuss the consequences of our findings for correlation distribution protocols and dimension witnesses.

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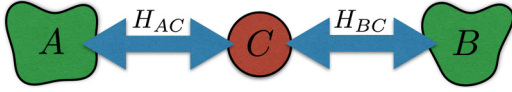


FIG. 1. Probe objects A and B individually interact with a mediator C but not with each other (A , B , and C could be open to their local environments). The coherent parts of the interactions are described by Hamiltonians H_{AC} and H_{BC} . We show how to infer nondecomposability of the temporal evolution based on a gain of correlation between A and B exceeding a certain threshold, which is a function of the dimensionality of C only.

II. RESULTS

A. General bounds

Consider the setup illustrated in Fig. 1. System C , with finite dimension d_C , is mediating interactions between higher-dimensional systems A and B . For simplicity we take $d_A = d_B > d_C$. We assume that there is no direct interaction between A and B , such that the Hamiltonian of the whole tripartite system is of the form $H_{AC} + H_{BC}$ (local Hamiltonians H_A , H_B , and H_C included). Our bounds follow from a generalization of the following simple observation. Consider, for the moment, the relative entropy of entanglement as the correlation quantifier [9]. If the evolution is decomposable, it can be written as $\Lambda_{BC}\Lambda_{AC}$, or vice versa. Therefore, it is as if particle C interacted first with A and then with B , a scenario similar to that in Refs. [10–15]. The first interaction can generate at most $\log_2(d_C)$ ebits of entanglement, whereas the second, in the best case, can swap all this entanglement. In the end, particles A and B gain at most $\log_2(d_C)$ ebits. The bound is indeed independent of the form of interactions. Furthermore, it is intuitively clear, as this is just the “quantum capacity” of the mediator.

Now let us consider correlation quantifiers obtained in the so-called “distance” approach [9,16]. The idea is to quantify correlation $Q_{X:Y}$ in state ρ_{XY} as the shortest distance $D(\rho_{XY}, \sigma_{XY})$ from ρ_{XY} to a set of states $\sigma_{XY} \in \mathcal{S}$ without the desired correlation property, i.e., $Q_{X:Y} \equiv \inf_{\sigma_{XY} \in \mathcal{S}} D(\rho_{XY}, \sigma_{XY})$. For example, the relative entropy of entanglement is given by the relative entropy of a state to the set of disentangled states [9]. It turns out that most such quantifiers are useful for the task introduced here. The conditions we require are that (i) \mathcal{S} is closed under local operations Λ_Y on Y , (ii) $D(\Lambda[\rho], \Lambda[\sigma]) \leq D(\rho, \sigma)$ (monotonicity), and (iii) $D(\rho_0, \rho_1) \leq D(\rho_0, \rho_2) + D(\rho_2, \rho_1)$ (triangle inequality). They are sufficient to prove the following theorem.

Theorem 1. Suppose a correlation $Q_{X:Y}$ satisfies properties (i)–(iii) listed above. If the evolution operator Λ_{ABC} is decomposable into $\Lambda_{BC}\Lambda_{AC}$, then

$$Q_{A:B}(t) \leq I_{AC:B}(0) + \sup_{|\psi\rangle} Q_{A:C}, \quad (1)$$

where $I_{AC:B}(0) = \inf_{\sigma_{AC} \otimes \sigma_B} D(\rho, \sigma_{AC} \otimes \sigma_B)$, ρ is the initial tripartite state, and the supremum of $Q_{A:C}$ is taken over pure states of AC .

Proof. The proof is given in Appendix A. ■

Note that although the relative entropy does not satisfy (iii) it still follows Theorem 1 (cf. Lemma 2 in Appendix A). Correlations between probe A and probe B are therefore

bounded by the maximal achievable correlation with the mediator, $\sup_{|\psi\rangle} Q_{A:C}$. The additional term $I_{AC:B}(0)$ reduces to the usual mutual information if $D(\rho_{XY}, \sigma_{XY})$ is the relative entropy distance [16] and characterizes the amount of total initial correlations between one of the probes and the rest of the system. Note that the bound is independent of time. This can be seen as a result of the effective description of such dynamics given by $\Lambda_{BC}\Lambda_{AC}$. The particle C is exchanged between A and B only once, independently of the duration of the dynamics.

