

Response in the violation of the Bell inequality to imperfect photon addition and subtraction in noisy squeezed states of light

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Violation of Bell inequality is a prominent detection method for quantum correlations present in composite quantum systems, both in finite and infinite dimensions. We investigate the consequence of the violation of local realism based on pseudo-spin operators when photons are added or subtracted in a single mode or in both the modes of the two-mode squeezed states of light in the presence of noise. In the noiseless situation, we show that for addition (subtraction) of photons in a single mode, there is an overall enhancement in the maximal violation, although we observe an interplay between monotonicity and nonmonotonicity in the violation of Bell inequality depending on the squeezing strength. Moreover, we report that for low squeezing or low number of photons added or subtracted, subtraction in both the modes can lead to higher violation of local realism than that in the case of addition. For any choice of parameters, such ordering is not seen if one compares their entanglement contents. In the event of a faulty twin-beam generator, we obtain a lower than expected squeezing in the state. In such a case, or in imperfect photon addition (subtraction) or under local noise, we find that the violation of local realism by the noise-affected two-mode squeezed states always decreases. Interestingly, however, we notice that photon addition (subtraction) can in general help to conquer the ill effects of noise by enhancing the violation of local realism or by transforming nonviolating states to violating ones, thereby acting as an activating agent.

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I. INTRODUCTION

Entangled quantum states [1] shared between multiple and distant partners have the potential of revolutionizing communication and computation schemes [2–9]. Historically, the existence of quantum entanglement was first pointed out in the seminal paper of Einstein, Podolsky, and Rosen (EPR) [10], which questioned whether the theory of quantum mechanics was “incomplete,” based on the assumptions of “locality” and “reality.” Bell formulated a mathematical inequality to be satisfied by any physical theory that is local and realistic and which can be violated by entangled quantum states [11]. With the development of quantum information science, violation of the Bell inequality turned out to be an experimental-friendly detection criterion for entangled states. Apart from its fundamental importance, violation of the Bell inequality has been proven to be the crucial ingredient in certain proofs of security of quantum cryptography [2,3].

Quantum information protocols like entanglement-based quantum key distribution [2], quantum dense coding [4], and quantum teleportation [5] were originally proposed for discrete variable systems and have been implemented, e.g., by using the polarization degree of freedom of photons [12]. However, the success probability of preparing entangled states in this way is very low, and at the same time, Bell-basis measurement, if required for certain processes, is not possible with linear optical elements [13], thereby making the overall success probability of protocols using photonic qubits even lower. It turns out that continuous-variable (CV) systems [14,15] can overcome certain difficulties, like Bell-basis detection, and

hence implementing quantum information processing tasks by using CV states in infinite-dimensional systems can be important. Specifically, they can be prepared with almost unit probability by using nonlinear interaction of a crystal with laser and can have only imperfections due to the varying intensity of laser light, resulting in a low squeezing parameter [14]. Therefore, studying the quantum nature of such CV systems plays a significant role in quantum information science and is the main goal of this article.

Gaussian states, having positive Wigner functions [16], are one of the most prominent examples of CV states and are advantageous for quantum communication and computation schemes [17]. Although the performance of these states clearly shows their nonclassical nature, Bell argued [18,19] that states with positive Wigner functions are naturally endowed with a hidden variable theory and hence would not violate a Bell inequality. Later, Banaszek and Wódkiewicz [20] pointed out that the positivity or negativity of Wigner function has a weak connection to violation of local realism and managed to construct a Bell expression out of parity-based operators to obtain violations for two-mode squeezed vacuum (Gaussian) states with positive Wigner functions. However, their technique had intrinsic optimization problems [21], and so even the EPR state, having maximal quantum correlation, does not violate the inequality maximally. In Ref. [22], an alternate approach was proposed using pseudospin operators (that are closely related to the parity operators), which is free from such optimization difficulties and can give the maximal violation in the case of EPR state (for a nice survey, see Ref. [23]). Bell inequality for two- and three-mode CV

systems were also analyzed in Ref. [24]. The pseudospin operators were later generalized [25,26] to calibrate the violation of local realism for other types of quantum correlated states of continuous variables. Moreover, the violation of Bell inequality for squeezed vacuum states have been tested experimentally using parity-type operators [27], which further motivates the study in this direction.

On the other hand, there exist several quantum information protocols, like entanglement distillation and quantum error correction [28,29], which cannot be performed by Gaussian states with Gaussian operations [30,31]. Therefore, over the past few years, active research has been carried out to investigate properties of non-Gaussian states. One of the simple methods to generate such states is to add or subtract photons [32,33] to or from the Gaussian states. These processes have also been demonstrated experimentally [34,35]. Moreover, it was shown that entanglement content of the two-mode photon-added (-subtracted) state is much higher than the corresponding two-mode squeezed vacuum states (TMSV) [32], thereby showing enhancement of entanglement due to photon addition or subtraction. Moreover, “degaussification” via photon addition and subtraction has also been proven to be useful in a variety of situations, like engineering of quantum states to attain hybrid entanglement [36] and for tackling boson sampling problems [37].

In this article, we investigate the violations of Bell inequality for photon-added and photon-subtracted two-mode squeezed vacuum states, both in noiseless and noisy scenarios, where violations of local realism are tested using the pseudospin operators. Before considering the imperfection, we first present the results in the case of single-mode operations without noise, specifically photon addition (subtraction) from a single mode of TMSV, and apart from some instances of diminution, we report an overall enhancement in the maximal violation of Bell inequality with added (subtracted) number of photons. However, we report some interesting nonmonotonic features, when odd or even numbers of photons are added (subtracted) to a single mode. The response of maximal violation of Bell inequality is also examined, when a given number of photons to be added or subtracted is distributed between the two modes. In particular, in a distributed scenario, we find that unlike entanglement, for a certain squeezing and a small number of added or subtracted photons, subtraction is better than addition according to their quantumness in terms of violation of local realism with pseudospin operators. Moreover, we compare the effect of distribution to single-mode operations and observe that for sufficiently high squeezing or number of added (subtracted) photons, the maximal violation for distributed operations displays a monotonic enhancement compared to that in case of single-mode operations.

An important aspect, which turns out to be crucial experimentally, is the role of the inevitable noise that creeps in the TMSV states [38] during preparation, transmission, and protocol implementation. We investigate the effects of noise on the violation of Bell inequality in two prototypical realistic scenarios: The states are affected by noise or the state generator is itself faulty, i.e., when instead of a TMSV state with certain squeezing, it prepares a state with lower squeezing. As expected, noise reduces the amount of quantum correlations present in these states and hence the amount of violation of

local realism. Interestingly, however, we show that the process of photon addition (subtraction) can enhance and in some cases *activate* the violations (cf. Ref. [24]). Specifically, we find that photon addition can transform certain nonviolating states to Bell-inequality-violating states, which we call the activation of violation of Bell inequality, as the word “activation” has been used in the literature for different processes [39]. In realistic scenarios, even the addition (subtraction) schemes of photons can be faulty due to mechanisms like dark counts [40,41] of the photodetectors. We also study the reaction of the pseudospin operator-based Bell inequality in the presence of both noisy and faulty scenarios and show that activation of violation due to the process of addition (subtraction) of photons is also possible even in the presence of two types of noise.

The paper is organized as follows. In Sec. II, we discuss about the two-mode squeezed vacuum states and the effect of photon addition or subtraction on it. The use of pseudospin operators for analyzing violations of Bell inequality is also explored here. In Sec. III, the case of single-mode operations is examined. Following it, in Sec. IV, we discuss how distributed operations affect the maximal violation. Furthermore, in Sec. IV A, we show that in contrast to single-mode operations, in the realm of distributed operations, photon addition is inequivalent to photon subtraction. In Sec. V, violations of Bell inequality are examined in more realistic scenarios, namely, in the presence of noise in Sec. V A and when the squeezed state generator is faulty in Sec. V D. Finally, in Sec. V E, we deal with the scenario of dark counts in the photon addition and subtraction mechanism, making them erroneous. In Sec. VI contains a conclusion. An appendix provides the proof of the maximization of the Bell expression.

II. FORMALISM

Study of Gaussian states lies at the heart of investigations with CV systems. In the state space of Gaussian states, the most general pure states are the displaced squeezed states [14,15]. Since we are interested in studying the quantum correlations of quantum states in CV system, and we know that the displacement operator does not alter the nonlocal properties of a state, in this article, without loss of generality, we consider the (undisplaced) two-mode squeezed vacuum (TMSV) state for our investigations. For squeezing strength r , the TMSV state can be represented as

$$|\psi_r\rangle = \sum_{n=0}^{\infty} c_n |n, n\rangle, \quad (1)$$

where $c_n = (1-x)^{\frac{1}{2}} x^{\frac{n}{2}}$ with $x = \tanh^2 r$, and $\{|n\rangle\}$ is the Fock basis consisting of the photon number states. The TMSV state, in the limit of infinite squeezing ($r \rightarrow \infty$), reduces to the well-known EPR state.

We can degaussify the TMSV state by simply adding (subtracting) photons locally in its two modes. It was shown that this degaussification process (photon addition or subtraction) leads to monotonic enhancement of entanglement [32]. In this paper, we analyze the effects of photon addition or subtraction on the violation of Bell inequality. The normalized state after adding k photons to the first mode and l photons to the second

mode of the TMSV state reads as

$$|\psi_r^{(k,l)}\rangle = \sum_{n=0}^{\infty} c_n^{(k,l)} |n+k, n+l\rangle, \quad (2)$$

where

$$c_n^{(k,l)} = \frac{x^{\frac{n}{2}}}{\sqrt{{}_2F_1(k+1, l+1, 1, x)}} \sqrt{\binom{n+k}{k} \binom{n+l}{l}}. \quad (3)$$

Here, ${}_2F_1$ is the Gauss hypergeometric function. Note that $c_n^{(0,0)} = c_n$ in Eq. (1). On the other hand, the normalized state after subtracting k and l photons from first and second modes respectively is given by

$$|\psi_r^{(-k,-l)}\rangle = \sum_{n=k}^{\infty} c_n^{(-k,-l)} |n-k, n-l\rangle, \quad (4)$$

where

$$c_n^{(-k,-l)} = \frac{x^{\frac{n-k}{2}}}{\sqrt{{}_2F_1(k+1, k+1, 1+k-l, x)}} \sqrt{\frac{\binom{n}{k} \binom{n}{l}}{\binom{n}{k}}}. \quad (5)$$

