Exponentially many entanglement and correlation constraints for multipartite quantum states

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We present a family of correlation constraints that apply to all multipartite quantum systems of finite dimension. The size of this family is exponential in the number of subsystems. We obtain these relations by defining and investigating the generalized state inversion map. This map provides a systematic way to obtain local unitary invariants of degree two in the state and is directly linked to the shadow inequalities proved by Rains [IEEE Trans. Inf. Theory **46**, 54 (2000)]. The constraints are stated in terms of linear inequalities for the linear entropies of the subsystems. For pure quantum states they turn into monogamy relations that constrain the distribution of bipartite entanglement among the subsystems of the global state.

I. INTRODUCTION

The discussion of entanglement monogamy started more than two decades ago [1]. Its first precise quantitative formulation was given by Coffman *et al.* in an *equality* for the distribution of entanglement among three qubits [2], whereas its weaker form, an inequality, subsequently was generalized by Osborne and Verstraete [3] to an arbitrary number of qubits. In the meantime, there have been numerous attempts to extend the results of Refs. [2,3] or to find alternative independent monogamy relations; see, e.g., Refs. [4–20]. Furthermore, it was found that correlations other than entanglement, such as nonlocality, also may obey monogamy relations [21–23].

Some authors consider monogamy of correlations an inherent property of quantum mechanics [24]; however, there are results that seem to challenge this point of view: (1) the fundamental monogamy relations by Coffman *et al.* [2] and Osborne and Verstraete [3] cannot straightforwardly be generalized to local dimensions higher than two [5], (2) faithfulness of entanglement measures and monogamy properties seem to be mutually exclusive [13], and (3) systematically including contributions of multipartite entanglement appears to be difficult [8,14]. This also raises the question of what the general form of a monogamy relation should be [8,10,13,18,24]. Generally, it is assumed that the terms characterizing different correlations have to be added, possibly after raising each term to some fixed power. Again, this seems to contradict the recent finding that general monogamy equalities, as well as inequalities, for any number of qubits [12] and even higher-dimensional systems [20] exist whose terms are summed with alternating signs.

In the present work, we adopt the viewpoint that *any* functional relation between quantifiers for different correlations (equality or inequality) may be considered a monogamy relation, simply because it constrains the free distribution of

these correlations among the parties of a multipartite system. The relevant point is that the terms in the relation are of physical significance. If, for example, all the terms are related to measures of entanglement in different subsets of the parties, one would call the correlation constraint a monogamy relation for entanglement, because it describes restrictions regarding the distribution of entanglement among the parties. An illuminating example that this approach is sensible is that certain correlation constraints in Refs. [12,20] hold for both pure and mixed states, but represent monogamy relations for entanglement only in the case of pure states.

It is natural to expect a variety of correlation constraints originating from the algebraic properties of the density matrix, such as the positivity of the state. Based on this intuition, our objective is to devise a method to systematically generate an entire family of correlation constraints. Our central results show that the positivity condition under certain mappings alone gives rise to an exponential number of independent correlation constraints, as well as to monogamy relations for entanglement in multipartite pure states of any number of parties and finite local dimension. Our method to derive these relations is based on and extends the so-called universal state inversion [25,26]. It turns out that the generalized inverter map is directly related to Rains' shadow inequalities [27,28], which, by virtue of these investigations, can be assigned a direct physical interpretation.

In Sec. II we explain universal state inversion by discussing relevant examples for entropy inequalities that can directly be derived from the inverted state. It is then straightforward to understand the definition and properties of the generalized state inversion map, which we introduce in Sec. III. After presenting our main results (Secs. IV and V) we discuss several routes of investigation that get input through our findings; these include detection of entanglement, derivation of

inequalities for the linear entropy, and the quantum marginal problem (Sec. VI).

II. ENTROPY INEQUALITIES FROM UNIVERSAL STATE INVERSION

In the following we consider normalized states of an N-partite system $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$, where $\mathcal{H}_j$ are Hilbert spaces with dim $\mathcal{H}_j = d_j$ $(j = 1, \ldots, N)$ and $\text{Tr}(\rho) = 1$. Let us start with bipartite systems, $N = 2$. We denote the global state by ρ_{12} while the reduced state of the first subsystem is $\rho_1 = \text{Tr}_2(\rho_{12})$, and analogously $\rho_2 = \text{Tr}_1(\rho_{12})$. In Ref. [26] it was shown that the operator

$$\tilde{\rho}_{12} = \mathbb{1}_{12} - \rho_1 \otimes \mathbb{1}_2 - \mathbb{1}_1 \otimes \rho_2 + \rho_{12} \geqq 0, \qquad (1)$$

is positive semidefinite. Here $\mathbb{1}_j$ is the identity operator acting on subsystem $j = 1, 2$, and $\mathbb{1}_{12}$ the one for the full system. By multiplying Eq. (1) by ρ_{12} and applying the trace as well as the definition for the linear entropy of subsystem j, $\tau_j = 2[1 - \text{Tr}(\rho_j^2)]$, one obtains the well-known subadditivity of linear entropy [26,29,30]

$$\tau_{12} \leqq \tau_1 + \tau_2. \qquad (2)$$

One recognizes the usefulness of the operator $\tilde{\rho}_{12}$, the result of the *universal state inversion map* applied to the state ρ_{12}. It arises through subsequently tracing out all of the subsets of parties and padding with identities, multiplying by (-1) per trace operation, and adding up the results. Analogously, universal state inversion for a three-party state ρ_{123} yields

$$\begin{aligned} 0 \leqq \tilde{\rho}_{123} = &\mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 - \rho_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 - \mathbb{1}_1 \otimes \rho_2 \otimes \mathbb{1}_3 \\ &- \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \rho_3 + \rho_{12} \otimes \mathbb{1}_3 + \rho_{13} \otimes \mathbb{1}_2 \\ &+ \mathbb{1}_1 \otimes \rho_{23} - \rho_{123} \\ \equiv &\mathbb{1} - \rho_1 - \rho_2 - \rho_3 + \rho_{12} + \rho_{13} + \rho_{23} - \rho_{123}. \end{aligned} \qquad (3)$$

In analogy with the operations above the inequality

$$\tau_1 + \tau_2 + \tau_3 + \tau_{123} \geqq \tau_{12} + \tau_{13} + \tau_{23} \qquad (4)$$

is found [20]. It resembles a symmetrized and reversed version of the strong subadditivity inequality for the von Neumann entropy S [31], which reads $S_{123} + S_2 \leqq S_{12} + S_{23}$. We mention that the analog of inequality (4) for von Neumann entropy was discussed by Hayden *et al.* [32], as a quantum extension to the so-called interaction information [33] and a desirable monogamy property in the context of holographic theories.