In a typical experimental situation the initial state can be prepared as completely uncorrelated $\rho = \rho_A \otimes \rho_B \otimes \rho_C$, in which case Theorem 1 simplifies and the bound is given solely in terms of the “correlation capacity” of the mediator:

$$Q_{A:B}(t) \leq \sup_{|\psi\rangle} Q_{A:C}. \quad (2)$$

Clearly, the same bound holds for initial states of the form $\rho = \rho_{AC} \otimes \rho_B$. In Appendix B we show that, with this initial state, Eq. (2) holds for any correlation quantifier that is monotonic under local operations Λ_{BC} , not necessarily based on the distance approach, e.g., any entanglement monotone.

For initial states that are close to $\rho = \rho_{AC} \otimes \rho_B$ one can utilize the continuity of the von Neumann entropy [17] and see that $I_{AC:B}(0)$ in Eq. (1) is indeed small. We can also ensure that the initial state is of the form $\rho = \rho_{AC} \otimes \rho_B$ by performing a correlation breaking channel on B first. One example of such a channel is a measurement in the computational basis followed by a measurement in some complementary (say Fourier) basis. This implements the correlation breaking channel $(\mathbb{1}_{AC} \otimes \Lambda_B)(\rho_{ABC}) = \rho_{AC} \otimes \frac{\mathbb{1}}{d_B}$ [18]. In this way, our method does not require any knowledge of the initial state and any operations on the mediator, similar in spirit to the detection of quantum discord of inaccessible objects in Ref. [19]. We now move to concrete correlation quantifiers and their correlation capacities.

B. Exemplary measures and bounds

We provide four correlation quantifiers which capture different types of correlations between quantum particles. All of them are shown to be useful in detecting nondecomposability.

Mutual information is a measure of total correlations [20] and is defined as $I_{X:Y} = S_X + S_Y - S_{XY}$, where, e.g., S_X is the von Neumann entropy of subsystem X . It can also be seen as a distance-based measure with the relative entropy as the distance and a set of product states $\sigma_X \otimes \sigma_Y$ as \mathcal{S} [16]. The supremum in Eq. (2) is attained by the state (recall that $d_A > d_C$),

$$|\Psi\rangle = \frac{1}{\sqrt{d_C}} \sum_{j=1}^{d_C} |a_j\rangle |c_j\rangle, \quad (3)$$

where $|a_j\rangle$ and $|c_j\rangle$ form orthonormal bases. One finds $\sup_{|\psi\rangle} I_{A:C} = 2 \log_2(d_C)$.

An interesting quantifier in the context of nonclassicality detection is the classical correlation in a quantum state. It is defined as mutual information of the classical state obtained by performing the best local von Neumann measurements on the original state ρ [21], i.e., $C_{X:Y} = \sup_{\Pi_X \otimes \Pi_Y} I_{X:Y}(\Pi_X \otimes \Pi_Y(\rho))$, where $\Pi_X \otimes \Pi_Y(\rho) = \sum_{xy} |xy\rangle \langle xy| \rho |xy\rangle \langle xy|$, and

$|x\rangle, |y\rangle$ form orthonormal bases. The supremum of mutual information over classical states of AC is $\log_2(d_C)$.

Quantum discord is a form of purely quantum correlations that is broader than quantum entanglement. It can be phrased as a distance-based measure. In particular, we consider the relative entropy of discord [16], also known as the one-way deficit [22]. It is an asymmetric quantity defined as $\Delta_{X|Y} = \inf_{\Pi_Y} S(\Pi_Y(\rho)) - S(\rho)$, where Π_Y is a von Neumann measurement conducted on subsystem Y . The relative entropy of discord is maximized by the state (3), for which we have $\sup_{|\psi\rangle} \Delta_{A|C} = \log_2(d_C)$.