Without any loss of generality, in this paper, we assume that $k \geq l$. If we restrict operations to a single mode, say, the first mode, the coefficients involved in $|\psi_r^{(\pm k, 0)}\rangle$ simplifies as

$$c_n^{(k,0)} = x^{\frac{n}{2}} (1-x)^{\frac{1+k}{2}} \sqrt{\binom{n+k}{k}} \quad (6)$$

and

$$c_n^{(-k,0)} = x^{\frac{n-k}{2}} (1-x)^{\frac{1+k}{2}} \sqrt{\binom{n}{k}}. \quad (7)$$

We consider Bell inequalities by using the following pseudospin operators [25], given by

$$S_q^z = \sum_{\substack{n=0 \\ 2n+q \geq 0}}^{\infty} |2n+q+1\rangle \langle 2n+q+1| - |2n+q\rangle \langle 2n+q|, \\ S_q^- = \sum_{\substack{n=0 \\ 2n+q \geq 0}}^{\infty} |2n+q\rangle \langle 2n+q+1| = (S_q^+)^{\dagger}, \quad (8)$$

where q is an integer. The correlation functions for an arbitrary state ρ , in terms of the pseudospin operators, are given by

$$E(\theta_a, \theta_b) = \text{Tr}[\rho \mathbf{S}_{q_1}^{\theta_a} \otimes \mathbf{S}_{q_2}^{\theta_b}], \quad (9)$$

where $\mathbf{S}_{q_j}^{\theta_j} = \cos \theta_j S_{q_j}^z + \sin \theta_j (S_{q_j}^- + S_{q_j}^+)$, $j = a, b$, with θ_j s being the settings of the measurements performed by both the parties, viz. a and b . Like the Clauser-Horne-Shimony-Holt (CHSH) version [42] of Bell inequality (Bell-CHSH inequality) in finite dimension, the Bell-CHSH expression

in this case based on the correlation functions, $E(\theta_a, \theta_b)$, in Eq. (9), also reads as

$$\chi_{\theta_a, \theta_b, \theta'_a, \theta'_b}^{q_1, q_2} = E(\theta_a, \theta_b) + E(\theta_a, \theta'_b) + E(\theta'_a, \theta_b) - E(\theta'_a, \theta'_b). \quad (10)$$

Our task is to maximize $\chi_{\theta_a, \theta_b, \theta'_a, \theta'_b}^{q_1, q_2}$ (which we refer to as χ without subscripts and superscripts) with respect to the settings specified by $\theta_a, \theta_b, \theta'_a, \theta'_b$, and the pair (q_1, q_2) . Note that, in the correlation function, constructed out of the pseudospin measurements [see Eq. (9)], we neglect any phase factors since they do not provide any additional information in the maximization of the Bell expression for the states considered here. Therefore, we are finally interested to study the properties of a physical quantity, given by

$$\chi^{\max} = \max_{\theta_a, \theta_b, \theta'_a, \theta'_b, q_1, q_2} \chi_{\theta_a, \theta_b, \theta'_a, \theta'_b}^{q_1, q_2}. \quad (11)$$

It turns out that the optimization over the q values can be performed easily by looking at the structure of the concerned state. Settling with the values of (q_1, q_2) , we are left with the optimization over the measurement settings $\{\theta_a, \theta_b, \theta'_a, \theta'_b\}$. The correlation function in Eq. (9), for the states considered in this article, typically is of the form

$$E(\theta_a, \theta_b) = \pm \cos \theta_a \cos \theta_b + \mathcal{K} \sin \theta_a \sin \theta_b, \quad (12)$$

where $0 \leq \mathcal{K} \leq 1$. For a TMSV state with squeezing parameter r , $\mathcal{K} = \tanh 2r$, and depending on the number of photons added or subtracted to the TMSV state, the $\mathcal{K}_{(\pm k, \pm l)}$ changes accordingly. Here, the subscript of \mathcal{K} denotes the number of photons added or subtracted from each mode of the TMSV state. The optimal measurement settings, which maximizes the violation of pseudospin-based Bell inequality considered in Eq. (10), is given by

$$\theta_a = 0, \theta_{a'} = \pi/2, \theta_b = -\theta_{b'} = -\theta, \quad (13)$$

where

$$\cos \theta = \frac{1}{\sqrt{1+\mathcal{K}^2}} \text{ and } \sin \theta = \frac{\mathcal{K}}{\sqrt{1+\mathcal{K}^2}}, \quad (14)$$

such that the maximal Bell-CHSH quantity, χ^{\max} , reduces to

$$\chi^{\max} = 2(\cos \theta + \mathcal{K} \sin \theta) = 2\sqrt{1+\mathcal{K}^2}. \quad (15)$$

The details of the optimization procedure are given in the appendix. Note that for two-qubit systems, the CHSH inequality constructed out of the usual spin operators (Pauli matrices) provides a necessary as well as sufficient condition for the violation of local realism with two settings of measurements having two outcomes. In the present paper, the pseudospin-operator-based CHSH inequality provides us a sufficient condition for the violation of local realism for CV systems. It is, however, not a necessary condition.

The reason is that for two-qubit states, the Pauli spin operators form a basis for all the operators in that space. In the two-qubit case, following the prescription given in Ref. [43] by Horodecki *et al.*, the maximization can be performed over all possible two-qubit operators, and hence the obtained maxima is the optimal among all the two-setting Bell inequalities. However, the pseudospin operators do not form a basis for all

dichotomic operators in the space of two-mode continuous-variable states. Therefore, the maximization involved here gives the maximal violation of Bell inequality for all possible dichotomic measurements in the subspace of pseudospin operators (all two-mode dichotomic operators which can be expanded in terms of the pseudospin operators), making the conditions of violation sufficient but not necessary.

Moreover, we want to mention here that the extraction of the necessary condition involves the maximization over all possible dichotomic observables in the continuous variable case, which is practically impossible since in this case, even the basis for constructing dichotomic observables is infinite. So, unlike two-qubit systems, for two-mode CV systems, to date there is no unique optimal inequality which can detect EPR-type nonlocality. However, there exist several results in the literature that strongly suggest that pseudospin operator-based inequality is one of the best options to detect nonlocality in two-mode CV systems. For example, the violation of Bell inequality for the two-mode squeezed vacuum state was studied in Ref. [20] using a Wigner-function-based CHSH inequality. In their analysis, even for the EPR state, their bound is not saturated to the Tsirelson's bound. Later it was shown that Tsirelson's bound by the EPR state can be achieved by the pseudospin operator-based inequality as shown in Ref. [22]. These operators were later generalized in various works [25,26] to find out the violation of other two-mode and multimode CV states [24].

In our work, we have used generalized pseudospin operator-based Bell inequalities which we believe to be sufficiently good for obtaining violation of local realism. The maximal violation obtained in this paper is in the pseudospin-operator space and provides a lower bound of the maximal violation which can be obtained by constructing all the Bell inequalities in CV systems.

The amount of enhancement in maximal violation of Bell inequality in the photon addition and subtraction process can be quantified as

$$\mathcal{G} = \frac{\chi^{\max}(|\phi\rangle) - \chi^{\max}(|\eta\rangle)}{\chi^{\max}(|\eta\rangle)}. \quad (16)$$

Here, we compute the enhancement in the maximal violation of local realism for $|\phi\rangle$ with respect to a given state $|\eta\rangle$. Typically, $|\eta\rangle$ is the TMSV state, while $|\phi\rangle$ is the same TMSV state after adding (subtracting) photons. Using these techniques, we first set out to investigate the effects of maximal violation of Bell inequalities due to single-mode operations.

III. SINGLE-MODE OPERATIONS

Let us first concentrate on the response of maximal violation of Bell inequality of TMSV states subject to addition and subtraction of photons in a single mode. We first note that, for single-mode operations, photon addition in one mode is equivalent to the photon subtraction from the other mode, as it can be easily shown that $|\psi_r^{(k,0)}\rangle = |\psi_r^{(0,-k)}\rangle$ by using Eqs. (2)–(5). Moreover, since $\chi^{\max}(|\psi_r^{(k,0)}\rangle) = \chi^{\max}(|\psi_r^{(0,k)}\rangle)$, we can easily see $\chi^{\max}(|\psi_r^{(k,0)}\rangle) = \chi^{\max}(|\psi_r^{(-k,0)}\rangle)$. Therefore, without any loss of generality, we only consider addition of photons in a single mode, say, the first mode, of the TMSV state in

this section. However, from an experimental point of view, subtraction is easier to realize than addition [34], since the latter process essentially requires an additional photon pumping apparatus. So, even if both processes are equivalent in terms of the maximal violation, experimentally, subtraction is preferred. Furthermore, in situations where addition and subtraction yields inequivalent maximal violation of Bell inequality, it would be noteworthy to find out regions in the relevant parameter space where photon subtraction gives a higher violation than photon addition. We will address this point in the succeeding section.

A. Addition and subtraction of arbitrary number of photons

The maximal violation of Bell inequality for the photon-added TMSV state, $|\psi_r^{(k,0)}\rangle$, in the first mode, has the form

$$\chi^{\max}(|\psi_r^{(k,0)}\rangle) = 2\sqrt{1 + \mathcal{K}_{(k,0)}^2}, \quad (17)$$

where

$$\mathcal{K}_{(k,0)} = 2 \sum_{n=0}^{\infty} c_{2n}^{(k,0)} c_{2n+1}^{(k,0)}, \quad (18)$$

by using Eqs. (2), (3), and (15). The (q_1, q_2) pair which yields this maximal value is $(k \bmod 2, 0)$.

Note that the structure of $\mathcal{K}_{(k,0)}$ remains same for both even and odd numbers of photon-added TMSV states. For $k \geq 2$, we obtain the expression for $\mathcal{K}_{(k,0)}$ as

$$\begin{aligned} \mathcal{K}_{(k,0)} &= 2(1-x)^{1+k} x^{\frac{1}{2}} \sum_{n=0}^{\infty} x^{2n} \prod_{i=2}^k (2n+i) \\ &\quad \times \sqrt{(2n+1)(2n+k+1)}. \end{aligned} \quad (19)$$

On the other hand, for the $k=0$ case, i.e., for the TMSV state, $\mathcal{K}_{(0,0)} = 2x^{1/2}/(1+x) = \tanh 2r$, while if a single photon is added, it takes the form as

$$\mathcal{K}_{(1,0)} = 2(1-x)^2 x^{\frac{1}{2}} \sum_{n=0}^{\infty} x^{2n} \sqrt{(2n+1)(2n+2)}. \quad (20)$$

Apart from the TMSV case, $\mathcal{K}_{(k,0)}$ and consequently χ^{\max} cannot be computed analytically for any $k \geq 1$, due to the presence of a square root in the sum involved. Therefore, we resort to approximate methods like series expansion and numerical techniques to compute these summations. For numerical calculations, we first evaluate the above summations up to $n=N$ terms. We then check whether the difference between χ^{\max} with partial sums up to N and $N+1$ terms falls below 10^{-10} . If this is the case, we conclude that the summation with N terms is sufficient.