Summarizing this introduction, one can use the positivity of the universal state inversion map [20,26,34,35]

$$\mathcal{I}(\rho) = \left\{ \prod_{j=1}^{N}[\text{Tr}_j(\cdot) \otimes \mathbb{1}_j - \text{id}] \right\} \rho \qquad (5)$$

to derive relevant inequalities, or *correlation constraints*, for arbitrary states of multipartite systems of any finite local dimension. In Eq. (5), id denotes the identity map. An alternative way of writing the map $\mathcal{I}(\rho)$ in terms of reduced states $\rho_S = \text{Tr}_{S^c}(\rho)$ is

$$\mathcal{I}(\rho) = \sum_{S \subseteq \{1,\ldots,N\}} (-1)^{|S|} \rho_S \otimes \mathbb{1}_{S^c}, \qquad (6)$$

where S is a set of subsystem indices, S^c is its complement $S^c = \{1, \ldots, N\} \setminus S$, and $|S|$ denotes the cardinality of S. From Eq. (6) it is evident that $\mathcal{I}(\rho)$ commutes with local unitary operations [20,26,34]. In what follows we will generalize the inversion map and obtain a powerful tool for the analysis of correlations in arbitrary finite-dimensional multiparty states.

III. GENERALIZED T-INVERTER

We obtain a more general form of the state inversion map, Eq. (5), by reversing the minus sign in some of the factors. Assume we retain a minus sign only for all those subsystem indices that are contained in $T \subseteq \{1, \ldots, N\}$, and the other factors come with a plus sign. Then the generalized *T-inversion map* $\mathcal{I}_T(\cdot)$ can elegantly be written as

$$\mathcal{I}_T(\rho) = \left\{ \prod_{j=1}^{N}[\text{Tr}_j(\cdot) \otimes \mathbb{1}_j + (-1)^{|T \cap \{j\}|}\text{id}] \right\} \rho \qquad (7)$$

$$= \sum_{S \subseteq \{1\ldots N\}} (-1)^{|S \cap T|}(\text{Tr}_{S^c} \rho) \otimes \mathbb{1}_{S^c}. \qquad (8)$$

The original state inverter Eq. (5) is found for $T = \{1, \ldots, N\}$.

Interestingly, there exists a representation of the map $\mathcal{I}_T(\cdot)$ in Kraus form, which we will derive now. First, let us consider the representation of a single factor in Eq. (7) acting on a d-level system. To this end, we note that for a complete basis of traceless Hermitian matrices, $\{h_m\}$ complemented by $h_0 \equiv \mathbb{1}$, with $\text{Tr}(h_m h_n) = d\delta_{mn}$, and a Hermitian operator A we have [20]

$$\text{Tr}(A)\mathbb{1} = \frac{1}{d} \sum_{m=0}^{d^2-1} h_m A h_m, \qquad (9)$$

$$A^T = \frac{1}{d} \sum_{m=0}^{d^2-1} h_m^T A h_m. \qquad (10)$$

With these relations it is easy to find the action of the jth factor in Eq. (7) on A_j (a Hermitian operator that acts on a d_j-dimensional Hilbert space),

$$\text{Tr}(A_j)\mathbb{1}_j - A_j = \frac{2}{d_j} \sum_{k<l}^{d_j-1} y_{kl} A_j^* y_{kl} \qquad (11)$$

$$\text{Tr}(A_j)\mathbb{1}_j + A_j = \frac{2}{d_j} \left[A_j^* + \sum_{k<l}^{d_j-1} x_{kl} A_j^* x_{kl} \right. \\ \left. + \sum_{k=1}^{d_j-1} z_k A_j^* z_k \right], \qquad (12)$$

where we have used the generalized Gell-Mann matrices [36].

In Eqs. (11) and (12) we clearly observe the Kraus form of the map. Note that it is applied to the complex conjugate A_j^*. Since all the factors in Eq. (7) commute, the Kraus form extends to the entire operator product, with the Kraus operators on the full system being tensor products of the single-system generators (for details see Appendix A). Thus, the map $\mathcal{I}_T(\cdot)$ on the full system can be written as $\mathcal{I}_T = \Lambda \circ K$

where Λ is the Kraus map and K is the complex conjugation. The existence of this representation proves the positivity of the generalized T-inversion map, Eqs. (7) and (8). We mention also that, on application of this map to Hermitian operators, the complex conjugation may be replaced by a transposition (see also below). In this sense, first transposing the state and subsequently applying $\mathcal{I}_T(\cdot)$ may be viewed as a generalization of the Werner-Holevo channel [37] to multipartite systems and is a completely positive map.

A consequence of the positivity of $\mathcal{I}_T(\cdot)$ is that, for two semidefinite positive operators M_1, M_2, we have [38]

$$\mathrm{Tr}[\, M_1 \, \mathcal{I}_T(M_2)\,] \;\geqq\; 0. \tag{13}$$

By inserting Eq. (8) and noting that $\mathrm{Tr}\,[M_1 \, \mathrm{Tr}_{S^c}(M_2)] = \mathrm{Tr}\,[\mathrm{Tr}_{S^c}(M_1)\,\mathrm{Tr}_{S^c}(M_2)]$ we obtain

$$\sum_{S \subseteq \{1,\dots,N\}} (-1)^{|S \cap T|} \mathrm{Tr}[\mathrm{Tr}_{S^c}(M_1)\,\mathrm{Tr}_{S^c}(M_2)] \geqq 0. \tag{14}$$

That is, as a by-product of the definition of generalized T-inversion we have derived Rains' shadow inequalities, Eq. (14) [27,28], which are an important tool for investigating the existence of quantum error correcting codes [39–43]. Indeed, to detect the nonexistence of codes and highly entangled states, different choices of subsets T (corresponding to different distribution of minus signs in the T-inverter) are necessary, depending on the parameters of the code or state under consideration. It is remarkable that the shadow inequalities are directly linked to generalized state inversion, which in turn is connected with correlation and entanglement distribution constraints, as we will show below. The shadow inequalities provide a quick alternative proof for the positivity of the generalized T-inversion map: Choose $M_1 = |\psi\rangle\langle\psi|$, $M_2 = \rho$ in Eq. (14), where $|\psi\rangle$ is an arbitrary finite-dimensional pure state, and ρ is an arbitrary state of the same multiparty system. This yields $\langle\psi|\mathcal{I}_T(\rho)|\psi\rangle \geqq 0$, implying positivity of $\mathcal{I}_T(\rho)$.