Our last example is negativity, a computable entanglement monotone [23]. For a bipartite system negativity is defined as $N_{X:Y} = (\|\rho^{T_X}\|_1 - 1)/2$, where $\|\cdot\|_1$ denotes the trace norm and ρ^{T_X} is a matrix obtained by partial transposition of ρ with respect to X . Negativity is maximized by the state (3), and the supremum reads $\sup_{|\psi\rangle} N_{A:C} = (d_C - 1)/2$.

Clearly, many other correlation quantifiers are suitable for our detection method because the assumptions behind Eqs. (1) and (2) are not demanding. In fact, one may wonder which correlations do not qualify for our method. A concrete example is the geometric quantum discord based on p -Schatten norms with $p > 1$, as it may increase under local operations on BC [24,25].

C. Violations

We now demonstrate, with concrete dynamics generated by noncommuting Hamiltonians, that the bounds derived can be violated. We next discuss the origin of this violation.

Consider a two-level atom C , i.e., $d_C = 2$, mediating interactions between two cavity fields A and B . A similar scenario has been considered and implemented, for example, in Refs. [5,8,26,27]. The interaction between the atom and each cavity field is taken to follow the Jaynes-Cummings model,

$$H = g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) + g(\hat{b}\hat{\sigma}_+ + \hat{b}^\dagger\hat{\sigma}_-), \quad (4)$$

where \hat{a} (\hat{b}) is the annihilation operator of field A (B), while $\hat{\sigma}_+$ ($\hat{\sigma}_-$) is the raising (lowering) operator of the two-level atom. For simplicity, we have assumed that the interaction strengths between the two-level atom and the fields are the same. Note that H is of the form $H_{AC} + H_{BC}$ with noncommuting components.

The resulting correlation dynamics are plotted in Fig. 2. Mutual information and negativity were calculated directly, whereas for the classical correlation and the relative entropy of discord, we provide the lower bounds $\tilde{C}_{A:B}$ and $-S_{A|B}$, respectively. $\tilde{C}_{A:B}$ is calculated as the mutual information of the state resulting from projective local measurements in the Fock basis (no optimization over measurements performed). The negative conditional entropy $-S_{A|B}$ is a lower bound on the distillable entanglement [28], which in turn is a lower bound on the relative entropy of entanglement $E_{A:B}$ [29]. Therefore, we note the chain of inequalities $-S_{A|B} \leq E_{A:B} \leq \Delta_{A|B} \leq I_{A:B}$, where the latter two inequalities follow from [16]. Already these lower bounds can beat the limit set by decomposable evolution, and therefore, all mentioned correlations can detect nondecomposability of the evolution. Since we consider closed systems, this infers noncommutativity

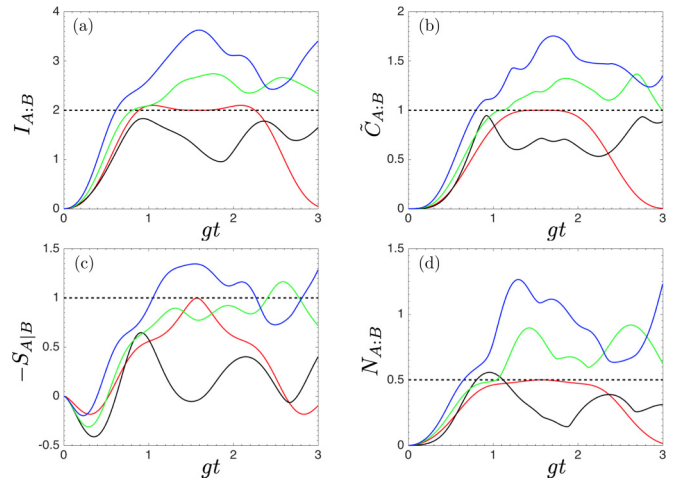


FIG. 2. Correlation dynamics with the Jaynes-Cummings model (solid curves) and the corresponding bounds for decomposable evolution (dashed lines). (a) Mutual information, (b) lower bound on the classical correlation (see the main text), (c) lower bound on the relative entropy of discord, and (d) negativity. In all cases, time is rescaled with the interaction strength g and the initial state of ABC is varied: $|110\rangle$ (red curves), $|101\rangle$ (black curves), $|210\rangle$ (green curves), and $|220\rangle$ (blue curves).

