Quantum Zeno effect at finite measurement strength and frequency

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The evolution of a system subject to measurement is restricted to Zeno subspaces of the measurement Hamiltonian in the limit of strong measurements k in a phenomenon known as the quantum Zeno effect (QZE). As the limit constrains QZE to the lowest orders of perturbation in $1/k$, we derive general expressions for the maximum probability leakage from Zeno subspaces for reversible interactions and leakage rates for irreversible interactions for both quantum decay and Lindblad measurement operators. We show that pulsed QZE can be expressed in the same Hamiltonian formulation as continuous QZE, and the two merge in the large-frequency f limit. We derive a nonperturbative expression for pulsed QZE at finite k and f, which reduces to previously known results for pulsed QZE at large k and continuous QZE at large f.

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I. INTRODUCTION

The quantum Zeno effect (QZE) was first proposed as a paradox in which a particle subject to quantum decay while being continuously observed never decays [1]. The analysis in [1] is based on the collapse postulate, where measurement corresponds to complete wave-function collapse described by a projection operator E, and the QZE arises as a mathematical consequence of an infinite sequence of projections E interspersed with Schrödinger evolution $U(t) = e^{-iHt/\hbar}$. As the interval between measurements tends to zero $\tau \to 0$ and the number of measurements N in time $T = \tau N$ tends to infinity, the evolution operator $V(T) = [EU(\tau)E]^N$ is shown to satisfy $V^\dagger(T)V(T) = E$. Then, if the initial state $\rho(0)$ is an eigenstate of the measurement operator $\mathrm{tr}[\rho(0)E] = 1$, the survival probability is unity at all times, $\mathrm{tr}[\rho(T)] = \mathrm{tr}[\rho(0)V^\dagger(T)V(T)] = 1$. It was argued that continuous observation is realized approximately by the tracks of an unstable charged particle in a bubble chamber.

More recently, understanding of the QZE has been improved [2–13]. It has been shown that continuous observation corresponds to a measurement timescale of the order of the characteristic (Zeno) time of initial quadratic evolution [8]. This time, $\tau_Z = \hbar\langle 1|(H - \langle 1|H|1\rangle)^2|1\rangle^{-1/2}$, obtained from the probability for no decay of an initial state $|\langle 1|e^{-iHt/\hbar}|1\rangle|^2$, is inversely related to the characteristic energy μ of the final-state spectrum [6]. Because the collision frequency is typically much smaller than the frequency corresponding to the characteristic energy of the decay continuum $\omega_c = \mu/\hbar$, the QZE has not been observed to date in its original setting of unstable particle decay [2,12] and can be altogether absent in systems with unbounded spectra [13]. However, the QZE has been observed in reversible and unitary interactions such as Rabi oscillations under a time-independent system Hamiltonian H_s, as well as interactions where the measurement itself is unitary [4]. Experimental verifications of the QZE have been reported in various settings, including continuous [14,15] and pulsed [16,17] measurements in systems exhibiting Rabi oscillations, and in continuous observations of tunneling [18]. The original formulation has also been refined, showing that evolution is not entirely frozen in the QZE limit, but rather restricted to quantum Zeno subspaces, which are the degenerate eigenspaces of the measurement Hamiltonian H_m [4].

In its original [1] and subsequent [2,16] formulations, pulsed QZE is modeled by periods of Schrödinger evolution and periods of complete collapse. In practice collapse due to measurement can be incomplete, and abrupt alternation between irreversible measurement and reversible system evolution is difficult to achieve. It has been shown that a so-called continuous QZE [4,8] occurs when a measurement interaction is characterized by continuous Schrödinger evolution leading to entanglement with a macroscopic system and decoherence of system states [19], which therefore evolve irreversibly. The formulation of the continuous QZE in terms of Schrödinger evolution alone removes the assumption of complete collapse and enables analysis of normal evolution and measurement simultaneously. Expectation values can be calculated without recourse to the collapse postulate, which is sometimes assumed to be a necessary ingredient of QZE [1,20]. Moreover, because a decrease in transition rates in response to increased interaction strength [4] is manifest already in simple interactions, devoid of the complexity of entanglement with a macroscopic system, the root of continuous QZE does not lie in measurement.

In this article we consider the continuous QZE as a perturbation of the Schrödinger equation by a system Hamiltonian H_s about the measurement Hamiltonian H_m [3,4] as the measurement strength k tends to infinity,

$$H = H_s + kH_m. \tag{1}$$

In Sec. II we show that due to the large-k limit, perturbation theory [20] provides a natural framework for analyzing the QZE, whereby in the QZE limit an unstable state $|\phi_0\rangle$ is stabilized as a nondegenerate eigenstate of the measurement Hamiltonian H_m. Generalizing this to Hamiltonians with degenerate eigenspaces, we show that as a consequence of

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perturbation theory, evolution is restricted to within each degenerate eigenspace of H_m, called the Zeno subspace [4], where evolution is governed by the projection of H_s onto the degenerate eigenspace of H_m. Extending this ideal case to practical situations with finite k, we derive a general expression for finite measurement Hamiltonians, showing the probability leakage from the Zeno subspaces vanishes as $1/k^2$ in the QZE limit. We also extend the analysis to irreversible interactions which are essential to macroscopic measurements responsible for the emergence of classicality [19]. Irreversibility results in a qualitative change, whereby the probability associated with any finite Zeno subspace vanishes with time for finite k; however, the probability leakage rate tends to zero as $1/k$ in the QZE limit. This applies to measurements with a single decoherence rate, for example, the transmon [21], where the QZE is manifest in an inhibition of jumps between the two eigenstates (QZE subspaces) with increasing decoherence [22].