Before presenting the results with series expansion, let us discuss the findings with the numerical method. Our analysis reveals an overall enhancement of the maximal violation of Bell inequality in terms of pseudospin operators of the TMSV state with moderate number of added photons (see Fig. 1). To put the amount of enhancement in a quantitative perspective, we calculate the gain, \mathcal{G} , as in Eq. (16), for some typical values of r and added number of photons, k , in the first mode, and it is summarized in Table I. As is clearly depicted in Fig. 1, there exists a critical value of the squeezing parameter, r ,

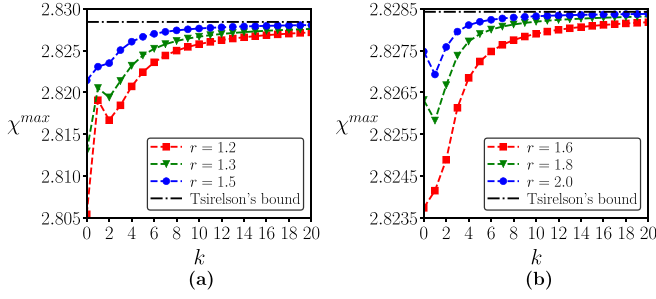


FIG. 1. Maximal violation of Bell inequality with respect to added (subtracted) number of photons, k , in the first mode. In panels (a) and (b), different values of the squeezing parameter, r , have been considered. In panel (a), we choose those values of r where χ^{\max} decreases when two or more than two photons are added, while in panel (b), whenever χ^{\max} shows decreasing nature, it occurs after an addition of a single photon. Clearly, such values of r are above 1.66 as found in Theorem 1. All quantities plotted are dimensionless.

beyond which photon addition may lead to a decrement in the maximal violation of Bell inequality, when either a single photon or more is added [comparing Figs. 1(a) and 1(b)]. Even when we are unable to compute χ^{\max} analytically, the approximate method helps us to obtain the critical value of r , r_c , where the violation decreases with addition of photons. In this respect, let us state the following theorem, which shows the special status of a single-photon addition.

Theorem 1: The maximal violation of Bell inequality based on pseudospin operators shows diminution in comparison to the TMSV state after the addition (subtraction) of a single photon for any finite squeezing parameter, r , beyond a critical value, $r_c \approx 1.66$.

Proof. We start by approximating the square root term $\sqrt{(2n+1)(2n+2)}$ in $\mathcal{K}_{(1,0)}$, given in Eq. (20). Let $X = 2n+1$ and $Y = 2n+2$. Now, using the identity, $(X+Y)^2 - (X-Y)^2 = 4XY$ and putting $(X-Y)^2 = 1$, we get

$$\begin{aligned} \sqrt{XY} &= \frac{X+Y}{2} \sqrt{1 - \frac{1}{(X+Y)^2}} \\ &\approx \frac{X+Y}{2} \left[1 - \frac{1}{2(X+Y)^2} - \frac{1}{8(X+Y)^4} \right]. \end{aligned} \quad (21)$$

Substitution of the value of $(X+Y)$ in Eq. (21) gives

$$\begin{aligned} \sqrt{(2n+1)(2n+2)} &\approx \left(2n + \frac{3}{2} \right) - \frac{1}{4(4n+3)} \\ &\quad - \frac{1}{16(4n+3)^3}. \end{aligned} \quad (22)$$

Under this approximation, $\mathcal{K}_{(1,0)}$ possesses a closed form in terms of known standard functions, given by

$$\begin{aligned} \mathcal{K}_{(1,0)}^{\text{approx}} &= 2 \frac{x^{1/2}}{1+x} \left[\frac{3+x^2}{2(1+x)} - (1-x)^2(1+x) \right. \\ &\quad \left. \times \left(\frac{{}_2F_1(3/4, 1, 7/4, x^2)}{12} + \frac{\Phi(x^2, 3, 3/4)}{2^{10}} \right) \right], \end{aligned} \quad (23)$$

where ${}_2F_1$ denotes the Gauss hypergeometric function [44] and Φ denotes the Lerch transcendent [45]. Note that the approximation used in Eq. (22) always leads to an

TABLE I. Percentage of gain, $\mathcal{G} \times 100$, for some typical values of the squeezing parameter, r , and added number of photons, k , with respect to the TMSV state.

$r \downarrow$ $k \rightarrow$	2	5	10	15
0.2	9.5	18.4	25.7	29.6
0.5	10.0	10.9	10.7	11.4
0.8	3.2	3.1	3.4	3.7
1.2	0.4	0.6	0.73	0.76

overestimation of $\sqrt{(2n+1)(2n+2)}$ and therefore $\mathcal{K}_{(1,0)}^{\text{approx}} > \mathcal{K}_{(1,0)}$. Now the quantity $\mathcal{K}_{(1,0)}^{\text{approx}} - \mathcal{K}_{(0,0)}$, and consequently $\mathcal{K}_{(1,0)} - \mathcal{K}_{(0,0)}$ and $\chi^{\max}(|\psi_r^{(1,0)}\rangle) - \chi^{\max}(|\psi_r\rangle)$, become negative when $x \gtrsim 0.86$, i.e., $r \gtrsim 1.66$ ($\approx r_c$) and asymptotically approach zero from below when $x \rightarrow \infty$. Since the process of approximation gives an upper bound of $\mathcal{K}_{(1,0)}$ (or $\chi^{\max}(|\psi_r^{(1,0)}\rangle)$), the diminution of maximal violation on adding a single photon persists even without the approximation. Furthermore, keeping up to second-order terms is justified, since the next term in the sum near $r = 1.66$ only makes a contribution of $O(10^{-6})$ to the sum. Such observation remains true for all the propositions in this and succeeding sections, and hence the proof.

Note that by numerical simulations, we find the above critical value r_c to be ≈ 1.66 . Although in Theorem 1 we have found the critical value of r , beyond which addition of a single photon always leads to diminution of the maximal violation of Bell inequality, increasing the number of added photons results in an overall enhancement of the maximal violation, as mentioned previously. However, the enhancement of the violation is accompanied by a seemingly generic nonmonotonic behavior with respect to the squeezing parameter, r . We find that there exists a range of the squeezing parameter, $1.42 < r < 1.66$, for which the maximal violation of local realism demonstrates a *monotonic* enhancement with respect to the added number of photons (see Fig. 1). Apart from the above specified range of the squeezing parameter, χ^{\max} displays a nonmonotonic behavior with the number of added photons (Fig. 1). Note however that the nonmonotonicity obtained for $r < 1.42$ with k is different than that of the photon-added state with $r > 1.66$. It is important to stress here that such a feature is absent in the case of entanglement [32]. We now ask whether the critical value of the squeezing parameter for which nonmonotonic to monotonic transition occurs in the behavior of the maximal violation can be found using the series expansion method.

We observe from our numerical results that when the squeezing parameter r is close to the critical value, 1.42, the transition is dictated by both the values of $\chi^{\max}(|\psi_r^{(1,0)}\rangle)$ and $\chi^{\max}(|\psi_r^{(2,0)}\rangle)$. The value of $\chi^{\max}(|\psi_r^{(1,0)}\rangle)$ has been calculated in Eq. (23) while the evaluation of $\chi^{\max}(|\psi_r^{(2,0)}\rangle)$ leads to the following proposition:

Proposition 2: Maximal violation of local realism based on pseudospin operators undergoes a transition from non-monotonic to monotonic behavior with respect to added number of photons in a single mode for $r \approx 1.42$.

Proof. By substituting $k = 2$ in Eq. (19), we obtain

$$\mathcal{K}_{(2,0)} = 2(1-x)^3 x^{\frac{1}{2}} \sum_{n=0}^{\infty} x^{2n} (2n+2) \sqrt{(2n+1)(2n+3)}. \quad (24)$$

We approximate $\sqrt{(2n+1)(2n+3)}$ in the same lines as in Eq. (21) and get

$$\sqrt{(2n+1)(2n+3)} \approx 2(n+1) - \frac{1}{4(n+1)} - \frac{1}{8(n+1)^3}. \quad (25)$$

Using the above expression, we get the approximate value of $\mathcal{K}_{(2,0)}$ as

$$\mathcal{K}_{(2,0)}^{\text{approx}} = \frac{(1-x)^3 x^{1/2}}{2} \left[\frac{8x^4}{(1-x^2)^3} - \frac{12x^2}{(1-x^2)^2} - \frac{7}{2(1-x^2)} - \frac{1}{32x^2} \text{Li}_2(x^2) \right], \quad (26)$$

where Li_2 is the polylogarithmic function of order 2. Now, $\mathcal{K}_{(2,0)}^{\text{approx}} - \mathcal{K}_{(1,0)}^{\text{approx}}$, and consequently $\chi_{\text{approx}}^{\max}(|\psi_r^{(2,0)}\rangle) - \chi_{\text{approx}}^{\max}(|\psi_r^{(1,0)}\rangle)$ becomes positive for $x \gtrsim 0.79$, i.e., $r \gtrsim 1.42$.

Remark 1. The nonmonotonic to monotonic transition and *vice versa* in maximal violation of local realism with the added number of photons happens because χ^{\max} first decreases and then starts increasing before saturating to a certain value with addition.

Remark 2. The criticalities in r are obtained by keeping up to the second-order terms in a series [Eqs. (22) and (25)]. We find that such approximations nicely match with the values obtained from the numerical simulations.

Remark 3. The critical squeezing parameters obtained in the above cases can, in principle, be observed in laboratories, as all the critical values of the squeezing parameter are below the maximal amount of experimentally generated squeezing, i.e., $r \approx 1.73$ [46].

Therefore, by combining the results from Theorem 1 and Proposition 2, we zero in on the squeezing parameter window for which the maximal violation shows monotonic enhancement on adding (subtracting) photons from a single mode using the series expansion method, which clearly agree with Fig. 1. To get more intuitive insights, we now look at the cases of addition (subtraction) of even or odd number of photons separately in the next subsection.