of the Jaynes-Cummings couplings. We also note another nonclassical feature of the studied dynamics: since Fig. 2 shows entanglement gain, according to Ref. [19] there must be quantum discord $D_{AB|C}$ during the evolution.

It is apparent that the detection is easier (faster and with more pronounced violation) with a higher number of photons in the initial states of the cavity fields. We offer an intuitive explanation. Consider, for example, $|mn0\rangle$ as the initial state of ABC . By defining $\hat{\xi} = (\hat{a} + \hat{b})/\sqrt{2}$, the Hamiltonian of Eq. (4) becomes $\sqrt{2}g(\hat{\xi}\hat{\sigma}_+ + \hat{\xi}^\dagger\hat{\sigma}_-)$ and it is straightforward to obtain the unitary evolution [30]. One finds that the quantum state of the fields oscillates incoherently between $\sum_{j=0}^{m+n} c_j(t)|j\rangle_A|m+n-j\rangle_B$ and $\sum_{j=0}^{m+n-1} d_j(t)|j\rangle_A|m+n-1-j\rangle_B$. Both of these states are superpositions of essentially $m+n$ biorthogonal terms giving rise to high entanglement and, therefore, also other forms of correlations.

Figure 2 illustrates that different correlation quantifiers have different detection capabilities and it is not clear at this stage whether there is a universal measure with which noncommutativity is detected, e.g., the fastest. For most initial states we studied mutual information detected noncommutativity the most rapidly, but there are exceptions, as shown by the black curve corresponding to the initial state $|101\rangle$. With this initial state the mutual information never violates its bound, but the negativity does.

III. DISCUSSION

Let us present the origin of the violation just observed. Since the total Hamiltonian is of the form $H_{AC} + H_{BC}$, the Suzuki-Trotter expansion of the resulting evolution is particularly illuminating,

$$e^{it(H_{AC}+H_{BC})} = \lim_{n \rightarrow \infty} (e^{i\Delta t H_{BC}} e^{i\Delta t H_{AC}})^n, \quad (5)$$

where $\Delta t = t/n$. If Hamiltonians do not commute, it is necessary to think about Eq. (5) as n sequences of pairwise interactions of C with A followed by C with B , each for a time Δt . Each pair of interactions can only increase correlations up to the correlation capacity of the mediator, but their multiple use allows the accumulation of correlations beyond what is possible with commuting Hamiltonians. Recall that, in the latter case, we deal with only one exchange of system C , independently of the duration of dynamics. We stress that Trotterization is just a mathematical tool and in the laboratory system C is continuously coupled to A and B . It is rather as if a virtual particle C were transmitted multiple times between A and B , interacting with each of them for a time Δt .

Our results imply that the noncommutativity (nondecomposability in general) is a desired feature of interactions in the task of correlation distribution, which is important for quantum information applications. As a contrasting physical illustration, we consider the strong dipole-dipole interactions in our field-atom-field example. The Hamiltonian reads

$$H' = g(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_+ + \hat{\sigma}_-) + g(\hat{b} + \hat{b}^\dagger)(\hat{\sigma}_+ + \hat{\sigma}_-), \quad (6)$$

with commuting components, i.e., $[H_{AC}, H_{BC}] = 0$. One can verify that the results of this model are in agreement with all the bounds we derived. Furthermore, we prove in Appendix C that, with this coupling, the state of AB at time t is effectively given by a two-qubit separable state. This makes $N_{A:B}(t) = 0$ and $I_{A:B}(t) \leq 1$. Note the counterintuitive result that strong interactions produce bounded correlations between the probes, while weak interactions (Jaynes-Cummings coupling) can increase the correlations above the bounds.