In Sec. III we show that pulsed QZE, described in the first paragraph, is based on the same phenomena as continuous QZE, so that in the limit of infinite measurement frequency, pulsed and continuous QZEs merge and dependence on the pulsing frequency vanishes. We derive a result for pulsed QZE with on-off pulsation in a two-state system and finite pulsing frequency and measurement strength, which reduces to known results for pulsed QZE for finite pulsing frequency and infinite measurement strength and continuous QZE for infinite frequency at finite measurement strength. Similar results are obtained for both a decay measurement Hamiltonian and the Lindblad measurement operator of [21]. An analysis of pulsed QZE based on the Schrödinger equation was presented in [23] for pulsed stimulated transitions in a two-level system, with one level decaying to a reservoir. In that case, the reservoir interaction was treated to first order in time, and the pulsation time was fixed as a function of Rabi frequency. In the present article the reservoir interaction is treated in the Weisskopf-Wigner approximation, and pulsation time is a free parameter.

Our analysis is restricted to bounded Hamiltonians, avoiding mathematical complications associated with singularities [24]; however, this does not present a serious constraint because many physical processes are approximately finite. This includes semibounded Hamiltonians generating irreversible decay, Eq. (17), where an Ohmic spectral density with cutoff frequency ω_c is typically introduced to ensure convergence, and reflects the fact that the theory becomes inaccurate at very high energies. The cutoff does not substantially influence the results because low-energy phenomenology is largely insensitive to high-energy behavior. In the same vein, to very good accuracy one can replace the continuous spectrum of the decay Hamiltonian by a discrete spectrum with energy spacing much less than the characteristic decay rate. In this way the decay Hamiltonian can be discretized and bound without a substantial impact on system evolution.

II. THE QZE WITH CONTINUOUS MEASUREMENTS

A. Interaction with finite systems

As a simple example, consider a three-state system in a Hilbert space spanned by the orthonormal basis $|\phi_i\rangle$,

$$|\Phi(t)\rangle = \sum_{i=1}^{3} a_i(t)|\phi_i\rangle, \tag{2}$$

in state $|\Phi(0)\rangle = |\phi_2\rangle$ prior to the interaction. The measurement system has a Hilbert space spanned by the orthonormal basis $|\varphi_i\rangle$,

$$|\Psi(t)\rangle = \sum_{j=1}^{N} b_j(t)|\varphi_j\rangle, \tag{3}$$

and is in state $|\Psi(0)\rangle = |\varphi_1\rangle$ prior to the interaction. An interaction Hamiltonian

$$H_{m1} = |\phi_1\rangle\langle\phi_2| \otimes |\varphi_2\rangle\langle\varphi_1| + |\phi_2\rangle\langle\phi_1| \otimes |\varphi_1\rangle\langle\varphi_2| \tag{4}$$

couples the two systems, producing an evolving entangled state correlating the system and measurement states, representing "measurement." The total state evolving under Hamiltonian kH_{m1}, where k is a real constant representing measurement strength, with $\hbar \equiv 1$, is

$$|\Phi(t)\rangle|\Psi(t)\rangle = \cos kt\,|\phi_2\rangle|\varphi_1\rangle + i\sin kt\,|\phi_1\rangle|\varphi_2\rangle. \tag{5}$$

Now add a system Hamiltonian H_s producing oscillations between system states $|\phi_3\rangle$ and $|\phi_2\rangle$,

$$H_s = \alpha(|\phi_2\rangle\langle\phi_3| + |\phi_3\rangle\langle\phi_2|) \otimes \mathbf{1}, \tag{6}$$

where $\mathbf{1} = \sum_j |\varphi_j\rangle\langle\varphi_j|$ is the identity operator acting on the measurement states $|\varphi_j\rangle$. Taking into account the expressions for H_s and H_{m1}, the total state evolving under $H = H_s + kH_{m1}$, with initial state $|\phi_3\rangle|\varphi_1\rangle$, has a three-dimensional solution space,

$$\begin{aligned}|\Phi(t)\rangle|\Psi(t)\rangle = {} & c_1(t)\,|\phi_3\rangle|\varphi_1\rangle + c_2(t)\,|\phi_2\rangle|\varphi_1\rangle \\ & + c_3(t)\,|\phi_1\rangle|\varphi_2\rangle,\end{aligned} \tag{7}$$

which entangles system and measurement states. The combined Hilbert space can be expanded in a new orthonormal basis, $|\Phi(t)\rangle \otimes |\Psi(t)\rangle = \sum_{j=1}^{3} c_j(t)|\eta_j\rangle$, so the first three basis states correspond to Eq. (7). A more detailed analysis is presented in the Appendix.

Since $|\eta_1\rangle = |\phi_3\rangle|\varphi\rangle$ is an eigenstate of H_{m1} (with eigenvalue 0), where $|\varphi\rangle$ is any measurement state, H_{m1} fulfills a requirement for a measurement of the occupation of state $|\phi_3\rangle$. The Schrödinger equation is solved by diagonalizing the Hamiltonian H in the three-dimensional space of Eq. (7), and for a system initially prepared in state $|\phi_3\rangle|\varphi_1\rangle$, we have

$$|c_1(t)|^2 = \left(\frac{k^2 + \alpha^2\cos\omega t}{k^2 + \alpha^2}\right)^2, \tag{8}$$

where $\omega = \sqrt{k^2 + \alpha^2}$. As the measurement strength k increases, the survival probability of the system state $|\phi_3\rangle$ changes from Rabi oscillations at $k = 0$ to unity at all times, while the magnitude of the oscillatory component diminishes and its frequency increases. This result is in accordance with what would be expected from the QZE.