There is another interesting property of the generalized T-inversion, which may be called "coarse graining." Consider, for example, the tripartite state ρ_{123} for which we may combine (coarse grain) the subpartitions 2 and 3 into a single partition, so that we end up with a bipartite state $\rho_{1(23)}$. Suppose we want to apply an inversion map with $T^{(\mathrm{coarse})} = \{(23)\}$ to the coarse-grained state $\rho_{1(23)}$ [i.e., $\mathcal{I}_{\{(23)\}}(\rho_{1(23)})$], can we build it from the inverted states on the fine-grained system? The answer is positive: One has to average over all those T-inverted states of the fine-grained system with the following rules for each set of subsystem indices grouped into a single party: (a) the sets T characterizing the fine-grained inversions have odd parity for those single-system indices appearing in $T^{(\mathrm{coarse})}$; (b) for the single-system indices not appearing in $T^{(\mathrm{coarse})}$ the parity in the fine-grained T sets has to be even. Hence, in our example $\mathcal{I}_{\{(23)\}}(\rho_{1(23)}) = \frac{1}{2}[\mathcal{I}_{\{2\}}(\rho_{123}) + \mathcal{I}_{\{3\}}(\rho_{123})]$. As special cases of this property we have for an N-partite state ρ

$$\mathbb{1} \,-\, \rho = \frac{1}{2^{N-1}} \sum_{T \subseteq \{1,\dots,N\},\ |T|\ \mathrm{odd}} \mathcal{I}_T(\rho), \tag{15a}$$

$$\mathbb{1} \,+\, \rho = \frac{1}{2^{N-1}} \sum_{T \subseteq \{1,\dots,N\},\ |T|\ \mathrm{even}} \mathcal{I}_T(\rho). \tag{15b}$$

We present a detailed proof in Appendix B.

IV. EXPONENTIALLY MANY CORRELATION CONSTRAINTS

Consider the special case $M_1 = M_2 = \rho$ of Eq. (13),

$$\mathrm{Tr}\,[\rho\,\mathcal{I}_T(\rho)] \geqq 0. \tag{16}$$

This observation gives rise to a notable set of constraints on the possible correlations in a multipartite state. We use the decomposition of the generalized inverted state in Eq. (8) as well as the definition of the linear entropy to expand Eq. (16) and find (for $T \neq \emptyset$)

$$
\begin{aligned}
0 &\leqq \sum_{S \subseteq \{1,\dots,N\}} (-1)^{|S \cap T|}\,\mathrm{Tr}\,[\rho\,\mathrm{Tr}_{S^c}(\rho)] \\
&= \sum_{S \subseteq \{1,\dots,N\}} (-1)^{|S \cap T|}\,\mathrm{Tr}\,(\rho_S^2) \\
&= \frac{1}{2} \sum_{\emptyset \neq S \subseteq \{1,\dots,N\}} (-1)^{|S \cap T|+1}\,\tau_S.
\end{aligned}
\tag{17}
$$

For each choice of $T \neq \emptyset$, this is a constraint for the correlations across the bipartite splits $S|S^c$ as quantified by the linear entropy, with different distribution of minus signs. Altogether these are $2^N - 1$ relations (the condition for $T = \emptyset$ is trivial in view of the fact that all subsystem purities are positive). We show in Appendix C that the right-hand sides of these inequalities are functionally independent.

The exponentially many correlation constraints in Eq. (17) constitute the first key result of our work. These are necessary conditions related to the quadratic local unitary invariants $\mathrm{Tr}\,(\rho_S^2)$ of any finite-dimensional multiparty quantum state. It is particularly satisfactory that these conditions do not originate from ad hoc assumptions regarding their functional form. Rather, they arise systematically through the definition and algebraic properties of generalized T-inversion.

V. MONOGAMY OF ENTANGLEMENT

A. Local unitary invariants $\mathscr{C}_T(\rho)$

The generalized T-inverter commutes with local unitaries, so that we can define the local unitary invariants [44,45]

$$\mathscr{C}_T(\rho) \;\equiv\; \sqrt{\mathrm{Tr}[\rho\,\mathcal{I}_T(\rho)]}. \tag{18}$$

As we have seen, these invariants are relevant because they generate the correlation constraints (17). Therefore, let us briefly mention some properties of $\mathscr{C}_T(\rho)$.

A direct consequence of the Kraus form of $\mathcal{I}_T(\cdot)$ is that, for pure states $\rho_\psi = |\psi\rangle\langle\psi|$, the local invariant $\mathscr{C}_T(\psi)$ vanishes whenever there is an odd number of factors with a minus sign in Eq. (7),

$$\mathscr{C}_T(\psi) \;=\; 0 \quad \text{for} \quad |T| \,\equiv\, 1 \pmod 2 \tag{19}$$

(we give a proof of this fact in Appendix D).

Furthermore, we recall that in Ref. [20] the distributed concurrence $C_D(\psi) \equiv \sqrt{\mathrm{Tr}\,[\rho_\psi\,\mathcal{I}_{\{1,\cdots,N\}}(\rho_\psi)]}$ was defined, which generalizes the well-known (bipartite) concurrence [26] and in certain cases is an entanglement monotone, that is, nonincreasing on average under stochastic local operations and communication. Because of the apparent analogies with Eq. (18) the question arises whether the other invariants

$\mathscr{C}_T(\psi)$ possibly are entanglement monotones. The answer is: None of the local invariants $\mathscr{C}_T(\psi)$ with $T \neq \{1, \dots, N\}$ can be an entanglement monotone. The proof will also be shown in Appendix D and essentially relies on the factorization property

$$\mathscr{C}_T(\rho_{\text{prod}}) = \mathscr{C}_{T_S}(\rho_S)\mathscr{C}_{T_{S^c}}(\rho_{S^c}) \qquad (20)$$

for product states $\rho_{\text{prod}} = \rho_S \otimes \rho_{S^c}$.