We also note an application of our bounds to estimate the dimension of the mediator; see, e.g., Refs. [31–33] for other dimension witnesses. For decomposable evolution, including discrete sequential operators considered in Refs. [10–15], the amount of correlation between the probes is bounded by the correlation capacity $\sup_{|\psi\rangle} Q_{A:C}$, which is a function of d_C . If one observes a $Q_{A:B}(t)$ value that is larger than the correlation capacity of a certain d_C , then the dimension of the mediator must be larger than d_C .

Finally, we wish to discuss a scenario where the three systems are open to their own *local* environments, as realized, e.g., in [8]. We take the evolution following the master equation in Lindblad form,

$$\dot{\rho} = -i[H_{AC} + H_{BC}, \rho] + \sum_{X=A,B,C} L_X \rho, \quad (7)$$

$$L_X \rho = \sum_k Q_k^X \rho Q_k^{X\dagger} - \frac{1}{2} \{Q_k^{X\dagger} Q_k^X, \rho\},$$

where the last term in (7) is the incoherent part of the evolution and L_X describes interactions of system X with its local environment, i.e., the operators Q_k^X act on system X only. We denote $\mathcal{L}_{AC} = -i[H_{AC}, \cdot] + L_A + L_C$ and $\mathcal{L}_{BC} = -i[H_{BC}, \cdot] + L_B$. One readily verifies that if $[H_{AC}, H_{BC}] = 0$ and $[L_C, H_{BC}] = 0$, we have commuting Lindblad operators $[\mathcal{L}_{AC}, \mathcal{L}_{BC}] = 0$. Note that, if one includes L_C in \mathcal{L}_{BC} instead, the second condition for commuting Lindblad operators now reads $[L_C, H_{AC}] = 0$. For commuting Lindbladians, the

corresponding evolution decomposes as $\Lambda_{BC}\Lambda_{AC}$, or vice versa. Therefore, our bounds apply accordingly. Their violation implies that either the Hamiltonians do not commute or the operators describing dissipative channels on C do not commute with H_{AC} and H_{BC} . In particular, if C is kept isolated so that its noise can be ignored, the violation of our bounds is solely the result of the noncommutativity of the Hamiltonians.

IV. CONCLUSIONS

We linked noncommutativity of interaction Hamiltonians (nondecomposability of time evolution in general) to the amount of correlations that can be created in the associated dynamics. This led us to a method for detection of nondecomposability of evolution in a scenario where subsystem C mediates interactions between A and B (all these objects can interact with their local environments). The method requires no explicit form of the evolution operators or knowledge of the initial state of the tripartite system. Nondecomposability is detected by observing violation of certain bounds on AB correlations, as measured by most correlation quantifiers. Furthermore, no operation on C is necessary at any time, which makes this strategy experimentally friendly. In particular, in addition to avoiding characterization of the interactions, the physics of C can remain largely unknown—only its dimension should be identified.

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APPENDIX A: PROOF OF THEOREM 1

For completeness let us begin with a useful lemma.

Lemma 1. For a measure of correlations $Q_{X:Y}$, between party X and party Y , that is nonincreasing under local operations on Y , the following property holds: $Q_{X:Y}$ is invariant under tracing-out of uncorrelated systems on the side of Y .

Proof. Since the correlation measure is nonincreasing under local operations on Y , tracing out an uncorrelated system on the side of Y can only decrease the correlation. However, if the correlation is strictly decreasing, then there is a reverse process, i.e., attaching the uncorrelated system back and, therefore, increasing the correlation. Hence, the correlation $Q_{X:Y}$ has to be invariant under tracing-out of uncorrelated systems on Y . In fact, this is true for all reversible operations.

Our main theorem is proven as follows.