We can obtain more general results using perturbation theory, which can be applied to Eq. (1) upon dividing by k and rescaling the time, with k^{-1} being a small parameter, or, equivalently, using H_s directly as the perturbing Hamiltonian, with powers of k scaling the eigenvalues of H_m in the perturbation expansion, ensuring its convergence. Using the latter approach, the first-order correction to the eigenvectors $|\chi_j\rangle$ of H is

$$|\chi_j\rangle = \left(1 - \frac{1}{2k^2}\sum_{n\neq j}\frac{|\langle\psi_n|H_s|\psi_j\rangle|^2}{(\lambda_j-\lambda_n)^2}\right) \times \left(|\psi_j\rangle + \frac{1}{k}\sum_{n\neq j}\frac{\langle\psi_n|H_s|\psi_j\rangle}{\lambda_j-\lambda_n}|\psi_n\rangle\right), \quad (9)$$

where λ_j and $|\psi_j\rangle$ are the eigenvalues and eigenvectors of H_m and the eigenvectors $|\chi_j\rangle$ are normalized to order k^{-2}. The eigenvalues μ_j of H to lowest nonzero order in k^{-1} are

$$\mu_j = k\lambda_j + \frac{1}{k}\sum_{n\neq j}\frac{|\langle\psi_n|H_s|\psi_j\rangle|^2}{\lambda_j-\lambda_n} \quad (10)$$

if H_m has no degenerate eigenvalues.

We can now evaluate the stability of a system initially in an eigenstate of the measurement Hamiltonian $|\psi_0\rangle$ but unstable under the action of the system Hamiltonian H_s, using an expansion of the solution in the eigenstates of the total Hamiltonian H,

$$|\Psi(t)\rangle = \sum_j e^{-i\mu_j t}|\chi_j\rangle\langle\chi_j|\psi_0\rangle, \quad (11)$$

and substituting from the perturbation theory expression (9). Then, the time-averaged probability of $|\Phi\rangle$ to order k^{-2} is

$$\overline{|\langle\psi_0|\Psi(t)\rangle|^2} = \sum_j |\langle\psi_0|\psi_j\rangle|^4 \quad (12)$$

$$= 1 - \frac{2}{k^2}\sum_{j\neq 0}|\langle\psi_0|H_s|\psi_j\rangle|^2\lambda_j^{-2}. \quad (13)$$

Since the maximum of $|\langle\psi_0|\Psi(t)\rangle|^2$ is unity and the oscillatory excursion of the probability is symmetric about the mean, the second term in Eq. (13) is half the maximum leakage of probability out of the unstable state $|\psi_0\rangle$. It is easily checked that Eq. (13) agrees with Eq. (8) when H_s is the two-state Rabi Hamiltonian. Consequently, in the general case, as k increases, the amplitude of probability oscillations decreases as k^{-2}, and state $|\psi_0\rangle$ stabilizes.

In the nondegenerate case of Eq. (11), evolution in the QZE is restricted to simple phase cycling in the one-dimensional eigenspaces of H_m, $|\psi_j(t)\rangle = e^{-ik\lambda_j t}|\psi_j\rangle$, with projection operators $P_j = |\psi_j\rangle\langle\psi_j|$. Similarly, an immediate result of degenerate perturbation theory is the restriction of evolution to multidimensional degenerate eigenspaces of H_m, with evolution in the jth eigenspace governed by the projection of the perturbing Hamiltonian H_s onto the eigenspace, $H_{sj} = P_j H_s P_j$, with $P_j = \sum_l |\psi_{jl}\rangle\langle\psi_{jl}|$, where l indexes the eigenspace of the jth eigenvalue [20]. This was obtained in a more complicated fashion in [4], where the degenerate eigenspaces of H_m are called quantum Zeno subspaces. We can immediately derive an expression for the evolution operator $U(t) = e^{-iHt}$ using the completeness of the eigenspace, $\sum_j P_j = 1$,

$$U(t) = \sum_{j,l} e^{-i(H_s+kH_m)t}|\phi_{jl}\rangle\langle\phi_{jl}| \quad (14)$$

$$= \sum_j e^{-ik\lambda_j t}\sum_l e^{-iH_{sj}t}|\phi_{jl}\rangle\langle\phi_{jl}| \quad (15)$$

$$= e^{-i(kH_m+\sum_j H_{sj})t}, \quad (16)$$

which was obtained in a more complicated fashion in [4].[1] Note that the zeroth order in k is neglected in the nondegenerate case because it leads to phase cycling and does not affect the stability. In the degenerate case, however, it couples the degenerate eigenstates, so the transition rate for a jth eigenstate of H_m, which is not simultaneously an eigenstate of H_{sj}, does not vanish in the QZE limit, $k \to \infty$. Quantitative results for finite k can be obtained from Eq. (16) as in the nondegenerate case.

B. Irreversible interactions representing classical measurements

Finite-measurement Hamiltonians [4] do not generate the usual properties of measurements in quantum theory. The latter typically correspond to classical irreversible processes with a definite outcome and are associated with an entropy increase [19]. The measurement operator E generating von Neumann projections in the Introduction has this property, while evolution for finite Hamiltonians is reversible and the system eventually returns to its initial state. To attain classicality, one introduces the thermodynamic limit, using a measurement apparatus whose number of degrees of freedom tends to infinity, leading to irreversible evolution of the system [11,19]. We now consider this limit of H_m, with the unstable state being an eigenvector. For such interactions the QZE differs qualitatively from the case at finite k, and there is not a limit on probability leakage from the unstable state, so the QZE subspaces of Eq. (16) decay.

1. Measurement by quantum decay

A simple model of irreversible measurement couples the system to an external field, resulting in the exponential decay of Weisskopf-Wigner (WW) theory, which registers a system transition by the irreversible emission of a quantum. Such a measurement applied to a two-state system was analyzed in [4,7] without accounting for Lamb shift. Here we establish notation by rederiving the two-state result and then obtain a general QZE expression for finite systems using the perturbation expansion of Eq. (10).