B. Exponentially many monogamy relations

Even though $\mathscr{C}_T(\psi)$ is not in general an entanglement monotone, Eq. (17) leads to exponentially many monogamy relations for the entanglement in the pure state $|\psi\rangle$, because for a pure state the linear entropy of subsystem S equals the squared concurrence over the bipartite split $S|S^c$, that is, $\tau_S = 2[1 - \text{Tr}(\rho_S^2)] \equiv C_{S|S^c}^2$.

Thus we have the second main result of our article, the $2^N - 1$ monogamy inequalities

$$0 \leqq \sum_{\emptyset \neq S \subset \{1, \dots, N\}} (-1)^{|S \cap T|+1} C_{S|S^c}^2(\psi), \qquad (21)$$

one inequality for each $\emptyset \neq T \subseteq \{1, \dots, N\}$, which are valid for any number of parties N and any finite local dimensions. These inequalities constrain the distribution of concurrence among the subsystems of any global pure state $|\psi\rangle$. They are related to the local unitary invariants of homogeneous degree two in the state. Note, however, that according to their definitions the invariants $\mathscr{C}_T(\psi)$ and the concurrence $C_{S|S^c}(\psi)$ are of homogeneous degree one in the state ρ_ψ, whereas the relations (21) [as well as Eq. (17)] are of homogeneous degree two, just as the results, e.g., in Refs. [2,3,12,20]. Again, there is no *ad hoc* assumption underlying these constraints, they naturally follow from the algebraic properties of the generalized T-inverter.

VI. APPLICATIONS

A. Entanglement detection

In Ref. [25] it was first observed that the reduction map $\Lambda(\rho) = \mathbb{1} - \rho$ is positive but not completely positive: For nonseparable bipartite states $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ we may have $(\mathbb{1} \otimes \Lambda)(\rho_{AB}) \not\geq 0$. Consequently, this map can be used to detect entanglement in the state ρ_{AB} (reduction criterion). Later, similar maps were studied [35,46–49]. As we have seen above, the state inversion map can be represented by a concatenation of a transposition and a subsequent Kraus channel. This elucidates that also the reduction criterion includes a partial transposition. While the reduction criterion is well known to detect fewer entangled states than the positive partial transpose criterion, the states it detects are guaranteed to be distillable [25]. In the spirit of Refs. [48,49] we may even further generalize the T-inversion map by introducing real numbers $0 \leqq \alpha_j, \beta_k \leqq 1$,

$$\mathcal{I}_T^{\{\alpha_j, \beta_k\}} = \prod_{j \in T}[\text{Tr}_j(\cdot)\mathbb{1}_j - \alpha_j \text{id}] \prod_{k \notin T}[\text{Tr}_k(\cdot)\mathbb{1}_k + \beta_k \text{id}], \qquad (22)$$

which again is a positive, but not completely positive, map. The strength of the corresponding entanglement criteria (in analogy with the bipartite case investigated in Refs. [48,49])

and the question how to choose the optimal α_j, β_k will be discussed in future work.

B. More entropy inequalities

Based on the positivity of generalized T-inversion, more relevant inequalities for the linear entropy can be derived. Note that the linear entropy is proportional to the Tsallis 2-entropy [29,50] and has a simple functional relation with the Rényi α-entropy for $\alpha = 2$ [51]. To date, only a few inequalities are known for the linear entropy [6,15,19,29,52,53].

Recall that for a bipartite state ρ_{12} the $\{12\}$-inverter leads to Eq. (2). Analogously, we may use the positivity of the $\{1\}$-inverter as well as the $\{2\}$-inverter, and find $\tau_{12} \geqq |\tau_1 - \tau_2|$, leading to the linear entropy analog of the Araki-Lieb triangle inequality [54] found by Zhang *et al.* [53]

$$|\tau_1 - \tau_2| \leqq \tau_{12} \leqq \tau_1 + \tau_2. \qquad (23)$$

Now let us go back to three-party states ρ_{123}, for which we have found Eq. (4). We note that the linear entropy analog of strong subadditivity, $\tau_2 + \tau_{123} \leqq \tau_{12} + \tau_{23}$, does *not* hold [52]; this is readily demonstrated by analyzing the three-qubit state $\rho_{\text{II}} = |\Phi_{12}^+\rangle\langle\Phi_{12}^+| \otimes \frac{1}{2}\mathbb{1}_3$, with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The reverse inequality does not hold either, as the state $\rho_{\text{III}} = |\Phi_{13}^+\rangle\langle\Phi_{13}^+| \otimes \frac{1}{2}\mathbb{1}_2$ shows. Yet we can add the relations for the coarse-grained $\mathscr{C}_{\{(12)\}}^2(\rho_{(12)3})$ and $\mathscr{C}_{\{(23)\}}^2(\rho_{1(23)})$, where partitions (12) and (23), respectively, are considered a single party, and obtain

$$\tau_1 + \tau_3 \leqq \tau_{12} + \tau_{23} + 2\tau_{123}. \qquad (24)$$

This relation is reminiscent of the weak monotonicity $S_1 + S_3 \leqq S_{12} + S_{23}$ for von Neumann entropies, which is equivalent to strong subadditivity [31,55]. Alternatively, we can purify the state ρ_{123} with a fourth party, use $\tau_{1234} = 0$, $\tau_{123} = \tau_4$ and $\tau_{23} = \tau_{14}$, and then relabel the parties $1 \leftrightarrow 2, 3 \leftrightarrow 4$, so that

$$\tau_2 + \tau_{123} \leqq \tau_{12} + \tau_{23} + 2\tau_3. \qquad (25)$$

The latter result shows the correction in a linear entropy inequality analogous to the standard strong subadditivity relation for the von Neumann entropy.

C. Compatibility of marginals

Finally we want to highlight the relation of our results with the quantum-marginal problem, that is, the question whether or not a given set of reduced states is compatible with a joint global state [56]. Clearly, our linear-entropy constraints (17) represent necessary conditions for the reduced states ρ_S to be compatible with the global state ρ. However, we can make statements even at the operator level.