Theorem 1. Consider a correlation measure $Q_{X:Y} \equiv \inf_{\sigma_{XY} \in \mathcal{S}} D(\rho_{XY}, \sigma_{XY})$ satisfying the following properties:

- (i) \mathcal{S} is closed under local operations Λ_Y on Y ;
- (ii) $D(\Lambda[\rho], \Lambda[\sigma]) \leq D(\rho, \sigma)$; and
- (iii) $D(\rho_0, \rho_1) \leq D(\rho_0, \rho_2) + D(\rho_2, \rho_1)$.

If the evolution operator Λ_{ABC} is decomposable into $\Lambda_{BC}\Lambda_{AC}$, then

$$Q_{A:B}(t) \leq I_{AC:B}(0) + \sup_{|\psi\rangle} Q_{A:C}, \quad (\text{A1})$$

where $I_{AC:B}(0) = \inf_{\sigma_{AC} \otimes \sigma_B} D(\rho, \sigma_{AC} \otimes \sigma_B)$, ρ is the initial tripartite state, and the supremum of $Q_{A:C}$ is taken over pure states of AC .

Proof. Properties (i) and (ii), and the definition of $Q_{X:Y}$ as the shortest distance, imply that $Q_{X:Y}$ is nonincreasing under local operations on Y . Accordingly, the property proven in Lemma 1 applies. We have

$$Q_{A:B}(t) \leq Q_{A:BC}(\Lambda_{BC}\Lambda_{AC}[\rho]) \quad (\text{A2})$$

$$\leq Q_{A:BC}(\Lambda_{AC}[\rho]) \quad (\text{A3})$$

$$\leq D(\Lambda_{AC}[\rho], \mu) \quad (\text{A4})$$

$$\leq D(\Lambda_{AC}[\rho], \Lambda_{AC}[\sigma_{AC}^0] \otimes \sigma_B^0) + D(\Lambda_{AC}[\sigma_{AC}^0] \otimes \sigma_B^0, \mu) \quad (\text{A5})$$

$$\leq D(\rho, \sigma_{AC}^0 \otimes \sigma_B^0) + D(\Lambda_{AC}[\sigma_{AC}^0] \otimes \sigma_B^0, \mu) \quad (\text{A6})$$

$$= I_{AC:B}(0) + Q_{A:BC}(\Lambda_{AC}[\sigma_{AC}^0] \otimes \sigma_B^0) \quad (\text{A7})$$

$$= I_{AC:B}(0) + Q_{A:C}(\Lambda_{AC}[\sigma_{AC}^0]) \quad (\text{A8})$$

$$\leq I_{AC:B}(0) + \sup_{|\psi\rangle} Q_{A:C}, \quad (\text{A9})$$

where the steps are justified as follows. In line (A2) we have used the fact that $Q_{X:Y}$ is nonincreasing under local operations on Y (tracing out C). Line (A3) follows, as $Q_{A:BC}$ is nonincreasing under local operation Λ_{BC} . The next line, (A4), utilizes the definition of $Q_{A:BC}$ as the shortest distance to the set of states $\mu \in \mathcal{S}_{A:BC}$. The inequality of (A5) follows from the triangle inequality (iii). Note that the first distance in (A5) does not depend on μ and at this point one can choose any σ_{AC}^0 and σ_B^0 . The inequality (A6) invokes property (ii). In (A7), we have chosen $\sigma_{AC}^0 \otimes \sigma_B^0$ as the closest product state to ρ and μ as a state in $\mathcal{S}_{A:BC}$ closest to $\Lambda_{AC}[\sigma_{AC}^0] \otimes \sigma_B^0$. Line (A8) uses the invariance of $Q_{A:BC}$ under tracing-out of the uncorrelated system σ_B^0 . For the final inequality, we note that a correlation measure that is nonincreasing under local operations on at least one side must be maximal on pure states [34].

Lemma 2. The conclusion in Theorem 1 still follows for the relative entropy as a distance measure.