We imagine that the emitted photon eigenspace is discrete, corresponding to a finite apparatus volume, where the continuous limit is approached as the volume tends to infinity. Then,

[1] Equation (15) is derived using the degenerate eigenvalue equation for the jth eigenspace $H_{sj}|\phi_{jl}\rangle = E_{jl}|\phi_{jl}\rangle$, where E_{jl} is the first-order energy correction, so $e^{-i(H_s+kH_m)t}|\phi_{jl}\rangle = e^{-ik\lambda_j t}e^{-iE_{jl}t}|\phi_{jl}\rangle = e^{-ik\lambda_j t}e^{-iH_{sj}t}|\phi_{jl}\rangle$. Equation (16) is derived using the properties of projection operators, $P_jP_k = H_{sj}H_{sk} = P_jH_{sk} = 0$ for $j \neq k$ and $[P_j, H_{sj}] = 0$, so that $\sum_j P_j e^{\lambda_j + H_{sj}} = e^{\sum_j \lambda_j P_j + H_{sj}}$, while $\sum_j \lambda_j P_j = H_m$.

the measurement eigenspace is that of Eq. (3), with j indexing the emitted photon energy ω_j and $N=\infty$.[2] The photon field interacts so a system transition $|\phi_1\rangle \rightarrow |\phi_0\rangle$ emits a photon in state $|\psi_j\rangle$ with coupling constant β_j. Including the photon self-energy at ground-state energy $-\Delta$, the measurement Hamiltonian becomes

$$H_m = \sum_j (\omega_j - \Delta)\mathbf{1} \otimes |\varphi_j\rangle\langle\varphi_j| + \sum_j \beta_j |\phi_0\rangle\langle\phi_1| \otimes |\varphi_j\rangle\langle\varphi_0| + \text{H.c.} \tag{17}$$

This interaction term entangles system and measurement states, as per Eq. (4). We set the initial state to $|\psi_0\rangle = |\phi_2\rangle|\varphi_0\rangle$, which is a nondegenerate eigenstate of H_m, so that the interaction is an irreversible measurement of the occupation of $|\phi_2\rangle$, in accordance with the property of classical measurements.

Adding a two-state Rabi Hamiltonian H_s generating the transition $|\phi_2\rangle \leftrightarrow |\phi_1\rangle$, we solve the Schrödinger equation using the state vector expansion

$$|\Phi(t)\rangle|\Psi(t)\rangle = a(t)|\phi_2\rangle|\varphi_0\rangle + b(t)|\phi_1\rangle|\varphi_0\rangle + \sum_{j\neq 0} c_j(t)|\phi_0\rangle|\varphi_j\rangle \tag{18}$$

to give the following relations between the expansion coefficients in Laplace space:

$$isa(s) - i = \alpha b(s), \tag{19}$$

$$isb(s) = \alpha a(s) + k\sum_j \beta_j c_j(s), \tag{20}$$

$$isc_j(s) = k(\omega_j - \Delta)c_j(s) + k\beta_j^* b(s). \tag{21}$$

The solution is

$$a(s) = \left[s + \alpha^2 \left(s + \sum_j \frac{k^2|\beta_j|^2}{s + ik(w_j - \Delta)} \right)^{-1} \right]^{-1} \tag{22}$$

$$\approx \left(s + \frac{\alpha^2}{s + k\Gamma/2 + ik\Delta_0} \right)^{-1}, \tag{23}$$

where we have used the Weisskopf-Wigner approximation [25] in Eq. (22), with decay rate $\Gamma = 2\pi\rho(\Delta)|\beta(\Delta)|^2$ and Lamb shift $\Delta_0 = \sum_j |\beta_j|^2/(\Delta - \omega_j)$.

Instead of solving the quadratic pole structure of $a(s)$, it is more instructive for what follows to note that the second term in parentheses in Eq. (23) tends to zero as $k\rightarrow\infty$, producing a dominant (amplitude squared ~ 1) pole at small s. Then, dropping terms in s^2, we immediately find

$$a(t) \approx \exp\left(\frac{-\alpha^2 t}{k\Gamma/2 + ik\Delta_0} \right), \tag{24}$$

so that for irreversible measurements the unstable state decays, but the rate tends to zero in the QZE limit. We shall now demonstrate this as a more general result of irreversible measurements using the eigenvalue perturbation expansion (10), which remains valid in the continuous limit as the eigenstates of H_m are orthogonal. In this case, although the unperturbed eigenvalues λ_j of H_m are real since H_m is Hermitian, decay arises from a sum over states vanishing irreversibly in the thermodynamic limit.

We expand the expression in Eq. (10) for the first-order correction to the zero eigenvalue λ_0 of the unstable state $|\phi_0\rangle$ over any orthonormal basis $|\eta_i\rangle$ spanning the system states, so that $\sum_i H_s|\eta_i\rangle\langle\eta_i| = H_s$. Then, since $\lambda_0 = 0$ for the unperturbed Hamiltonian H_m, the corresponding eigenvalue μ_0 of the total Hamiltonian H, corrected to first order due to the perturbation H_s, is

$$\mu_0 = -\frac{1}{k}\sum_{n\neq 0}\lambda_n^{-1}\left|\sum_i \langle\psi_0|H_s|\eta_i\rangle\langle\eta_i|\psi_n\rangle\right|^2 \tag{25}$$

$$= -\frac{1}{k}\sum_{i,j}\langle\eta_j|H_s|\psi_0\rangle\langle\psi_0|H_s|\eta_i\rangle \sum_{n\neq 0}\langle\eta_i|\psi_n\rangle\lambda_n^{-1}\langle\psi_n|\eta_j\rangle \tag{26}$$

$$= -\frac{1}{k}\sum_{i,j}\langle\eta_j|H_s|\psi_0\rangle\Lambda_{ij}\langle\psi_0|H_s|\eta_i\rangle, \tag{27}$$

where $\Lambda_{ij} \equiv \sum_{n\neq 0}\langle\eta_i|\psi_n\rangle\lambda_n^{-1}\langle\psi_n|\eta_j\rangle$ is a matrix dependent only on the measurement apparatus, while the vector $\alpha_i \equiv \langle\psi_0|H_s|\eta_i\rangle$ depends only on the system. If the measurement is irreversible, the amplitude of at least one state $|\eta_i\rangle$ decays with time under the action of H_m. We now illustrate the consequences of this for the example of the Rabi Hamiltonian H_s of Eq. (6).