B. Even-odd dichotomy

To find out the reason behind such dependence on squeezing parameter of maximal violation, we now study separately the TMSV states when even (odd) number of photons are added. The intuition for such investigation comes from the fact the χ^{\max} depends on the (q_1, q_2) pair, which is different for odd and even numbers of photons. Let us first restrict ourselves to addition of even number of photons from the first mode and study violations of Bell inequality with respect to the number of photons added for fixed values of the squeezing parameter. From numerical simulations, we find when only even numbers of photons are added from a particular mode

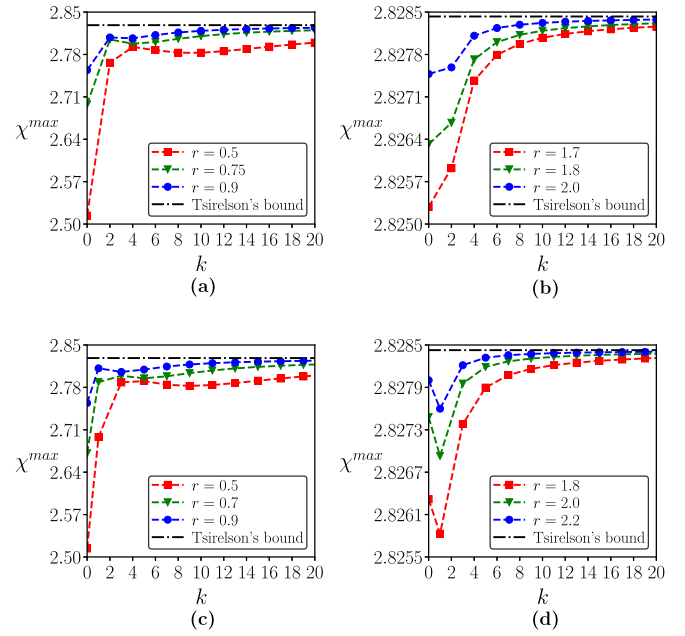


FIG. 2. Even vs odd. (Upper panel) χ^{\max} against even number of added photons in a single mode and (lower panel) χ^{\max} with odd k . All quantities plotted are dimensionless.

[Figs. 2(a) and 2(b)], the maximal violation shows monotonic enhancement for $r \gtrsim 0.94$. We now show the same by using the series expansion method.

Proposition 3: The maximal violation of Bell inequality undergoes a transition from nonmonotonic to monotonic behavior with respect to an even number of photons added in a single mode when $r \gtrsim 0.92$.

Proof. The insight, obtained from numerical simulation, as reflected in Fig. 2(a), tells us that when r is close to 0.92, the nonmonotonicity can be observed in the diminution of χ^{\max} after adding at least four photons to the system. Hence, unlike Theorem 1 and Proposition 2, the quantity of interest now becomes $\chi^{\max}(|\psi_r^{(4,0)}\rangle) - \chi^{\max}(|\psi_r^{(2,0)}\rangle)$. The approximation of $\chi^{\max}(|\psi_r^{(2,0)}\rangle)$ has already been done in Eq. (26), and hence we are left with the approximation of $\chi_{\text{max}}(|\psi_r^{(4,0)}\rangle)$. Expression of $\chi^{\max}(|\psi_r^{(4,0)}\rangle)$ is obtained by substituting $k = 4$ in Eq. (19), where the square root term $\sqrt{(2n+1)(2n+5)}$ can be approximated as

$$\sqrt{(2n+1)(2n+5)} \approx (2n+3) - \frac{2}{(2n+3)} - \frac{2}{(2n+3)^3}. \quad (27)$$

This approximation allows $\mathcal{K}_{(4,0)}$ and thereby $\chi^{\max}(|\psi_r^{(4,0)}\rangle)$ to be written in terms of known functions. Using Eqs. (26) and (27), we find that $\mathcal{K}_{(4,0)}^{\text{approx}} - \mathcal{K}_{(2,0)}^{\text{approx}}$ and consequently $\chi_{\text{approx}}^{\max}(|\psi_r^{(4,0)}\rangle) - \chi_{\text{approx}}^{\max}(|\psi_r^{(2,0)}\rangle) \gtrsim 0$, which implies $x \gtrsim 0.51$ or $r \gtrsim 0.92$.

We now move to the situation where odd numbers of photons are added. Interestingly, a qualitatively different picture emerges in this case compared to the even-photon addition [Figs. 2(c) and 2(d)]. In particular, there exists only a region in the squeezing parameter, namely $1.23 < r < 1.66$, where we get monotonic behavior of χ^{\max} with odd numbers of added

photons, k . Here, we should also comment that the natures of nonmonotonicity for $r < 1.23$ and $r > 1.66$ are different, as also seen in Fig. 1. When $r > 1.66$, the nonmonotonicity is reflected by a decrement in the maximal violation of Bell inequality after addition of a single photon, as already mentioned in Theorem 1, while for $r < 1.23$, the nonmonotonicity is observed after addition higher number of photons [see Fig. 2(c)]. However, the feature of overall enhancement of the violation for higher values of k compared to the TMSV state persists both for even as well as odd k . We again employ the series expansion method for obtaining the lower bound on r .

Proposition 4: When only odd numbers of photons are added (subtracted) to a single mode of the TMSV state, the maximal violation undergoes a transition from nonmonotonic to monotonic behavior at $r \approx 1.23$.

Proof. Again our numerical results help us to identify k in $\mathcal{K}_{(k,0)}$ relevant to prove this proposition. In this case, we notice that the quantity $\chi^{\max}(|\psi_r^{(3,0)}\rangle) - \chi^{\max}(|\psi_r^{(1,0)}\rangle)$ is appropriate. The square root term $\sqrt{(2n+1)(2n+4)}$ in $\mathcal{K}_{(3,0)}$ [see Eq. (19)] can be approximately written as

$$\sqrt{(2n+1)(2n+4)} \approx (2n+5/2) - \frac{9}{8(2n+5/2)} - \frac{81}{27(2n+5/2)^3}. \quad (28)$$

This approximation allows $\mathcal{K}_{(3,0)}$ to be written in terms of known hypergeometric and transcendental functions. We find $r \gtrsim 1.23$, for which $\chi_{\text{approx}}^{\max}(|\psi_r^{(3,0)}\rangle) - \chi_{\text{approx}}^{\max}(|\psi_r^{(1,0)}\rangle) \gtrsim 0$.

Remark. There exists a region, $1.23 < r < 1.42$, where both additions of even and odd numbers of photons lead to monotonic enhancement of maximal Bell violation, although the combined curve shows nonmonotonicity. This can be understood by noting the following fact. Even if both even and odd operations give monotonic violation of Bell inequality, it does not guarantee that their combined effect would be monotonic. Hence, individual monotonicity is a necessary but not a sufficient condition for combined monotonicity. However, nonmonotonic maximal Bell violation in individual cases ensures the violation for the state with arbitrary number of added photons to be nonmonotonic.

We know that the violation of Bell inequality by quantum states quantifies the content of quantum correlations present in these states. Another way to quantify quantum correlation is the amount of entanglement possessed by these states. By comparing the results obtained here with the entanglement content [32] of photon-added (photon-subtracted) TMSV states, we observe that the monotonic relationship of these quantities for pure two-qubit states in finite dimension is no longer true for pure states in the continuous variable case provided the Bell test is performed with pseudospin operators.

IV. DISTRIBUTED PHOTON ADDITION AND SUBTRACTION

In this section, we go beyond the realm of single-mode operations and analyze the effect of local operations on both modes. Specifically, for a given number of photons to be added, instead of dumping them in a single mode, we distribute them in two modes and examine the effects of

distribution on violations of Bell inequality based on pseudospin operators. Similar operations are considered in case of photon subtraction, which in this case is different from photon addition. For addition, the correlation function of the state given in Eq. (3), in terms of pseudospin operators, also takes the form as in Eq. (12) with

$$\mathcal{K}_{(k,l)} = 2 \times \max \left[\sum_{n=0}^{\infty} c_{2n}^{(k,l)} c_{2n+1}^{(k,l)}, \sum_{n=0}^{\infty} c_{2n+1}^{(k,l)} c_{2n+2}^{(k,l)} \right], \quad (29)$$

where $c_n^{(k,l)}$ are given in Eq. (3). Note that the maximization in the above equation arises due to the optimization involved in the (q_1, q_2) duo. In the case of photon subtraction, we can rewrite the state given in Eq. (5) as $|\psi_r^{(-k,-l)}\rangle = \sum_{n=0}^{\infty} c_{n+k}^{(-k,-l)} |n, n+k-l\rangle$. For this case,

$$\mathcal{K}_{(-k,-l)} = 2 \times \max \left[\sum_{n=0}^{\infty} c_{2n+k}^{(-k,-l)} c_{2n+1+k}^{(-k,-l)}, \sum_{n=0}^{\infty} c_{2n+1+k}^{(-k,-l)} c_{2n+2+k}^{(-k,-l)} \right], \quad (30)$$

where $c_n^{(-k,-l)}$ are represented in Eq. (5). The corresponding maximal violation of Bell inequality for both addition and subtraction reads as

$$\chi^{\max}(|\psi_r^{(\pm k, \pm l)}\rangle) = 2\sqrt{1 + \mathcal{K}_{(\pm k, \pm l)}^2}. \quad (31)$$

Note that in general $\mathcal{K}_{(k,l)} \neq \mathcal{K}_{(-k,-l)}$ and hence $\chi^{\max}(|\psi_r^{(k,l)}\rangle)$ and $\chi^{\max}(|\psi_r^{(-k,-l)}\rangle)$ are typically different. However, when $k=l$, we notice that $|\psi_r^{(k,k)}\rangle$ and $|\psi_r^{(-k,-k)}\rangle$ have the same Schmidt coefficients [32], and consequently, $\chi^{\max}(|\psi_r^{(k,k)}\rangle) = \chi^{\max}(|\psi_r^{(-k,-k)}\rangle)$. Therefore, when $k \neq l$ ($k \neq 0$), there exists a disparity in the maximal violation of Bell inequality for distributed photon-added and photon-subtracted states (see Figs. 3 and 4). We discuss this inequivalence in this section.

A. Inequivalence of addition and subtraction

In Sec. III, we argued that both addition and subtraction of photons from a single mode yields the same maximal violation. As pointed out earlier, this equivalence breaks down in the case of distributed photon addition and subtraction (Figs. 3 and 4). This inequivalence prompts a natural question: In terms of maximal violation under distribution, which one is better, addition or subtraction?

To answer the above question, for a fixed squeezing parameter r and for fixed total number of photons added (subtracted) in both the modes, $k+l$, with k and l being the added (subtracted) photons from the first and second modes respectively, we investigate the behavior of χ^{\max} with respect to k . Extensive numerical analysis reveals the following qualitative trends of χ^{\max} under distributed operations (certain exemplary scenarios are depicted in Figs. 3 and 4).

(1) In cases of both addition and subtraction, for low values of squeezing parameter r and the total number of added (subtracted) photons, $k+l$, we observe that the maximal violation usually decreases with the number of photons added in the first mode, k , which sometimes leads to the nonmonotonicity of χ^{\max} against k . Moreover, in the distributed case, the value of χ^{\max} occasionally turns out to be smaller compared to that of the single-mode operations.