Proof. Let us begin with an identity,

$$\begin{aligned} S(\rho||\sigma_X \otimes \sigma_Y) &= \text{tr}(\rho \log_2 \rho - \rho \log_2 \sigma_X \otimes \sigma_Y) \\ &= \text{tr}(\rho \log_2 \rho - \rho \log_2 \rho_X \otimes \rho_Y) \\ &\quad + \text{tr}(\rho \log_2 \rho_X \otimes \rho_Y - \rho \log_2 \sigma_X \otimes \sigma_Y), \\ &= S(\rho||\rho_X \otimes \rho_Y) + S(\rho_X||\sigma_X) + S(\rho_Y||\sigma_Y), \end{aligned} \quad (\text{A10})$$

where ρ_X and ρ_Y are the marginals of ρ and we have used, for example, relation $\text{tr}(\rho \log_2 \sigma_X \otimes \sigma_Y) = \text{tr}(\rho_X \log_2 \sigma_X) + \text{tr}(\rho_Y \log_2 \sigma_Y)$.

Although relative entropy satisfies (ii) [35], it is well known not to follow (iii). Therefore, starting from (A3), we

have

$$\begin{aligned} Q_{A:BC}(\Lambda_{AC}[\rho]) &= \inf_{\mu \in \mathcal{S}_{A:BC}} S(\Lambda_{AC}[\rho]||\mu) \quad (\text{A11}) \\ &\leq S(\Lambda_{AC}[\rho]||\mu_{AC} \otimes \mu_B) \quad (\text{A12}) \end{aligned}$$

$$\begin{aligned} &= S(\Lambda_{AC}[\rho]||\rho'_{AC} \otimes \rho'_B) \\ &\quad + S(\rho'_{AC}||\mu_{AC}) + S(\rho'_B||\mu_B) \quad (\text{A13}) \end{aligned}$$

$$= I_{AC:B}(\Lambda_{AC}[\rho]) + Q_{A:C}(\rho'_{AC}) \quad (\text{A14})$$

$$\leq I_{AC:B}(0) + \sup_{|\psi\rangle} Q_{A:C}, \quad (\text{A15})$$

where ρ'_{AC} and ρ'_B are marginals of $\Lambda_{AC}[\rho]$. The steps above are justified as follows. Line (A12) follows for any state of the form $\mu_{AC} \otimes \mu_B \in \mathcal{S}_{A:BC}$. We have used identity (A10) in line (A13). The equality (A14) uses the definition of mutual information as the relative entropy from a state to its marginals [16]. We have also chosen μ_{AC} as a state in $\mathcal{S}_{A:C}$ closest to ρ'_{AC} and $\mu_B = \rho'_B$. The last line follows as mutual information is non-increasing under local operation Λ_{AC} and the correlation $Q_{A:C}$ achieves the supremum on pure states.

APPENDIX B: PROOF OF EQ. (2) FOR CORRELATIONS ONLY MONOTONIC UNDER LOCAL OPERATIONS Λ_{BC}

Theorem 2. Suppose the initial state has the form $\rho = \rho_{AC} \otimes \rho_B$. If the evolution operator is decomposable into $\Lambda_{BC}\Lambda_{AC}$, then

$$Q_{A:B}(t) \leq \sup_{|\psi\rangle} Q_{A:C} \quad (\text{B1})$$

for all correlation measures, Q , nonincreasing under local operations Λ_{BC} .

Proof. For initial states of the form $\rho_{AC} \otimes \rho_B$ we have the chain of arguments

$$\begin{aligned} Q_{A:B}(t) &\leq Q_{A:BC}(t) \leq Q_{A:BC}(\Lambda_{AC}[\rho]) \\ &= Q_{A:C}(\Lambda_{AC}[\rho]) \leq \sup_{|\psi\rangle} Q_{A:C}, \end{aligned} \quad (\text{B2})$$

where the steps are justified as follows. Since the action of tracing out (the, in general, correlated) system C is a local operation on BC , we obtain the first inequality. The second inequality follows as the correlation is nonincreasing under Λ_{BC} . As we start with the initial state $\rho_{AC} \otimes \rho_B$ and Λ_{AC} does not act on B , system B stays uncorrelated in $\Lambda_{AC}[\rho]$. Using Lemma 1, we have the equality. Finally, the correlation $Q_{A:C}$ is again maximal on pure states.