Writing the decay of the unstable states as $e^{-\gamma_i t} = \langle\eta_i|e^{-itH_m}|\eta_i\rangle$ for some γ_i and expanding the exponential of H_m in terms of its eigenstates $e^{-iH_m t} = \sum_n e^{-i\lambda_n t}|\psi_n\rangle\langle\psi_n|$ give $e^{-\gamma_i t} = \sum_n e^{-i\lambda_n t}|\langle\psi_n|\eta_i\rangle|^2$. Multiplying both sides by the convergence factor $e^{-\epsilon t}$, integrating from zero to infinity, and letting $\epsilon\rightarrow 0$, we find $\gamma_i^{-1} = \sum_n(\epsilon + i\lambda_n)^{-1}|\langle\psi_n|\eta_i\rangle|^2$, which has a real component if the spectrum of H_m has support at the origin, $\lambda_n \rightarrow 0$. Comparing the expression for γ_i^{-1} with the expression for Λ_{ij}, we have $\Lambda_{ii} = i\gamma_i^{-1}$, while $\Lambda_{i\neq j} = 0$ since a WW Hamiltonian does not couple system states. We thus find

$$\mu_0 = -\frac{1}{k}\sum_i \frac{|\alpha_i|^2}{\gamma_i}. \tag{28}$$

For the Rabi Hamiltonian H_s there is a single unstable state which decays due to H_m, $|\eta_1\rangle = |\phi_1\rangle|\varphi_0\rangle$, while the initial state is $|\psi_0\rangle = |\phi_2\rangle|\varphi_0\rangle$. Then the system vector has a single nonzero element $\langle\psi_0|H_s|\eta_1\rangle = \alpha$. Using the WW decay coefficient $\gamma_1 = \Gamma/2 + i\Delta_0$, we recover Eq. (24) from Eq. (28).

2. Measurement by Lindblad operators

We now consider the more general case where the measurement operator obeys the Lindblad equations describing a Markovian, no-memory interaction with the environment [11]. An example of such an equation can be derived by considering the measurement Hamiltonian as a sum of products of operators with $\hat{H}_m$ acting just on the system space and H_m' acting just on the apparatus space, $H_m = \sum_i \hat{H}_m^i \otimes H_m'^i$ [25].

[2]The angular variable in WW decay gives a multiplicity factor which can be absorbed into a redefinition of the energy eigenstate amplitudes.

Choosing for clarity the case when the sum contains a single term and carrying out a trace over the apparatus states, one derives [25,26] a Lindblad equation for the reduced density matrix,

$$i\frac{d\rho}{dt} = [H_s, \rho] - i\gamma k[\hat{H}_m, [\hat{H}_m, \rho]], \tag{29}$$

where 2γ is the decoherence rate. Here ρ acts only on the system space, so that the first term on the right-hand side represents evolution in the absence of measurement, while the second term embodies the action of the measurement on the system in terms of the projection of the interaction Hamiltonian onto the measurement space $\hat{H}_m$. Such an equation of motion with a two-state H_s was recently used to analyze the QZE in a transmon [21].

We now transform this equation to the eigenspace of operator $\hat{H}_m$, where $\hat{H}_m$ is diagonal with eigenvalues λ_a, and denote the transformed system Hamiltonian by H_s'. In this basis, Eq. (29) becomes

$$i\frac{d\rho_{ab}}{dt} = [H_s', \rho]_{ab} - i\gamma k(\lambda_a - \lambda_b)^2\rho_{ab}, \tag{30}$$

where this type of equation for reduced matrix evolution is typical at small decoherence rates [19]. At large k, H_s' represents a small perturbation about a diagonal density matrix, which solves Eq. (29) when $H_s' = 0$, while for $k = 0$ Eq. (30) generates Rabi oscillations in the absence of measurement. As before, the unstable state $|\psi_0\rangle$ is an eigenstate of $\hat{H}_m$, and so in the QZE limit, $k \to \infty$, the density matrix $\rho(t) = |\psi_0\rangle\langle\psi_0|$ is a solution of Eq. (30). Therefore, for large k we search for a solution of the form

$$\rho(t) = \rho_{00}(t)|\psi_0\rangle\langle\psi_0| + k^{-1}\rho'(t), \tag{31}$$

where $\rho_{00}(0) = 1$. Substituting this trial solution into Eq. (30), we find

$$ik\frac{d}{dt}\rho_{00}(t) = \sum_a [H_{0a}\,\rho_{a0}'(t) - \rho_{0a}'(t)H_{a0}], \tag{32}$$

$$i\frac{d}{dt}\rho_{a0}'(t) = kH_{a0}\,\rho_{00}(t) - i\gamma k\lambda_a^2\rho_{a0}'(t), \tag{33}$$

where we have dropped terms of zeroth order in k on the right-hand side of Eq. (33) and for clarity of notation replaced H_s' by the symbol H. Transforming to Laplace space and solving the set of simultaneous equations, we find an expression for the probability of the unstable state $|\psi_0\rangle$

$$\rho_{00}(s) = \left(s + 2\sum_a \frac{|H_{a0}|^2}{s + \gamma k\lambda_a^2}\right)^{-1}, \tag{34}$$

which is similar in form to Eq. (24). In particular, as $k \to \infty$, the right term in Eq. (34) generates a dominant low-frequency pole, at which $a(s)$ can be evaluated by neglecting the s dependence in the denominator of the second term. We thus find

$$\rho_{00}(t) \approx \exp\left(-\frac{2t}{\gamma k}\sum_a \lambda_a^{-2}|H_{a0}|^2\right), \tag{35}$$

so that as for measurement by decay, for a Lindblad measurement operator the decay rate of the unstable state vanishes in proportion to $1/k$ in the QZE limit.