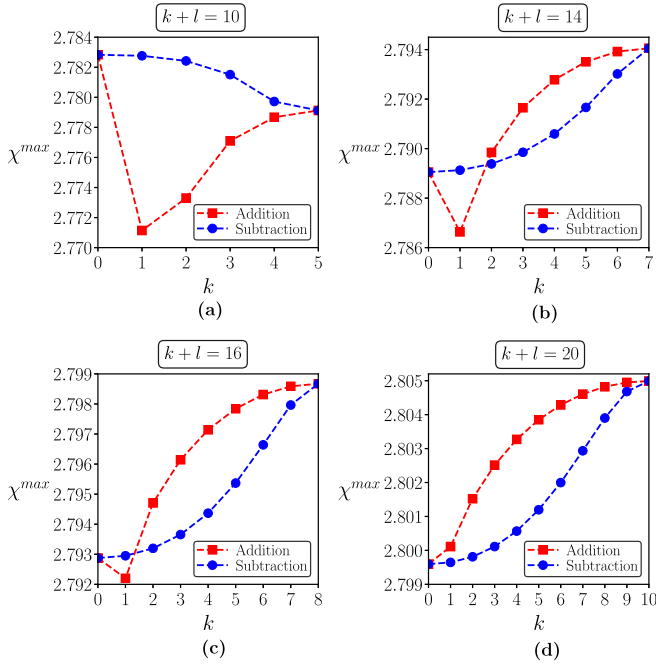


FIG. 3. Inequivalence of distributed addition and subtraction of photons. Here $r = 0.5$. The abscissa represents the number of photons added or subtracted from the first mode for a given total number of photons, $k + l$, where k and l represent the photons added (subtracted) in the first and the second modes, respectively. Furthermore, it highlights a relationship between nonmonotonicity and the relative performance of distributed addition (subtraction) in terms of their maximal violation of Bell inequality. Both the axes are dimensionless.

(2) For low to intermediate values of r and $k + l$, interestingly, we find that distributed subtraction gives more violation compared to distributed addition for some specific values of r and $k + l$.

(3) The traits of diminution and nonmonotonicity get completely washed away to monotonic enhancement of maximal violation for sufficiently high r or by increasing the total number of added or subtracted photons $k + l$ or both. In this parameter regime, distributed addition typically yields a higher violation compared to distributed subtraction.

(4) For distributed addition, the transition from nonmonotonicity and diminution to monotonic enhancement of maximal violation usually requires higher values of squeezing, r , or, total number of photons, $k + l$, compared to the distributed subtraction case.

The observations are in sharp contrast to the results obtained in the case of entanglement [32], where distribution always leads to monotonic enhancement of entanglement for both addition and subtraction. Furthermore, distributed addition is shown to ubiquitously outperform distributed subtraction in terms of the entanglement content (cf. Ref. [47]). As argued above, this is no more true in the case of violation of the Bell inequality. Moreover, a careful survey in the space of squeezing parameter and total number of photons added or subtracted indicates that the outcome of this duel (addition vs subtraction) has a one-to-one correspondence with monotonicity of maximal violation upon distribution of the added or subtracted photons. The general trend being when maximal

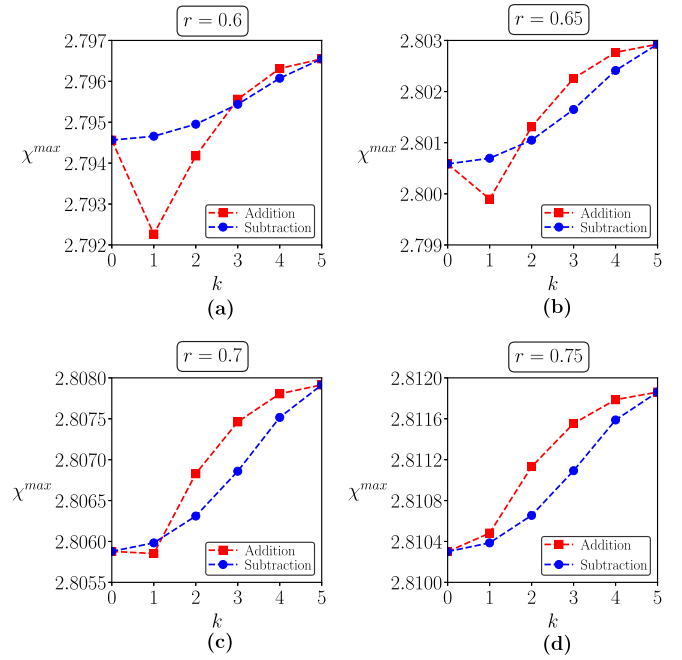


FIG. 4. Washing away of diminution and nonmonotonicity in the maximal violation with increasing squeezing parameter. The total number of photons added (subtracted) is always fixed to 10. Here, the abscissa, k , is the added number of photons in the first mode. Other details are the same as in Fig. 3.

violation for distributed addition shows nonmonotonicity or diminution, subtraction prevails, which as pointed out earlier occurs for low to intermediate values of r and $k + l$.

V. VIOLATIONS OF BELL INEQUALITY IN REALISTIC SITUATIONS

The cases considered so far are ideal, as the TMSV states were not reckoned to be tampered by any noise due to environmental interactions and the twin beam generator was assumed to be without any imperfections. However, in laboratories, the presence of noise and faulty machines are generic [38]. In this section, we address these issues, and focus on imperfect (noisy and faulty) scenarios which reduce the maximal violation of Bell inequality, and in some cases, even makes the system nonviolating. We show how even single-mode operations, namely addition or subtraction of photons, can enhance violation of Bell inequality in these scenarios and sometimes can even *activate* violation for states which ceased to violate Bell inequalities in the presence of noise or imperfections.

Here we consider two major sources of imperfections that can have detrimental effect on the maximal violation of Bell inequality: (1) We consider the case of a general local noise model and examine its effect on the violation of Bell inequality for TMSV states. We then analyze enhancement and/or activation of the violation via photon addition or subtraction, giving examples for specific cases of local thermal and Gaussian noise. We also repeat the same analysis for a classically correlated noise model. (2) We assume that there is a faulty twin beam generator, resulting a TMSV state with squeezing different than the desired one and perform the same

investigations, like effects on violation on local realism due to states with defects, as in the case of noisy states. (3) We consider the situation where the photon addition and subtraction procedures are themselves faulty because of features like dark counts [40,41] of the photodetectors employed during the photon addition and subtraction procedures.

A. Noise in states

We now look at the TMSV states, tampered by noise, and study their robustness against such mixing in terms of its ability to violate the Bell inequality based on the pseudospin operators. The violation is computed in two distinct scenarios: (i) when the probability with which the noise gets mixed with the TMSV state is known and (ii) when the information about the mixing probability is absent. In the first case, for a given p , the maximal violation of the Bell inequality is evaluated, while in the second one, the settings chosen for optimizing the violation of the Bell inequality is same as the one with vanishing p . In both the cases, we analyze the effects of photon addition and subtraction on the violation of Bell inequality.

B. Local noise

We consider a general local noise model, where the noisy state reads as

$$\rho = (1 - p)|\psi_r\rangle\langle\psi_r| + p\left(\sum_{n=0}^{\infty} \mu_n |n\rangle\langle n| \otimes \sum_{m=0}^{\infty} \nu_m |m\rangle\langle m|\right), \tag{32}$$

where $0 \leq p \leq 1$, $|\psi_r\rangle$ is the TMSV state with squeezing parameter r , and $\sum_{n=0}^{\infty} \mu_n = \sum_{m=0}^{\infty} \nu_m = 1$. The correlation function for ρ , following Eq. (9), in terms of the pseudospin operators is given by

$$E(\theta_a, \theta_b) = A\left(\cos \theta_a \cos \theta_b + \frac{B}{A} \sin \theta_a \sin \theta_b\right), \tag{33}$$

with

$$A = (1 - p) + p\left(\sum_{n=0}^{\infty} (-1)^n \mu_n\right)\left(\sum_{m=0}^{\infty} (-1)^m \nu_m\right), \tag{34}$$

$$B = (1 - p) \tanh 2r. \tag{35}$$

In practical situations, the knowledge of p , i.e., whether any error have acted or not, may be elusive. Therefore, two situations may arise: (i) the value of p is known and (ii) it is unknown. The maximum value of the Bell expression for the state ρ , when the mixing probability, p , is known, is given by [see Eqs. (14) and (15)]

$$\chi_p^{\max}(\rho) = 2\sqrt{A^2 + B^2}. \tag{35}$$

When the knowledge about p is absent, one might proceed with the optimal measurement setup for the TMSV state, $|\psi_r\rangle$, and calculate the violation. Bell expression for such a setting of the state given in Eq. (32) reads as

$$\chi_{\mathcal{R}}^{\max}(\rho) = 2\left(\frac{A + \mathcal{K}_{(0,0)}B}{\sqrt{1 + \mathcal{K}_{(0,0)}^2}}\right) = 2\left(\frac{A + B \tanh 2r}{\sqrt{1 + \tanh^2 2r}}\right). \tag{36}$$

When the value of p is known, we have $\chi_p^{\max}(\rho) > 2$, when

$$p < 1 - \frac{1}{a^2 + b^2} [a(a - 1) + \sqrt{a(a - ab^2 + 2b^2)}], \tag{37}$$

where $a = 1 - [\sum_{n=0}^{\infty} (-1)^n \mu_n][\sum_{m=0}^{\infty} (-1)^m \nu_m]$ and $b = \tanh 2r$. When the knowledge about the value of p is absent, then $\chi_{\mathcal{R}}^{\max}(\rho) > 2$, if

$$p < \frac{\sqrt{1 + b^2}(\sqrt{1 + b^2} - 1)}{2 + b^2 - a}. \tag{38}$$

We now explore the possibilities of enhancement and/or activation of the violation via addition (subtraction) of photons to one of the modes of such noisy states. We assume, without any loss of generality, that the single-mode operations are performed in the first mode. The normalized state when k photons are added in the first mode of ρ , given in Eq. (32), can be represented as

$$\tilde{\rho}_k = (1 - p)|\psi_r^{(k,0)}\rangle\langle\psi_r^{(k,0)}| + p\left(\sum_{n=0}^{\infty} \tilde{\mu}_n^k |n+k\rangle\langle n+k| \otimes \sum_{m=0}^{\infty} \nu_m |m\rangle\langle m|\right). \tag{39}$$

where

$$\tilde{\mu}_n^k = \frac{\mu_n \binom{n+k}{k}}{\sum_{t=0}^{\infty} \mu_t \binom{t+k}{k}}. \tag{40}$$