In order to see this, consider again a three-party state ρ_{123}. Butterley *et al.* [57] found for three qubits that, given a set of two-body marginals ρ_{12}, ρ_{23}, and ρ_{13}, compatibility with a joint state ρ_{123} requires positivity of the operator

$$\Delta = \mathbb{1} - \rho_1 - \rho_2 - \rho_3 + \rho_{12} + \rho_{13} + \rho_{23} \geqq 0. \qquad (26)$$

By comparing with Eq. (3) we note that this follows immediately for arbitrary local dimensions from the positivity of $\tilde{\rho}_{123} \equiv \mathcal{I}_{\{123\}}(\rho_{123})$, and hence $\mathcal{I}_{\{123\}}(\rho_{123}) + \rho_{123} \geqq 0$. Now,

invoking generalized T-inversion for odd integer $|T|$, e.g., $\mathcal{I}_{\{1\}}(\rho_{123}) + \rho_{123} \geqq 0$ gives

$$\mathbb{1} - \rho_1 + \rho_2 + \rho_3 - \rho_{12} - \rho_{13} + \rho_{23} \geqq 0, \quad (27)$$

and analogous relations for $\mathcal{I}_{\{2\}}(\rho_{123})$ and $\mathcal{I}_{\{3\}}(\rho_{123})$. This construction generalizes to an arbitrary number of parties: From $\mathcal{I}_T(\rho) + \rho \geqq 0$ for odd $|T|$, one obtains exponentially many independent operator constraints for the compatibility of quantum marginals [58] (cf. also Ref. [59], where the condition for $T = \{1, \ldots, N\}$ was considered). Note that these constraints are necessary, but not sufficient conditions for the marginals to belong to a joint global state.

VII. CONCLUSIONS

We have extended the theory of universal quantum state inversion by defining the generalized T-inversion map $\mathcal{I}_T$. This map turns out to be a unifying building block for various aspects of quantum correlations in finite-dimensional multiparty systems: It brings together the theory of multipartite entanglement and entanglement monogamy with a formalism originating from quantum error correcting codes, and also the quantum marginal problem. Thereby it elucidates the common algebraic origin of the different physical properties investigated in these fields. Most prominently, it provides a systematic way to generate and explore correlation constraints and monogamy relations for entanglement in composite systems of arbitrary finite local dimension. We mention the immediate application of the constraints in Eq. (21) in excluding the existence of absolutely maximally entangled states for certain party numbers and local dimensions [43].

APPENDIX A: KRAUS FORM OF THE GENERALIZED T-INVERSION MAP

The procedure to derive the Kraus representation of the generalized T-inversion map is analogous to that for the universal state inverter; cf. Ref. [20]. We use the definitions of the generalized Gell-Mann matrices in d dimensions

$$\mathsf{x}_{kl} = \sqrt{\frac{d}{2}}(|k\rangle\langle l| + |l\rangle\langle k|),$$

$$\mathsf{y}_{kl} = \sqrt{\frac{d}{2}}(-\mathrm{i}|k\rangle\langle l| + \mathrm{i}|l\rangle\langle k|),$$

$$\mathsf{z}_l = \sqrt{\frac{d}{l(l+1)}}\left(-l|l\rangle\langle l| + \sum_{k=0}^{l-1}|k\rangle\langle k|\right),$$

where $(0 \leq k < l < d)$. Now we relabel them as follows:

$$\mathsf{h}_0 = \mathbb{1}, \quad (A1)$$

$$\mathsf{h}_{l^2+2k} = \mathsf{x}_{kl}, \quad (A2)$$

$$\mathsf{h}_{l^2+2k+1} = \mathsf{y}_{kl}, \quad (A3)$$

$$\mathsf{h}_{l^2+2l} = \mathsf{z}_l. \quad (A4)$$

Hence, the expressions for the factors of the inverter, Eqs. (11) and (12), take the form

$$\mathrm{Tr}(A_j)\mathbb{1}_j - A_j = \frac{2}{d_j}\sum_{k=0}^{d_j-2}\sum_{l=k+1}^{d_j-1}\mathsf{h}_{l^2+2k+1} \, A_j^* \, \mathsf{h}_{l^2+2k+1},$$
$$(A5)$$

$$\begin{aligned}\mathrm{Tr}(A_j)\mathbb{1}_j + A_j &= \frac{2}{d_j}\Bigg[\sum_{k=0}^{d_j-2}\sum_{l=k+1}^{d_j-1}\mathsf{h}_{l^2+2k} \, A_j^* \, \mathsf{h}_{l^2+2k}\\ &\quad + \sum_{k=0}^{d_j-1}\mathsf{h}_{k^2+k} \, A_j^* \, \mathsf{h}_{k^2+k}\Bigg]\\ &= \frac{2}{d_j}\sum_{k=0}^{d_j-1}\sum_{l=k}^{d_j-1}\mathsf{h}_{l^2+2k} \, A_j^* \, \mathsf{h}_{l^2+2k}. \quad (A6)\end{aligned}$$

As before, A_j denotes a Hermitian operator acting on a d_j-dimensional Hilbert space.

The generalized T-inversion map for the state ρ of an N-partite system has N factors as in the preceding equalities; the sign in front of the identity in the jth factor depends on the presence of the index j of the respective subsystem in T:

$$\mathcal{I}_T(\rho) = \left\{\prod_{j=1}^{N}[\mathrm{Tr}_j(\cdot)\otimes\mathbb{1}_j + (-1)^{|T\cap\{j\}|}\mathrm{id}]\right\}\rho, \quad (A7)$$

where id represents the identity map. In order to rewrite Eq. (A7) we introduce the compact notation

$$k = (k_1, \ldots, k_N),$$

$$t = (t_1, \ldots, t_N), \quad t_j = \frac{1}{2}[1 - (-1)^{|\{j\}\cap T|}],$$

$$\mathsf{h}_{l^2+2k+t} = \mathsf{h}_{l_1^2+2k_1+t_1}\otimes\mathsf{h}_{l_2^2+2k_2+t_2}\otimes\cdots\otimes\mathsf{h}_{l_N^2+2k_N+t_N},$$

$$\sum_{kl} \equiv \sum_{k_1l_1}\cdots\sum_{k_Nl_N},$$

where the jth tensor factor in h_{l^2+2k+t} acts only on the jth subsystem and the index ranges in the summation $\sum_{k_jl_j}$ need to be chosen as in Eq. (A5) if $j \in T$, or as in Eq. (A6) otherwise. Then we can write

$$\mathcal{I}_T(\rho) = \frac{2^N}{\prod_{j=1}^{N}d_j}\sum_{kl}\mathsf{h}_{l^2+2k+t} \, \rho^* \, \mathsf{h}_{l^2+2k+t}, \quad (A8)$$

which is the desired Kraus form of the generalized T inverter.