Since Eq. (30) is a general decoherence result for a Schrödinger interaction with a measurement system in the thermodynamic limit, Eq. (35) is a general continuous QZE result at finite measurement strength which follows from the Schrödinger equation.

III. PULSED QZE

The original formulation of the QZE, as described in the Introduction, is based on pulsed measurements, allowing periods of unitary evolution between events of instantaneous collapse. This contrasts with the manifestations of QZE considered above, which are based on time-independent system H_s and measurement H_m Hamiltonians. Although the two modes of QZE have sometimes been regarded as being of separate origin, it is easily shown that the pulsed QZE does not essentially differ from the continuous case. For a pulsed measurement Eq. (1) becomes

$$H = H_s + kf(t)H_m, \tag{36}$$

for which the Schrödinger equation can be integrated to give

$$|\Phi(t)\rangle = \exp\left(-itH_s - ikH_m\int_0^t dt' f(t')\right)|\Phi(0)\rangle. \tag{37}$$

Since the pulsing is periodic, $f(t)$ can be Fourier expanded, $f(t) = c_0 + \sum_n c_n \sin n\omega t$, so that the evolution operator in Eq. (37) reads

$$U(t) = \exp\left\{-itH_s - ik\left[c_0\,t + \omega^{-1}\sum_n c_n(1-\cos n\omega t)/n\right] \times \sum_j \lambda_j P_j\right\}, \tag{38}$$

where we have used $H_m = \sum_j \lambda_j P_j$. In the limit of very fast measurements $k\lambda_j\omega^{-1} \to 0$, the time dependence of $f(t)$ is lost, and only the constant term remains. Then, the problem reduces to the continuous QZE previously analyzed, with $f(t)$ replaced by its time-averaged mean. This is demonstrated in the special case considered below.

Continuous QZE is typically analyzed as a function of measurement strength k, with the unstable state decaying exponentially, which, using (24) and neglecting the Lamb shift, gives

$$|a(T)|^2 = \exp(-4\alpha^2 T/k\Gamma), \tag{39}$$

where the measurement strength k determines the decay rate in the QZE limit. In contrast, pulsed QZE assumes $k \to \infty$ *a priori*, so collapse is instantaneous on the measurement timescale, and the measurement frequency $1/\tau$ determines the decay rate in the QZE limit, $\tau \to 0$ [2]. The latter also gives rise to exponential decay of the unstable state, which we can see in the case of the Rabi Hamiltonian of Eq. (6). For an alternating sequence of unitary evolutions, $a(\tau) = \cos\alpha t$, interspersed with T/τ measurements in time T, we have

$$|a(\tau)|^{2T/\tau} \approx \cos^{2T/\tau}(\alpha\tau) \approx \exp(-\alpha^2\tau T). \tag{40}$$

This well-known result [2,8] assumes a dual limit. The measurement interval must be short compared to the characteristic time of quadratic evolution $\tau \ll \alpha^{-1}$, enabling the approximation $\cos^2 \alpha\tau \approx (1-\alpha^2\tau^2)$ to be used, and the measurement interval must be large compared to the collapse timescale $\tau \gg (k\Gamma)^{-1}$, so the decay rate is independent of $k\Gamma$. Comparing the functional dependence of the unstable state survival probability in the continuous and pulsed QZE limits, Eqs. (39) and (40), shows the decay rates are equal when $\tau k\Gamma = 4$. This was proposed in [8] as a relationship between the characteristic response time for continuous measurement, $4(k\Gamma)^{-1}$, and the characteristic measurement rate for pulsed observations, τ.

Equation (40) for pulsed QZE is rather problematic because of the assumed dual limit. For practical purposes one wants an expression valid for finite measurement strengths, which reduces to Eq. (40) for $(k\Gamma)^{-1} \ll \tau \ll \alpha^{-1}$ and to the continuous QZE of Eq. (39) for $\tau \ll (k\Gamma)^{-1} \ll \alpha^{-1}$ when $f(t)$ is replaced by its mean value. In this way the unspecified instantaneous measurement which collapses the state in the Introduction [1] is replaced by the mechanism of the Schrödinger evolution of Sec. II B, introducing measurements of finite strength and duration into the pulsed QZE.