When k photons are subtracted from ρ , we have

$$\tilde{\rho}_{-k} = (1 - p)|\psi_r^{(-k,0)}\rangle\langle\psi_r^{(-k,0)}| + p\left(\sum_{n=0}^{\infty} \tilde{\mu}_n^{-k} |n\rangle\langle n| \otimes \sum_{m=0}^{\infty} \nu_m |m\rangle\langle m|\right), \tag{41}$$

$$= (1 - p)|\psi_r^{(0,k)}\rangle\langle\psi_r^{(0,k)}| + p\left(\sum_{n=0}^{\infty} \tilde{\mu}_n^{-k} |n\rangle\langle n| \otimes \sum_{m=0}^{\infty} \nu_m |m\rangle\langle m|\right)$$

with

$$\tilde{\mu}_n^{-k} = \frac{\mu_{n+k} \binom{n+k}{k}}{\sum_{t=0}^{\infty} \mu_{t+k} \binom{t+k}{k}}. \tag{42}$$

Here, the forms of $|\psi^{\pm k, \pm l}\rangle$ are given in Eqs. (2) and (4). The correlation functions corresponding to states in Eqs. (39) and (41) have the same structure as Eq. (34), and the corresponding maximal Bell inequality violation, when p is known, is given by

$$\chi_p^{\max}(\tilde{\rho}_{\pm k}) = 2\sqrt{A_{\pm k}^2 + B_{\pm k}^2}. \tag{43}$$

For addition of photons in the first mode, with the optimal (q_1, q_2) pair, A_{+k} and B_{+k} take the following forms,

$$A_{+k} = (1 - p) + p\left(\sum_{n=0}^{\infty} (-1)^n \tilde{\mu}_n^k\right)\left(\sum_{m=0}^{\infty} (-1)^m \nu_m\right), \tag{44}$$

$$B_{+k} = (1 - p)\mathcal{K}_{(k,0)},$$

and in the case of photon subtraction,

$$A_{-k} = (1-p) + (-1)^k p \left[\left(\sum_{n=0}^{\infty} (-1)^n \tilde{\mu}_n^{-k} \right) \times \left(\sum_{m=k \bmod 2}^{\infty} (-1)^m v_m \right) \right],$$

$$B_{-k} = (1-p) \mathcal{K}_{(-k,0)}. \quad (45)$$

When the knowledge about p is absent, the measurement settings which are optimal for the k photon-added TMSV state are employed. The maximal Bell expression for such a setting for the photon-added and photon-subtracted noisy state, $\tilde{\rho}_{\pm k}$, is as follows:

$$\chi_{\mathcal{R}}^{\max}(\tilde{\rho}_{\pm k}) = 2 \left(\frac{A_{\pm k} + \mathcal{K}_{(\pm k,0)} B_{\pm k}}{\sqrt{1 + \mathcal{K}_{(\pm k,0)}^2}} \right)$$

$$= 2 \left(\frac{A_{\pm k} + \mathcal{K}_{(k,0)} B_{\pm k}}{\sqrt{1 + \mathcal{K}_{(k,0)}^2}} \right). \quad (46)$$

Note that unlike in the noiseless scenario, in the presence of local noise, the maximal violations for single-mode addition and subtraction are structurally different [see Eqs. (44) and (45)]. We now consider two special cases where the local noises considered in Eq. (32) are thermal and Gaussian. In both these cases, the system ceases to violate the Bell inequality after a critical value of p , even when the value of p is known.

1. Local thermal noise

Let us first consider the scenario of local thermal noise. In this situation, the resulting state, $\rho^{\beta_1 \beta_2}$, is the admixture of TMSV state with the thermal noise having inverse temperatures, $\beta_1 = \frac{1}{k_B T_1}$ and $\beta_2 = \frac{1}{k_B T_2}$, for first and second modes respectively with k_B being the Boltzman constant and t_i , $i = 1, 2$ being the temperature of the i th mode. The local thermal noise parameters are given by

$$\mu_n = (1 - e^{-\beta_1}) e^{-\beta_1 n},$$

$$v_m = (1 - e^{-\beta_2}) e^{-\beta_2 m}. \quad (47)$$

For these choices of noise parameters, we obtain

$$A = (1-p) + p \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2}, \quad (48)$$

while B remains the same as in Eq. (34). The maximal violation, when the value of p is known, reduces to [see Eq. (35)]

$$\chi_p^{\max}(\rho^{\beta_1 \beta_2}) = 2 \sqrt{(1-p)^2 \tanh^2 2r + \left\{ (1-p) + p \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2} \right\}^2}.$$

$$(49)$$

The range of mixing probability, p , for which $\rho^{\beta_1 \beta_2}$ violates Bell inequality, is given in Eq. (37), where a and b are now as

follows:

$$a = 1 - \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2},$$

$$b = \tanh 2r. \quad (50)$$

When the value of p is unknown, following Eq. (36), the violation is given by

$$\chi_{\mathcal{R}}^{\max}(\rho^{\beta_1 \beta_2}) = \frac{2}{\sqrt{1 + \tanh^2 2r}} \left[(1-p)(1 + \tanh^2 2r) + p \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2} \right]. \quad (51)$$

Using Eqs. (38) and (50), we have $\chi_{\mathcal{R}}^{\max}(\rho^{\beta_1 \beta_2}) > 2$ when

$$p < \frac{\sqrt{1 + \tanh^2 2r} (\sqrt{1 + \tanh^2 2r} - 1)}{1 + \tanh^2 2r - \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2}}. \quad (52)$$

Clearly, the parameter space, in which violation of Bell inequality occurs in the p -unknown scenario, is smaller compared to the case when p is known. For example, if β_1, β_2 , and r are taken to be 3.5, and 1.25 respectively, we obtain violation for $p < 0.633$ when p is known and for $p < 0.526$ with p being unknown. The distinction becomes more pronounced in the low-temperature limit of the noise. When $\beta_1, \beta_2 \rightarrow \infty$, knowledge of p guarantees that the state keeps violating Bell inequality for all values of $p < 1$. On the contrary, when the knowledge of the value of p is absent, the state in the above limit violates Bell inequality only when $p < \frac{\sqrt{1 + \tanh^2 2r} (\sqrt{1 + \tanh^2 2r} - 1)}{\tanh^2 r}$. For the EPR state, this bound reduces to $p < 2 - \sqrt{2}$. On the other hand, in the high-temperature limit ($\beta_1, \beta_2 \rightarrow 0$), the violation becomes insensitive to the knowledge of p , and in both the cases, the state violates Bell inequality for $p < 1 - 1/\sqrt{1 + \tanh^2 2r}$, which reduces to $p < 1 - 1/\sqrt{2}$ for the EPR state. This is reminiscent of the result involving continuous-variable Werner state in Ref. [48]. Note here that although the properties of states with known p have been studied before, the situation when p is unknown, although very relevant, has hardly been investigated before.

Let us first analyze how the photon addition and subtraction processes affect the violation of Bell inequality when p is known. When k photons are added to these states, A_{+k} in Eq. (44) becomes

$$(1-p) + p \tanh^{k+1} \frac{\beta_1}{2} \tanh \frac{\beta_2}{2}, \quad (53)$$

while B_{+k} is same as given in Eq. (44). When an even number of photons are subtracted, k , we have $A_{-k} = A_{+k}$ and $B_{-k} = B_{+k}$, leading to the same χ^{\max} for photon-added and photon-subtracted states. However, for the case of subtracting an odd number of photons, B_{-k} remains same but $A_{-k} = (1-p) - p \tanh^{k+1} \frac{\beta_1}{2} \{ \tanh \frac{\beta_2}{2} - (1 - e^{-\beta_2}) \}$. The above expressions clearly indicate that in the presence of local thermal noise, addition and subtraction of photons are equivalent. However, when odd numbers of photons are involved, addition performs better than subtraction in terms of the maximal violation. Therefore, we restrict ourselves to single-mode

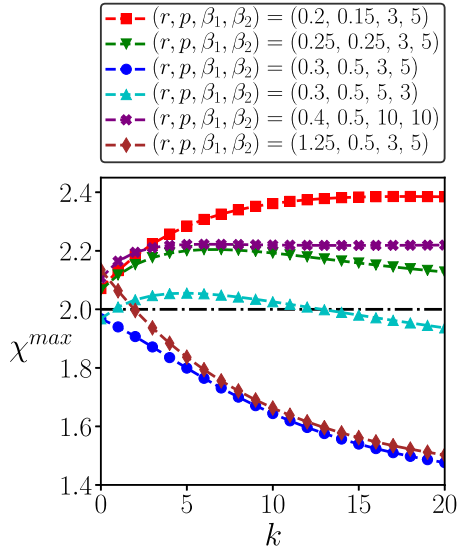


FIG. 5. Variation of the maximal violation of Bell inequality for photon-added TMSV states mixed with local thermal noise against number of added photons k . We choose different values of (r, p, β_1, β_2) to make the observation more prominent. Both axes are dimensionless.

operations only involving photon addition. Nevertheless, similar analysis can also be carried out for photon subtraction.

Note that when photons are added to $\rho^{\beta_1\beta_2}$, B_k ($\sim \mathcal{K}_{(k,0)}$) shows overall enhancement, while A_k monotonically decreases following the decrement of the term $\tanh^{k+1} \frac{\beta_1}{2}$ with k in its expression. Therefore, the maximal value of Bell expression, $\chi_p^{\max} = 2\sqrt{A_k^2 + B_k^2}$, is not guaranteed to increase after adding photons and is determined by the competing enhancement and decrement of B_k and A_k respectively. In Fig. 5, we plot χ_p^{\max} for various values of system parameters for known p , which encapsulates the following patterns:

(1) For low values of p , the noisy state is essentially close to the TMSV state, and therefore we get enhancement in the Bell expression on addition of photons (see the curve with $(r, p, \beta_1, \beta_2) = (0.2, 0.15, 3, 5)$ in Fig. 5).

(2) For low values of the squeezing parameter, r , the overall gain $\mathcal{G}(|\psi_r^{(k,0)}\rangle)$ on addition of photons to the TMSV state is large (see Table I). Now, in the low- to intermediate-

temperature regime, when p is small, the increase of $\mathcal{K}_{(k,0)}$ (due to the high gain) dominates, and therefore the overall violation increases and ultimately saturates for high values of k .