From Eq. (A8) we readily see the product property of the inverter on product states $\rho_{\mathrm{prod}} = \rho_S \otimes \rho_{S^c}$, which is at the

origin of Eq. (19):

$$\mathcal{I}_T(\rho_S \otimes \rho_{S^c}) = \mathcal{I}_{T_S}(\rho_S) \otimes \mathcal{I}_{T_{S^c}}(\rho_{S^c}). \qquad \text{(A9)}$$

Of course, this property is also evident from the product representation of generalized T-inversion, Eq. (7) and Eq. (A7).

APPENDIX B: COARSE GRAINING OF THE GENERALIZED T-INVERSION MAP

If we have a multipartite system with N local parties, we can choose to "coarse grain" the state by combining some of the parties, say, n, into a single system. We will show now how the generalized T-inverter can be assembled from the inverters of the "fine-grained" system. In principle, the N local systems can be combined to k coarse-grained parties, where $1 < k < N$. Due to the product structure of the generalized T-inversion map it is evident that it suffices to understand how several parties can be combined to a *single* party; the more general case of several coarse-grained parties is obtained by applying the rules found for single-party coarse graining to each combined party separately. We denote the

coarse-graining of the multi-party into a single-party state by

$$\rho_{1,\dots,n} \quad \longrightarrow \quad \rho_{(1,\dots,n)}$$

and the inverter on the combined system $\mathcal{I}_{T^{(1)}}(\rho_{(1,\dots,n)})$, as opposed to that of the fine-grained system, $\mathcal{I}_T(\rho_{1,\dots,n})$. Clearly, $T^{(1)}$ can equal $\emptyset$ or $\{1\}$, corresponding to the two possible signs of the single-system inverter. We will show now that

$$\mathcal{I}_{T^{(1)}}(\rho_{(1,\dots,n)}) = \frac{1}{2^{n-1}} \sum_{\substack{S \subseteq \{1,\dots,n\}, \\ |S| \equiv |T^{(1)}| \,(\mathrm{mod}\,2)}} \mathcal{I}_S(\rho_{1,\dots,n}), \qquad \text{(B1)}$$

which means, in order to obtain the coarse-grained inversion, one has to add all the fine-grained inversion operators whose parity coincides with that of the desired coarse-grained operator. For example, a minus inversion on a coarse-grained three-party system, where $T^{(1)} = \{1\}$, is obtained via

$$\mathcal{I}_{T^{(1)}}(\rho_{(123)}) = \tfrac{1}{4}[\mathcal{I}_{\{1\}}(\rho_{123}) + \mathcal{I}_{\{2\}}(\rho_{123})$$
$$+ \mathcal{I}_{\{3\}}(\rho_{123}) + \mathcal{I}_{\{123\}}(\rho_{123})].$$

The proof is by induction. The case $n = 1$ is trivial, as $\frac{1}{2^0} \sum'_{S \subseteq \{1\}} \mathcal{I}_S = \mathcal{I}_{T^{(1)}}$, because the sum over S contains only two terms $\emptyset$ and $\{1\}$, and we take into account (denoted by the prime) only the term $|S| \equiv |T^{(1)}|$ (mod 2). Now we assume correctness of Eq. (B1) for n parties and find for $(n + 1)$-party coarse graining

$$
\begin{aligned}
\mathcal{I}_{T^{(1)}}[\rho_{(1\dots(n+1))}] &= \mathrm{Tr}[\rho_{(1,\dots,(n+1))}]\mathbb{1} + (-1)^{|T^{(1)}|}\rho_{(1,\dots,(n+1))} \\
&= \tfrac{1}{2}(\mathrm{Tr}[\rho_{(1,\dots,(n+1))}]\mathbb{1} + (-1)^{|T^{(1)}|}\rho_{(1,\dots,n)} \otimes \mathbb{1}_{(n+1)} + \mathbb{1}_{1,\dots,n} \otimes \rho_{(n+1)} + (-1)^{|T^{(1)}|}\rho_{(1,\dots,(n+1))} + \\
&\quad + \mathrm{Tr}[\rho_{(1,\dots,(n+1))}]\mathbb{1} - (-1)^{|T^{(1)}|}\rho_{(1,\dots,n)} \otimes \mathbb{1}_{(n+1)} - \mathbb{1}_{1,\dots,n} \otimes \rho_{(n+1)} + (-1)^{|T^{(1)}|}\rho_{(1,\dots,(n+1))}) \\
&= \tfrac{1}{2}\{\mathcal{I}_{T^{(1)}}(\cdot) \otimes [\mathrm{Tr}_{(n+1)}(\cdot) \otimes \mathbb{1}_{(n+1)} + \mathrm{id}]\rho_{(1,\dots,n)(n+1)} \\
&\quad + \mathcal{I}_{\{1\}\setminus T^{(1)}}(\cdot) \otimes [\mathrm{Tr}_{(n+1)}(\cdot) \otimes \mathbb{1}_{(n+1)} - \mathrm{id}]\rho_{(1,\dots,n)(n+1)}\},
\end{aligned} \qquad \text{(B2)}
$$

where we can now make use of Eq. (B1):

$$
\begin{aligned}
\mathcal{I}_{T^{(1)}}[\rho_{(1,\dots,(n+1))}] &= \frac{1}{2}\left(\frac{1}{2^{n-1}} \sum_{\substack{S \subseteq \{1,\dots,n\}, \\ |S| \equiv |T^{(1)}| \,(\mathrm{mod}\,2)}} \mathcal{I}_{S \cap \emptyset}(\rho_{1,\dots,(n+1)}) + \frac{1}{2^{n-1}} \sum_{\substack{S \subseteq \{1,\dots,n\}, \\ |S| \equiv |T^{(1)}|+1 \,(\mathrm{mod}\,2)}} \mathcal{I}_{S \cap \{(n+1)\}}(\rho_{1,\dots,(n+1)}) \right) \\
&= \frac{1}{2^n} \sum_{\substack{S \subseteq \{1,\dots,(n+1)\}, \\ |S| \equiv |T^{(1)}| \,(\mathrm{mod}\,2)}} \mathcal{I}_S(\rho_{1,\dots,(n+1)}),
\end{aligned} \qquad \text{(B3)}
$$

which concludes the proof for $(n + 1)$.