We now perform such an analysis for the Rabi Hamiltonian H_s of Eq. (6) and the decay measurement Hamiltonian H_m of Eq. (17). The expansion coefficients satisfy equations similar to (19)–(21) but are now more conveniently solved in the time domain due to the time dependence of $f(t)$

$$i\frac{da}{dt} = \alpha\, b(t), \tag{41}$$

$$i\frac{db}{dt} = \alpha\, a(t) + kf(t)\sum_j \beta_j c_j(t), \tag{42}$$

$$i\frac{dc_j}{dt} = kf(t)(\omega_j - \Delta)c_j(t) + kf(t)\beta_j^* b(t), \tag{43}$$

with the initial conditions $a(t) = 1$, $b(0) = 0$, and $c_j(0) = 0$. The measurement system amplitudes $c_j(t)$ can be eliminated from Eq. (42) using Eq. (43). Applying the WW approximation in the time domain [27], we obtain

$$i\frac{db}{dt} = \alpha\, a(t) - i\frac{k\Gamma}{2}f(t)b(t), \tag{44}$$

where Γ is the WW decay rate and we have neglected the Lamb shift for subsequent clarity. Differentiating Eq. (41) and solving it simultaneously with Eq. (44), we obtain a differential equation for the unstable state amplitude $a(t)$,

$$\frac{d^2a}{dt^2} + \frac{k\Gamma}{2}f(t)\frac{da}{dt} + \alpha^2 a(t) = 0, \tag{45}$$

with the initial conditions $a(0) = 1$ and $\dot{a}(0) = 0$. Transforming the dependent variable as $a(T) = \exp[-\int_0^T s(t)dt]$, which automatically satisfies the first initial condition, we find

$$\frac{ds}{dt} + \frac{k\Gamma}{2}f(t)s(t) - \alpha^2 = 0, \tag{46}$$

where we have neglected the nonlinear term $s^2(t)$ since $s(t) \ll k\Gamma$ follows from Eq. (40) as a consequence of $\alpha \ll k\Gamma$. We have checked by numerical simulation that the nonlinear term has no noticeable effect on the solution to Eq. (46) in the entire parameter space where $a(t)$ is dominated by decay rather than Rabi oscillations.

Equation (46) can be solved analytically for $s(t)$; however, since we are not interested in the details of short transient behavior around $t = 0$ when $a(t)$ evolves quadratically with time, but rather in the long-term exponential decay, it is simpler and more transparent to proceed as follows. For correspondence with standard pulsed QZE, where measurement is varied in an on-off fashion, we let $f(t)$ be a square wave with period τ and amplitude 2, so that its mean is unity. Then the long-term behavior of Eq. (46) is that of relaxation oscillations with constant amplitude, where $s(t)$ is entirely positive due to the α^2 term. There is a period of linear growth satisfying $ds/dt - \alpha^2 = 0$ for time $\tau/2$ and exponential decay satisfying $ds/dt + k\Gamma s(t) - \alpha^2 = 0$. Solving both differential equations for $s(t)$ and setting the growth and decay amplitudes equal in the steady state, we obtain the following expression for $s_m = \max[s(t)]$:

$$s_m = \alpha^2\left(\frac{1}{k\Gamma} + \frac{\tau}{2}\frac{1}{1-e^{-k\Gamma\tau/2}}\right). \tag{47}$$

We now integrate $s(t)$ over the periods exhibiting linear growth and decay to yield the decay coefficient of $|a(t)|^2$,

$$2\int_0^T s(t)dt \approx 2T\frac{1}{\tau}\int_0^\tau s(t)dt \tag{48}$$

$$= \frac{\alpha^2 T}{k\Gamma}\left[3 + \frac{k\Gamma\tau}{4}\coth\left(\frac{k\Gamma\tau}{4}\right)\right]. \tag{49}$$

The same result is obtained in a more laborious fashion by solving Eq. (46) exactly and then dropping the transient behavior.

We now investigate the two limits of Eq. (49). In the limit of infinite measurement frequency so that $k\Gamma\tau \to 0$, the second term in brackets tends to unity, and Eq. (49) tends to the decay coefficient of Eq. (39), corresponding to continuous QZE for an $f(t)$ with a mean of unity. This behavior was predicted above on the basis of Eq. (38). In the other extreme, $k\Gamma\tau \to \infty$, corresponding to sufficiently strong measurements that the complete collapse of Eq. (40) is a valid approximation, the measurement strength cancels in Eq. (49), and the decay coefficient tends to $\alpha^2\tau/4$. The factor of 4 difference from Eq. (40) is to be expected since under a square wave on-off measurement, unitary evolution lasts half the period, $\tau/2$, and the appropriate limit in the collapse model is $\cos^{2T/\tau}(\alpha\tau/2) \approx e^{-\alpha^2\tau T/4}$.

Thus, Eq. (49) has the expected limiting behavior and extends the well-known result of Eq. (40) to measurements of finite strength and duration, when complete collapse is not achieved during the measurement interval.

Equation (49) was derived with the assumption of the decay measurement Hamiltonian of Eq. (17), but we now show that it is more general and applies to Lindblad operator measurements, Eq. (29). To see this, we write the Rabi Hamiltonian as $H_s = \alpha\,\sigma_x$, and without loss of generality let $\hat{H}_m = \sigma_z$, where σ_i are the Pauli matrices. The equation of motion becomes

$$i\frac{d\rho}{dt} = \alpha[\sigma_x, \rho] - ik\gamma f(t)[\sigma_z, [\sigma_z, \rho]]. \tag{50}$$

We expect a system initially in an eigenstate of σ_z to display the QZE and so calculate the expectation value of this state as $z = tr(\sigma_z \rho)$. Multiplying Eq. (50) by σ_z and σ_y and taking the trace using the relation $\mathrm{tr}(A[B, C] = \mathrm{tr}([A, B]C)$ as well as the commutation relations of the Pauli matrices $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, we find

$$\dot{z} = 2\alpha y, \tag{51}$$

$$\dot{y} = -2\alpha z - 4k\gamma f(t)y. \tag{52}$$

Solving simultaneously, we obtain Eq. (45) for z, with double the oscillation frequency since $z(t)$ corresponds to the amplitude squared, $z = \rho_{00} - \rho_{11} \approx \rho_{00}$, and with $\Gamma = 4\gamma$ since 2γ is the decay rate of the amplitude.