(3) There exists regions in the parameter space where we can have activation of violation of the Bell inequality, i.e., the state which is originally non-Bell-violating violates local realism after adding k photons. See the plot with $(r, p, \beta_1, \beta_2) = (0.3, 0.5, 5, 3)$ in Fig. 5. Also note that, in this situation, the value of the Bell expression initially increases with the number of added photons, but it starts decreasing after some time, as the decrement of the term $\tanh^{k+1} \frac{\beta_1}{2}$ in A_k becomes dominating.

(4) For high values of the squeezing parameter, r , the value of $\mathcal{K}_{(k,0)}$ does not change substantially. It is reflected in the low gain percentages for highly squeezed TMSV states in Table I. Therefore, the Bell expression decreases monotonically with k . Similarly, for high values of p , the Bell expression can decrease, as $\tanh^{k+1} \frac{\beta_1}{2}$ in A_k dominates.

For the case of unknown p , we observe qualitatively the same features as in the scenario for which p is known but with reduced values of the maximal violation.

2. Local Gaussian noise

We now admix the TMSV state with local Gaussian noise, denoted by $\rho^{\sigma_1\sigma_2}$ having coefficients

$$\begin{aligned} \mu_n &= \frac{2}{1 + \vartheta_3(0, e^{-\sigma_1^{-2}})} e^{-n^2/\sigma_1^2}, \\ \nu_n &= \frac{2}{1 + \vartheta_3(0, e^{-\sigma_2^{-2}})} e^{-n^2/\sigma_2^2} \end{aligned} \quad (54)$$

where σ_1 and σ_2 are the relevant noise parameters, and ϑ_n denotes the Jacobi θ function of order n [49]. In this case,

$$A = (1 - p) + p \frac{1 + \vartheta_4(0, e^{-\sigma_1^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_1^{-2}})} \frac{1 + \vartheta_4(0, e^{-\sigma_2^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_2^{-2}})}, \quad (55)$$

and B remains the same as in Eq. (34).

Like in the case of thermal noise, for a given p , the maximal violation of Bell inequality using Eq. (35), takes the form as

$$\chi_p^{\max}(\rho^{\sigma_1\sigma_2}) = 2 \sqrt{(1 - p)^2 \tanh^2 2r + \left\{ (1 - p) + p \frac{1 + \vartheta_4(0, e^{-\sigma_1^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_1^{-2}})} \frac{1 + \vartheta_4(0, e^{-\sigma_2^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_2^{-2}})} \right\}^2}. \quad (56)$$

In the case of local Gaussian noise, when p is unknown, the violation, following Eq. (36) reads as

$$\chi_R^{\max}(\rho^{\sigma_1\sigma_2}) = \frac{2}{\sqrt{1 + \tanh^2 2r}} \left[(1 - p)(1 + \tanh^2 2r) + p \frac{1 + \vartheta_4(0, e^{-\sigma_1^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_1^{-2}})} \frac{1 + \vartheta_4(0, e^{-\sigma_2^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_2^{-2}})} \right]. \quad (57)$$

In such a situation, $\chi_R^{\max}(\rho^{\sigma_1\sigma_2}) > 2$ holds for

$$p < \frac{\sqrt{1 + \tanh^2 2r}(\sqrt{1 + \tanh^2 2r} - 1)}{1 + \tanh^2 2r - \frac{1 + \vartheta_4(0, e^{-\sigma_1^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_1^{-2}})} \frac{1 + \vartheta_4(0, e^{-\sigma_2^{-2}})}{1 + \vartheta_3(0, e^{-\sigma_2^{-2}})}}. \quad (58)$$

C. Classically correlated noise

Instead of uncorrelated noise considered in Eq. (32), we now move to classically correlated local noise model and examine Bell inequality violations for these states. Such a

state can be represented as

$$\tilde{\rho} = (1-p)|\psi_r\rangle\langle\psi_r| + p \sum_{n=0}^{\infty} C_n |n, n\rangle\langle n, n|. \quad (59)$$

The correlation functions for $\tilde{\rho}$, from Eq. (9), in terms of the pseudospin operators are given by

$$E(\theta_a, \theta_b) = \cos \theta_a \cos \theta_b + \tilde{\mathcal{K}}_{(0,0)} \sin \theta_a \sin \theta_b, \quad (60)$$

with $\tilde{\mathcal{K}} = (1-p)\mathcal{K}_{(0,0)} = (1-p)\tanh 2r$. For known p , $\chi_p^{\max}(\tilde{\rho}) = 2\sqrt{1 + \tilde{\mathcal{K}}^2}$. In this situation, it is easy to see that $\chi_p^{\max} > 2$ for any values of $p < 1$, and for any finite values of the squeezing parameter, r .

On the other hand, the maximal Bell expression takes the form

$$\begin{aligned} \chi_{\mathcal{R}}^{\max}(\tilde{\rho}) &= 2 \left(\frac{1 + \mathcal{K}_{(0,0)}\tilde{\mathcal{K}}_{(0,0)}}{\sqrt{1 + \mathcal{K}_{(0,0)}^2}} \right) \\ &= 2 \left(\frac{1 + (1-p)\tanh^2 2r}{\sqrt{1 + \tanh^2 2r}} \right) \end{aligned} \quad (61)$$

with unknown p . Under this assumption about the uncertainty in the error estimation and detection, we observe criticalities in the values of r and p , beyond which the system ceases to violate the Bell inequality based on pseudospin operators. For $r \rightarrow 0$, we find that $\chi_{\mathcal{R}}^{\max}(\tilde{\rho}) \leq 2$ for $p \geq 1/2$. Therefore, for $p < 1/2$, the noisy state violates the Bell inequality for any finite squeezing, even when the value of p is not known. However, if $p > 1/2$, the state given in Eq. (59) starts violating the Bell inequality only when

$$2r \geq \tanh^{-1} \frac{\sqrt{1 - 2(1-p)}}{1-p}. \quad (62)$$

Note that if $p \geq 2 - \sqrt{2}$, even the EPR state, i.e., the TMSV state with $r \rightarrow \infty$, does not violate a Bell inequality in this setting. So we get a criticality in the squeezing parameter, r , given by the above equation, when $1/2 \leq p \leq 2 - \sqrt{2}$.

When we add or subtract photons to a single mode of the state given in Eq. (59), we have $\tilde{\mathcal{K}}_{(\pm k, 0)} = (1-p)\mathcal{K}_{(k, 0)} = (1-p)\mathcal{K}_{(-k, 0)} = \tilde{\mathcal{K}}_{(k, 0)}$. Now, if p is known, the maximal violation of Bell inequality simply reads

$$\chi_p^{\max}(\tilde{\rho}_{\pm k}) = 2\sqrt{1 + \tilde{\mathcal{K}}_{(k, 0)}^2}. \quad (63)$$

From the above expression, it is clear that photon addition (subtraction) always leads to an overall enhancement in the violation of Bell inequality, which will be dictated by the change in $\mathcal{K}_{(k, 0)}$ with respect to k . In the absence of any knowledge about p , the violation is given by

$$\chi_{\mathcal{R}}^{\max}(\tilde{\rho}_{\pm k}) = 2 \left(\frac{1 + \mathcal{K}_{(k, 0)}\tilde{\mathcal{K}}_{(k, 0)}}{\sqrt{1 + \mathcal{K}_{(k, 0)}^2}} \right). \quad (64)$$

Again, upon addition (subtraction) of photons in one mode, the above expression can be increased. Specifically, for $1/2 \leq p \leq 2 - \sqrt{2}$ and $r < \tanh^{-1} [\sqrt{1 - 2(1-p)}/(1-p)]$, the violation can be *activated* via photon addition or subtraction in a single mode.

Interestingly, note that for any noise with the same structure ($\sum C_n |n, n\rangle\langle n, n|$) as given in Eq. (59), the Bell expressions for known or unknown values of p do not depend of the values of C_n .

D. Faulty twin beam generator

Up to now, we consider the scenario where the state is affected by noise. There can be a situation where the twin beam generator is typically imperfect, and as a result of internal imperfection and losses, it may end up in generating TMSV states with less squeezing than it is ought to. To put things in a quantitative perspective, we assume that a twin beam generator which is labeled to produce states with squeezing r does so with an unknown r' . Of course, $r' < r$. The correlators are calculated via measurements performed with pseudospin operators oriented in the optimal direction for the TMSV state with squeezing parameter r . The maximal violation obtained in such a situation, following Eqs. (14) and (15), is given by

$$\begin{aligned} \chi_r^{\max}(|\psi^{r'}\rangle) &= 2(\cos \theta + \tanh 2r' \sin \theta), \\ &= 2 \frac{1 + \tanh 2r' \tanh 2r}{\sqrt{1 + \tanh^2 2r}}. \end{aligned} \quad (65)$$

We have $\chi_r^{\max}(|\psi^{r'}\rangle) \leq 2$ when

$$\tanh 2r' \leq \frac{1}{\tanh 2r} (\sqrt{1 + \tanh^2 2r} - 1). \quad (66)$$

The equality holds when $r' = r_c$, where r_c is the critical value of r' for a given r , below which the state fails to show any violation. For the EPR state, the critical value of r' saturates to a finite value $r_c^\infty = \frac{1}{2} \operatorname{arctanh}(\sqrt{2} - 1) \approx 0.22$. Although r_c might seem to be a small value even for the EPR state, for experimentally relevant squeezing parameters, r_c is comparable to r . For example, r_c for $r = 0.75$ approximately reads 0.203172. We want to analyze the effects of adding or subtracting photons from a single mode when r' falls below r_c , i.e., when the state does not violate Bell inequality based on pseudospin operators.