APPENDIX C: FUNCTIONAL INDEPENDENCE OF THE CORRELATION CONSTRAINTS

In order to prove the functional independence for the right-hand sides of the constraints in Eq. (17), we need to show that if for all ρ,

$$\mathscr{D}(\rho) \equiv \sum_{T \subseteq \{1,\dots,N\}} \alpha_T \mathscr{C}_T^2(\rho) = 0, \qquad \text{(C1)}$$

then $\alpha_T = 0$ for all T.

We demonstrate this by constructing a family of states $\rho(S)$, so that $\alpha_T = 0$ is necessary in order to fulfill Eq. (C1). Consider for all subsets $S \subseteq \{1,\dots,N\}$ the state

$$\rho(S) = \bigotimes_{k=1}^N \rho_k \quad \text{where}$$

$$\rho_k = \begin{cases} |0\rangle\langle 0| & \text{for } k \in S \\ \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) & \text{otherwise} \end{cases}. \qquad \text{(C2)}$$

A straightforward calculation gives

$$\mathscr{C}_T^2(\rho(S)) = \begin{cases} 0 & \text{if } S \cap T \neq \emptyset \\ \frac{4^{|S|}3^{N-|S|-|T|}}{2^N} & \text{if } S \cap T = \emptyset \end{cases}. \qquad \text{(C3)}$$

Now we have $\rho(\{1,\dots,N\}) = \bigotimes_k |0\rangle\langle 0|$, so that $\mathscr{D}(\rho(\{1,\dots,N\})) = \alpha_\emptyset 2^N = 0$, and hence $\alpha_\emptyset = 0$. Next we consider $S = \{1,\dots,(k-1),(k+1),\dots,N\}$. The only

nonzero invariants for this $\rho(S)$ are $\mathscr{C}_\emptyset$ and $\mathscr{C}_{\{k\}}$. But we have already found that $\alpha_\emptyset = 0$, hence also $\alpha_{\{k\}} = 0$. By recursively applying the same reasoning we conclude that $\alpha_S = 0$ for all $S \subseteq \{1, \ldots, N\}$.

Thus, we have proven independence of the $\mathscr{C}_T^2(\rho)$ for mixed states. However, since we explicitly make statements also for pure states [see Eq. (21)], it is desirable to show independence also for pure states. For this purpose, some preliminary observations are helpful. First, we note that we may split the summation in Eq. (C1)

$$\sum_{T \subseteq \{1,\ldots,N\}} = \sum_{T'=\emptyset,\{1\}} \sum_{T'' \subseteq \{2,\ldots,N\}}, \tag{C4}$$

and $T = T' \cup T''$. Further, we have from Eq. (A7)

$$\mathcal{I}_{\emptyset \cup T''}(\psi) + \mathcal{I}_{\{1\} \cup T''}(\psi) = 2\, \mathbb{1}_1 \otimes \mathcal{I}_{T''}(\mathrm{Tr}_1\,[|\psi\rangle\langle\psi|]),$$

and therefore

$$\mathscr{C}_{\emptyset \cup T''}^2(\psi) + \mathscr{C}_{\{1\} \cup T''}^2(\psi) = 2\mathscr{C}_{T''}^2(\mathrm{Tr}_1\,[|\psi\rangle\langle\psi|]). \tag{C5}$$

Moreover, on the left-hand side of Eq. (C5) only one of the terms can be nonzero, because the other term has an odd number of minus signs in the inverter [that is, $|T| = |T' \cup T''| \equiv 1 \pmod 2$] and therefore $\mathscr{C}_T^2(\psi) = 0$, as we will prove in Appendix D.

With the preceding remarks we conclude

$$\mathscr{D}(\psi) \equiv \sum_{T \subseteq \{1,\ldots,N\}} \alpha_T \mathscr{C}_T^2(\psi)$$
$$= 2 \sum_{T'' \subseteq \{2,\ldots,N\}} \alpha_{T''} \mathscr{C}_{T''}^2(\mathrm{Tr}_1\,[|\psi\rangle\langle\psi|]), \tag{C6}$$

that is, we have reduced the N-qudit problem for pure states $|\psi\rangle$ to an $(N-1)$-qudit problem for $\mathrm{Tr}_1\,(|\psi\rangle\langle\psi|)$. Hence, in principle we can use the proof for mixed states. The only remaining task is to construct a family of pure states $|\psi(S)\rangle$ for which $\mathrm{Tr}_1(|\psi(S)\rangle\langle\psi(S)|)$ has properties analogous to those of $\rho(S)$; see Eq. (C2). Note that $S \subseteq \{2, \ldots, N\}$. An example for such a state is

$$|\psi(S)\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle_1 \bigotimes_{k \in S^c} |0\rangle_k + |1\rangle_1 \bigotimes_{l \in S^c} |1\rangle_l \right] \otimes \bigotimes_{m \in S} |0\rangle_m,$$

for which we find

$$\mathscr{C}_T^2(\psi(S)) = \begin{cases} 0 & \text{if } S \cap T \neq \emptyset \\ \delta_{0,|T|} 2^{N-1} + 2^{|S|} & \text{if } S \cap T = \emptyset \end{cases}. \tag{C7}$$

With this, the proof can be completed as above for mixed states.

APPENDIX D: PROPERTIES OF THE LOCAL UNITARY INVARIANTS $\mathscr{C}_T(\psi)$

First, let us prove Eq. (19). To this end, consider for pure states $\rho = |\psi\rangle\langle\psi|$ a term in the sum of Eq. (A8) when $|T| = m$ is an odd integer. Without loss of generality we may assume that the minus sign occurs in the first m parties, so that

$$\mathsf{h}_{l^2+2k+t}\, \rho^*\, \mathsf{h}_{l^2+2k+t} = \mathsf{h}_{l^2+2k+t}|\psi^*\rangle\langle\psi^*|\, \mathsf{h}_{l^2+2k+t}$$

and

$$\mathsf{h}_{l^2+2k+t}$$
$$= \mathsf{y}_{k_1 l_1} \otimes \cdots \otimes \mathsf{y}_{k_m l_m} \otimes \mathsf{h}_{l_{m+1}^2+2k_{m+1}} \otimes \cdots \otimes \mathsf{h}_{l_N^2+2k_N},$$

where the last $(N-m)$ tensor factors are of x type or diagonal. Those latter operators do not change under transposition or conjugation; therefore, in what follows, we will not write them explicitly.