IV. CONCLUSIONS

We have shown that the QZE can advantageously be understood as a perturbative effect of H_s about the measurement Hamiltonian H_m in an approach which readily generates quantitative results. A bound for probability leakage from the eigenspaces of H_m, an instance of which is the initial unstable state, can be calculated in perturbation theory and tends to zero as a function of measurement strength. For H_m in the thermodynamic limit, which properly describes classical irreversible measurements, application of perturbation theory leads to unlimited probability leakage from the eigenspaces at finite measurement strength, but the rate of leakage tends to zero as measurement strength increases. Perturbative expressions are obtained in decoherence theory based on Lindblad operators with factorization into system and measurement components. The pulsed QZE mode originally envisaged in [1] is shown to be based on the same features of the Schrödinger equation as continuous QZE, with collapse not essential for its manifestation. In particular, it is shown that pulsed QZE merges smoothly into continuous QZE as the rate of measurement increases, and an analytic result is obtained extending the usual pulsed QZE result to finite measurement strengths and durations.

APPENDIX

In the von Neumann measurement scheme, a measurement interaction of a system and measurement apparatus results in an entangled state $|\Psi\rangle$ in the Hilbert space of the tensor product of the system and measurement Hilbert spaces. If the system and measurement spaces are spanned by bases $|\phi_i\rangle$ and $|\varphi_j\rangle$, respectively, we have

$$|\Psi\rangle = \sum_{i=1}^{M} a_i(t)|\phi_i\rangle \otimes \sum_{j=1}^{N} b_j(t)|\varphi_j\rangle.$$

An interaction Hamiltonian

$$H_m = \sum (H_m)_{lnij}|\phi_l\rangle\langle\phi_i| \otimes |\varphi_n\rangle\langle\varphi_j|$$

representing a measurement entangles these two Hilbert spaces. The system Hamiltonian H_s becomes $H_s \otimes \mathbf{1}$ in the total Hilbert space, where $\mathbf{1} = \sum_j |\varphi_j\rangle\langle\varphi_j|$ is the identity in measurement space. We can expand $|\Psi\rangle$ in terms of a new one-dimensional basis $|\eta_l\rangle$,

$$|\Psi\rangle = \sum_{l=1}^{NM} c_l(t)|\eta_l\rangle,$$

with $|\eta_l\rangle = |\phi_i\rangle|\varphi_j\rangle$, $c_l(t) = a_i(t)b_j(t)$, and index l defined, for example, by $l = i + Mj$. The measurement Hamiltonian in the new basis is then a two-dimensional matrix coupling states $|\eta_l\rangle$,

$$H_m \equiv \sum_{ln} (H_m)_{ln}|\eta_l\rangle\langle\eta_n|.$$

Such a representation of measurement is used in [4].

As a simple example, let $M = 3$ and $N = 2$, so the total Hilbert space is six-dimensional, with basis $\{|\phi_3\rangle|\varphi_1\rangle, |\phi_2\rangle|\varphi_1\rangle, |\phi_1\rangle|\varphi_1\rangle, |\phi_3\rangle|\varphi_2\rangle, |\phi_2\rangle|\varphi_2\rangle, |\phi_1\rangle|\varphi_2\rangle\}$. Let the system Hamiltonian H_s be the Rabi Hamiltonian of Eq. (6) and the measurement Hamiltonian generate transitions between system states $|\phi_2\rangle$ and $|\phi_1\rangle$ correlated to transitions between $|\varphi_1\rangle$ and $|\varphi_2\rangle$ in the measurement system, Eq. (4). Then, the total Hamiltonian $H = H_s + kH_m$ couples the states $\{|\phi_3\rangle|\varphi_1\rangle, |\phi_2\rangle|\varphi_1\rangle, |\phi_1\rangle|\varphi_2\rangle\} \equiv |\eta_l\rangle$, which are decoupled from the remaining states. If only states $|\eta_l\rangle$ are initially populated, we can write the total state $|\Psi\rangle$ as a sum over the amplitudes of these three states, $|\Phi\rangle = \sum_{l=1}^{3} c_l|\eta_l\rangle$. In the new basis $|\eta_l\rangle$, the total Hamiltonian H is represented by a two-dimensional matrix acting on a three-component vector consisting of the amplitudes of the three basis states c_l,

$$H_s + kH_{m1} \equiv \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & k \\ 0 & k & 0 \end{pmatrix}.$$

The solution to the Schrödinger equation for the initial occupation of state $|\phi_3\rangle|\varphi_1\rangle$ is then given by Eq. (8).

However, not all interactions H_m generating QZE-like effects produce entanglement between the system and the measurement apparatus, in which case such effects cannot be attributed to measurement. Consider a system with $M = 2$ coupled to a measurement apparatus with $N = 2$, resulting in a four-dimensional Hilbert space. The system Hamiltonian generating Rabi oscillations is

$$H_s = \alpha(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|) \otimes (|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|),$$

and two states decouple if the interaction Hamiltonian affects only the self-energy of a single state

$$H_{m2} = 2\,|\phi_1\rangle\langle\phi_1| \otimes |\varphi_1\rangle\langle\varphi_1|.$$

In the basis $\{|\phi_2\rangle|\varphi_1\rangle, |\phi_1\rangle|\varphi_1\rangle\}$, the total Hamiltonian is represented by

$$H_s + kH_{m2} \equiv \begin{pmatrix} 0 & \alpha \\ \alpha & 2k \end{pmatrix}.$$

The initial state $|\phi_2\rangle|\varphi_1\rangle$ is an eigenstate of H_{m2} and therefore becomes stable in the limit $k \to \infty$, which is confirmed by the no-decay probability

$$|c_1(t)|^2 = \frac{k^2 + \alpha^2\cos^2\omega t}{k^2 + \alpha^2},$$

where $\omega = \sqrt{k^2 + \alpha^2}$. This exhibits the same qualitative features as Eq. (8), but in the absence of entanglement.

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