Let us check whether the range of squeezing parameter which shows nonviolation can be changed if one adds (subtracts) photons even in a single mode. In this case, for single-mode operations, photon addition remains equivalent to subtraction, since the fault in the generator just reduces the squeezing parameter of the TMSV state, and thus the equivalence argument goes through. So we add k photons to the first mode of the obtained squeezed state with squeezing r' . As before, we use the optimal measurement settings for k photon-added TMSV state with squeezing parameter r . Note that r is the labeled value of squeezing that the twin beam generator is intended to produce. The maximal violation, so obtained in this scenario, is given by

$$\chi_r^{\max}(|\psi_{r'}^{(\pm k, 0)}\rangle) = 2 \left(\frac{1 + \mathcal{K}_{(k, 0)}^{r'} \mathcal{K}_{(k, 0)}}{\sqrt{1 + \mathcal{K}_{(k, 0)}^2}} \right), \quad (67)$$

where $\mathcal{K}_{(k, 0)}$ is in Eq. (17), and is calculated for the TMSV state with squeezing parameter r , while $\mathcal{K}_{(k, 0)}^{r'}$ represents the same thing for the state $|\psi_{r'}^{(k, 0)}\rangle$. We consider some representative states for which r' falls below the critical value as

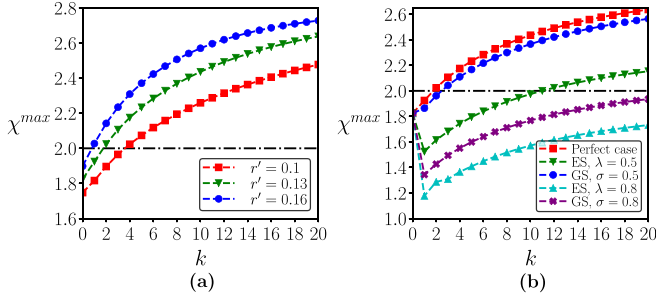


FIG. 6. Activation of the violation of pseudospin operator-based Bell inequality. We choose $r = 0.75$, such that the corresponding critical value of r' is $r_c = 0.203172$. (a) Activation via deterministic addition (subtraction) of photons in single mode for three values of $r' < r_c$. (b) The dual effects of imperfect addition (subtraction) of photons in single mode and faulty TMSV state for $r' = 0.13$, and for different values of λ and σ (see discussions in Sec. VE). Both the axes are dimensionless.

given in Eq. (66). We show that it is possible to *activate* the violation for these states by using single mode operations. See Fig. 6(a).

In the entire analysis, we have assumed that the addition or subtraction process is error free. In the next subsection, the same situations will be re-examined, when the photon addition or subtraction process is itself imperfect.

E. Imperfections in photon addition and subtraction mechanism

The indeterminacy in addition (subtraction) of photons can be attributed to variety of reasons, like dark counts [40,41] of the detector, etc., and hence can lead to decrement in violation of local realism. In this subsection, we consider two distinct models of imperfections in the added or subtracted number of photons. First, for a given k number of added (subtracted) photons, we assume that the state is to be mixed with $k - 1, k - 2, \dots, k - m$ ($m \leq k$) number of photon-added (photon-subtracted) states with probabilities which follows the exponential suppression (ES). Hence, such

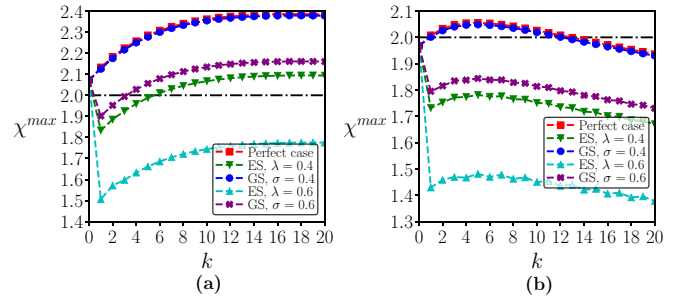


FIG. 7. Effects of imperfect photon addition process on TMSV states with local thermal noise, (a) with $(r, p, \beta_1, \beta_2) = (0.2, 0.15, 3, 5)$ and (b) with $(r, p, \beta_1, \beta_2) = (0.3, 0.5, 5, 3)$. See Fig. 5 for the perfect photon addition case for these choices of system-noise parameters. We chose these two values to highlight the consequences of imperfect photon addition on situations of enhancement (a) and activation (b) of χ^{\max} . Both the axes are dimensionless.

that the effective state becomes

$$\bar{\rho}_{\pm k} = \sum_{i=0}^m p_i \rho_{\pm|k-i|}. \quad (68)$$

Here, m is the cutoff on the maximal discrepancy in the photon number during the addition (subtraction) procedure, $\rho_{\pm l}$ represents a state with l number of added (subtracted) photons, and p_i are the mixing probabilities, which decrease according to exponential law. The second scenario considered in this paper is where the probabilities are Gaussian. Specifically, the exponential probabilities, for a given m , are given by

$$p_i = \frac{e^{-i/\lambda}}{\sum_{i=0}^m e^{-i/\lambda}}, \quad (69)$$

whereas for Gaussian suppression (GS), the probability takes the form as

$$p_i = \frac{e^{-i^2/\sigma^2}}{\sum_{i=0}^m e^{-i^2/\sigma^2}}. \quad (70)$$

Here, λ and σ give the measures of dispersion for these imperfect additions (subtractions).

1. Noisy states

In Sec. VA, we have discussed the effects of noise on the violation of Bell inequality for the TMSV states and the role of photon addition and subtraction to improve the situation. Specifically, we have discussed the cases of local noises (thermal and Gaussian) and a classically correlated noise. In this subsection, we study the effects of faulty addition (subtraction) of photons on the Bell expression, when the noise probability, p , is known.

When the photon addition scheme on TMSV states with local noise suffers exponential suppression, and when we know the value of p , the violation of Bell inequality is given by

$$\chi_{ES}^{\max} = \frac{2(\sum_{n=0}^m e^{-n/\lambda})^{-1}}{\sqrt{A_k^2 + B_k^2}} \left[\sum_{i=0,2,4,\dots \leq m} e^{-i/\lambda} (A_k A_{k-i} + B_k B_{k-i}) - \sum_{j=1,3,5,\dots \leq m} e^{-j/\lambda} A_k A_{k-j} \right]. \quad (71)$$

The corresponding violation for GS reads as

$$\chi_{GS}^{\max} = \frac{2(\sum_{n=0}^m e^{-n^2/\sigma^2})^{-1}}{\sqrt{A_k^2 + B_k^2}} \left[\sum_{i=0,2,4,\dots \leq m} e^{-i^2/\sigma^2} (A_k A_{k-i} + B_k B_{k-i}) - \sum_{j=1,3,5,\dots \leq m} e^{-j^2/\sigma^2} A_k A_{k-j} \right]. \quad (72)$$

Here, A_k and B_k are given in Eqs. (44). In this imperfect addition scenario, for low values of noise parameters, the enhancement [Fig. 7(a)] and activation [Fig. 7(b)] in the maximal violation persists, with lower values compared to the perfect addition scenario (see Fig. 5). The domain of activation also naturally shrinks in this imperfect case, see Fig. 7.

2. Faulty twin beam generator

We now study the response on violation of Bell inequality under coupled imperfect scenario. In particular, along with imperfect photon addition, the twin beam generator produces the TMSV state with r' , instead of r . In the case of ES,

$$\chi_{ES}^{\max} = \frac{2(\sum_{n=0}^m e^{-n/\lambda})^{-1}}{\sqrt{1 + \mathcal{K}_{(k,0)}^2}} \left[\sum_{i=0,2,4,\dots \leq m} e^{-i/\lambda} (1 + \mathcal{K}_{(k-i,0)}^{r'}) \mathcal{K}_{(k,0)} - \sum_{j=1,3,5,\dots \leq m} e^{-j/\lambda} \right], \quad (73)$$

while for GS, we get

$$\chi_{GS}^{\max} = \frac{2(\sum_{n=0}^m e^{-n^2/\sigma^2})^{-1}}{\sqrt{1 + \mathcal{K}_{(k,0)}^2}} \left[\sum_{i=0,2,4,\dots \leq m} e^{-i^2/\sigma^2} (1 + \mathcal{K}_{(k-i,0)}^{r'}) \mathcal{K}_{(k,0)} - \sum_{j=1,3,5,\dots \leq m} e^{-j^2/\sigma^2} \right]. \quad (74)$$

To take one concrete example, we restrict m to be equal to k , choose different values of λ and σ , and examine the consequence of faulty photon addition procedure on the Bell expression [see Fig. 6(b)]. We observe that for low values of dispersions (λ and σ), the Bell expression, which initially does not violate, always increases with varying numbers of added photons, k , leading to activation of the violation. However, if the dispersions are large, in both exponential and Gaussian cases, the Bell expression initially shows a decrement in its value and can finally be enhanced or activated after adding sufficiently high number of photons. There can also exist scenarios where this activation is not possible at all, even after adding a large number of photons [see Fig. 6(b)].

VI. CONCLUSION

Violation of Bell inequalities by quantum systems establishes the existence of correlations beyond the classical ones. For finite-dimensional quantum systems, violation of Bell inequalities have been studied more thoroughly in comparison to the same for continuous-variable (infinite-dimensional) systems. In the field of continuous-variable systems, there was a long-standing debate, started by John Bell, as to whether states with positive Wigner functions would violate a Bell inequality. It was resolved conclusively by constructing Bell expressions using parity operators, later using pseudospin operators, and demonstrating violation for certain entangled states with positive Wigner function. In this paper, we used pseudospin operators to examine the violation of Bell inequality for photon-added and photon-subtracted two-mode squeezed vacuum (TMSV) states, where addition and subtraction are performed either in a single mode or in both modes. We found that unlike entanglement, the amount of violation of pseudospin operator-based Bell inequality by photon-subtracted state can be higher than that of the photon-added ones.

We have further studied the effects of local noise (specifically, local thermal and local Gaussian noise) on the maximal violation of Bell inequality for the TMSV states and computed the parameter ranges for which the noisy TMSV state abstains from violating the considered Bell inequality. We demonstrated that under such circumstances, single-mode operations like photon addition can *activate* violation. We repeated the same drill of investigations with the goal of activation in the

case of a faulty twin-beam generator for generating TMSV states and imperfections in photon addition or subtraction process. We reported here that in both the scenarios, the answer is affirmative, i.e., the activation is possible, thereby transforming nonviolating states to violating ones.

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APPENDIX: MAXIMIZATION OF BELL EXPRESSION

Let us now discuss the method for obtaining χ^{\max} , by performing maximization over the settings, i.e., $\theta_a, \theta_b, \theta'_a, \theta'_b$. The correlation matrix, $T_{ij} = \langle S_{q_1}^i \otimes S_{q_2}^j \rangle$, where $i, j = x, y, z$ and q_1 and q_2 are chosen appropriately depending on the structure of the state. S_q^x and S_q^y are simply given by $S_q^+ + S_q^-$ and $-i(S_q^+ - S_q^-)$ respectively, where S_q^+ and S_q^- are given in Eq. (8). The T (correlation) matrix for photon-added (subtracted) TMSV states can be expressed as $T = \text{diag}(\mathcal{K}, -\mathcal{K}, 1)$. The two highest eigenvalues of the matrix $T^\dagger T$ are 1 and \mathcal{K}^2 respectively. Now, following the argument as given in Ref. [43], the maximal violation of Bell inequality is given by $2 \times \sqrt{\mathcal{M}(T^\dagger T)}$, where $\mathcal{M}(T^\dagger T)$ represents the sum of the two largest eigenvalues of $T^\dagger T$. The same in this case reads $2\sqrt{1 + \mathcal{K}^2}$. This completes the proof of the expressions in Eqs. (15) and (35).

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