Now consider the corresponding term in the expansion of

$$\mathscr{C}_T^2(\psi) = \mathrm{Tr}(|\psi\rangle\langle\psi|\,\mathcal{I}_T(\psi))$$

(for the sake of compactness we will drop also the symbol for the tensor product), we find

$$\mathrm{Tr}(|\psi\rangle\langle\psi|\,\mathsf{h}_{l^2+2k+t}|\psi^*\rangle\langle\psi^*|\,\mathsf{h}_{l^2+2k+t})$$
$$= \langle\psi|\mathsf{y}_{k_1 l_1} \cdots \mathsf{y}_{k_m l_m} \cdots |\psi^*\rangle\langle\psi^*|\,\mathsf{y}_{k_1 l_1} \cdots \mathsf{y}_{k_m l_m} \cdots |\psi\rangle.$$

That is, each term in the expansion of $\mathscr{C}_T^2(\psi)$ can be written as

$$|\langle\psi|\mathsf{y}_{k_1 l_1} \cdots \mathsf{y}_{k_m l_m} \cdots |\psi^*\rangle|^2.$$

But we have

$$\langle\psi|\mathsf{y}_{k_1 l_1} \cdots \mathsf{y}_{k_m l_m} \cdots |\psi^*\rangle = \langle\psi^*|\mathsf{y}_{k_1 l_1}^* \cdots \mathsf{y}_{k_m l_m}^* \cdots |\psi\rangle^*$$
$$= -\langle\psi|\mathsf{y}_{k_1 l_1} \cdots \mathsf{y}_{k_m l_m} \cdots |\psi^*\rangle$$

for an odd number of y-type operators; therefore each term in the expansion of $\mathscr{C}_T^2(\psi)$ vanishes, and

$$\mathscr{C}_T(\psi) = 0 \tag{D1}$$

for odd $|T| = m$.

Let us finally prove that $\mathscr{C}_T(\psi)$ cannot be an entanglement monotone if $T \neq \{1, \ldots, N\}$. First, we remark that we need to consider only $T \neq \emptyset$ because $\mathscr{C}_\emptyset^2(\psi)$ is a sum of local purities, and therefore neither $\mathscr{C}_\emptyset(\psi)$ nor $\mathscr{C}_\emptyset^2(\psi)$ can be entanglement monotones (they are maximized on product states). Similarly, a $\mathscr{C}_T(\psi)$ from a T-inverter with at least two plus signs in the product Eq. (A7) cannot be a monotone either. To see this, we note that the local invariant obeys the product property on product states $\rho_{\mathrm{prod}} = \rho_S \otimes \rho_{S^c}$:

$$\mathscr{C}_T(\rho_{\mathrm{prod}}) = \mathscr{C}_{T_S}(\rho_S)\mathscr{C}_{T_{S^c}}(\rho_{S^c}), \tag{D2}$$

[which immediately follows from Eq. (A9)]. Now consider a state $|\psi_1\rangle = |\psi_S\rangle \otimes |\psi_{S^c}\rangle$ such that S^c contains all the parties with a plus sign in the inverter (i.e., $T_S = T$). For a fully separable state $|\psi_{S^c}\rangle$, $\mathscr{C}_T(\psi_1)$ will then have a larger value than if we had chosen an entangled state for ψ_{S^c}. Consequently, $\mathscr{C}_T(\psi)$ cannot be an entanglement monotone. The remaining case is that of a single plus sign in the inverter. There, we need to have at least *two* minus signs [otherwise $\mathscr{C}_T(\psi) = 0$ because of Eq. (D1)]. It is then easy to find counterexamples for the monotone assumption. Consider, e.g., a system of three parties (123) with local dimensions $d_j \geqq 2$, $T = \{2, 3\}$, and the state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, so that $\mathscr{C}_T(\psi_1) = 1$. If we apply a two-outcome positive operator-valued measure (POVM) $\{A_1, A_2\}$ to the first qudit with $A_{1,2} = |\pm\rangle\langle\pm| + \frac{1}{\sqrt{2}}\sum_{j=2}^{d_1-1}|j\rangle\langle j|$, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, the resulting state is a

tensor product of a pure state of the first party and a Bell-type state of the other two qudits, so that the average of $\mathscr{C}_T$ for the two outcomes gives $\sqrt{2} > \mathscr{C}_T(\psi_2)$, in contradiction with the monotone assumption. Finally, for the case of a single plus sign and a larger (even) number $k > 2$ of minus signs in the inverter we can construct an analogous counterexample $|\psi_3\rangle = |\psi_2'\rangle \otimes \bigotimes_{l=1}^{\frac{k}{2}-1} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$; here $|\psi_2'\rangle$ is a state of three parties, where the first party is the one with the plus sign in the inverter, in analogy with $|\psi_2\rangle$. The corresponding two-outcome POVM acts on that first party in $|\psi_2'\rangle$. Note that instead of the tensor product of Bell-type states in $|\psi_3\rangle$ we could have used any other state for which the inverter with only minus signs does not vanish. This concludes the

proof that $\mathscr{C}_T(\psi)$ cannot be an entanglement monotone for $T \neq \{1, \dots, N\}$.

For completeness we mention that for $T = \{1, \dots, N\}$, N even, the distributed concurrence $\mathscr{C}_T(\psi)$ is an entanglement monotone only in the following cases (recall that for odd N we have $\mathscr{C}_T \equiv 0$):

(a) $N = 2$, d_j arbitrary. Then $\mathscr{C}_T(\psi)$ coincides with the well-known concurrence for bipartite states.

(b) $d = 2$, N arbitrary. This case corresponds to the polynomial invariant $|H(\psi)| = |\langle\psi|\sigma_2^{\otimes N}|\psi^*\rangle|$, which is the straightforward generalization of Wootters' two-qubit concurrence to N-qubit states.

(c) $N > 2$, $d_j \leqq 3$. Also in those cases, $\mathscr{C}_T(\psi)$ is an entanglement monotone, as was proven in Ref. [20].

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prefactor of $2^{N/2}$. We do not call $\mathscr{C}_T(\Psi)$ a concurrence, because it is not an entanglement monotone for pure states whenever $T \neq \{1, \ldots, N\}$; see Appendix D.

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