APPENDIX C: PROOF OF SEPARABILITY VIA DIPOLE-DIPOLE COUPLING FOR PARTICULAR INITIAL STATES

Let us define $\hat{\xi} = (\hat{a} + \hat{b})/\sqrt{2}$. The dipole-dipole Hamiltonian, Eq. (6), is reformulated as $H' = \sqrt{2}g(\hat{\xi} + \hat{\xi}^\dagger)\hat{\sigma}_x$, where $\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$ and $[\hat{\xi}, \hat{\xi}^\dagger] = \mathbb{1}$. The unitary evolution operator

is given by

$$\begin{aligned}\hat{U}_t &= e^{-iH't} \\ &= \frac{1}{2}[(\mathbb{1} - \hat{\sigma}_x)e^{i\sqrt{2}gt(\hat{\xi} + \hat{\xi}^\dagger)} + (\mathbb{1} + \hat{\sigma}_x)e^{-i\sqrt{2}gt(\hat{\xi} + \hat{\xi}^\dagger)}] \\ &= \frac{1}{2}[(\mathbb{1} - \hat{\sigma}_x)\hat{D}_a(\alpha)\hat{D}_b(\alpha) + (\mathbb{1} + \hat{\sigma}_x)\hat{D}_a(-\alpha)\hat{D}_b(-\alpha)],\end{aligned}\quad (\text{C1})$$

where $\alpha =igt$ and, e.g., $\hat{D}_a(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$. Given an initial state $|mn0\rangle$, the state at time t reads

$$\begin{aligned}|\psi_t\rangle &= \frac{1}{4}[(d_{++}^{(mn)}|D_+^{(m)}, D_+^{(n)}\rangle + d_{--}^{(mn)}|D_-^{(m)}, D_-^{(n)}\rangle)|0\rangle \\ &\quad - (d_{+-}^{(mn)}|D_+^{(m)}, D_-^{(n)}\rangle + d_{-+}^{(mn)}|D_-^{(m)}, D_+^{(n)}\rangle)|1\rangle],\end{aligned}$$

where

$$\begin{aligned}d_{\pm\pm}^{(mn)} &= 2\sqrt{[1 \pm e^{-2|\alpha|^2}L_m(4|\alpha|^2)][1 \pm e^{-2|\alpha|^2}L_n(4|\alpha|^2)]}, \\ |D_\pm^{(n)}\rangle &= \frac{1}{\sqrt{d_{\pm\pm}^{(nn)}}}[\hat{D}(\alpha) \pm \hat{D}(-\alpha)]|n\rangle.\end{aligned}$$

Note that $\langle D_+^{(n)}|D_-^{(n)}\rangle = 0$ and $\langle D_\pm^{(n)}|D_\pm^{(n)}\rangle = 1$. $L_n(|\alpha|^2)$ is the Laguerre polynomial, which comes from the relation $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}L_n(|\alpha|^2)$. After tracing-out of the atomic mode C , the state of the fields is effectively given by a two-qubit state,

$$\frac{1}{16} \begin{pmatrix} (d_{++}^{(mn)})^2 & 0 & 0 & d_{++}^{(mn)}d_{--}^{(mn)} \\ 0 & (d_{+-}^{(mn)})^2 & d_{++}^{(mn)}d_{--}^{(mn)} & 0 \\ 0 & d_{++}^{(mn)}d_{--}^{(mn)} & (d_{-+}^{(mn)})^2 & 0 \\ d_{+-}^{(mn)}d_{-+}^{(mn)} & 0 & 0 & (d_{--}^{(mn)})^2 \end{pmatrix}, \quad (\text{C2})$$

which is positive under partial transposition and, hence, separable [36,37]. The same result follows for initial state $|mn1\rangle$.

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