

Control of spin-exchange interaction between alkali-earth-metal atoms via confinement-induced resonances in a quasi-(1+0)-dimensional system

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(Received 1 May 2018; published 29 October 2018)

A nuclear-spin-exchange interaction exists between two ultracold fermionic alkali-earth-metal-like atoms in the electronic 1S_0 state (g state) and 3P_0 state (e state) and is an essential ingredient for the quantum simulation of the Kondo effect. We study the control of this spin-exchange interaction for two atoms simultaneously confined in a quasi-one-dimensional (quasi-1D) tube, where the g atom is freely moving in the axial direction while the e atom is further localized by an additional axial trap and behaves as a quasi-0D impurity. In this system, the two atoms experience effective 1D spin-exchange interactions in both even and odd partial-wave channels, whose intensities can be controlled by the characteristic lengths of the confinements via confinement-induced resonances (CIRs). In a previous work, we studied this problem with a simplified pure 1D model [Y. Cheng *et al.*, *Phys. Rev. A* **96**, 063605 (2017)]. In the present work, we go beyond that pure 1D approximation. We model the transverse and axial confinements by harmonic traps with finite characteristic lengths $a_\perp$ and a_z , respectively, and exactly solve the quasi-1D–quasi-0D scattering problem between these two atoms. Using the solutions, we derive the effective 1D spin-exchange interaction and investigate the locations and widths of the even-odd wave CIRs for our system. It is found that when the ratio $a_z/a_\perp$ is larger, the CIRs can be induced by weaker confinements, which are easier to be realized experimentally. The comparison between our results and a recent experiment by L. Riegger *et al.* [*Phys. Rev. Lett.* **120**, 143601 (2018)] shows that the two experimentally observed resonance branches of the spin-exchange effect are due to an even-wave CIR and an odd-wave CIR, respectively. Our results are advantageous for the control and description of either the effective spin-exchange interaction or other types of interactions between ultracold atoms in quasi-(1+0)-dimensional systems.

DOI: [10.1103/PhysRevA.98.043627](https://doi.org/10.1103/PhysRevA.98.043627)

I. INTRODUCTION

In recent years the ultracold gases of alkali-earth-metal-like atoms, e.g., Ca, Sr, and Yb, have attracted much attention [1–16]. One important application of this system is the quantum simulation for the Kondo effect [17] which is induced by the spin-exchange between localized impurities and itinerant fermions [18–24]. The following two features of alkali-earth-metal-like atoms play a critical role in this quantum simulation.

(i) An alkali-earth-metal-like atom has not only a stable electronic orbital ground state, i.e., the 1S_0 state (g state), but also a very long-lived electronic orbital excited state, i.e., the 3P_0 state (e state) [Fig. 1(a)]. These two states have different ac polarizabilities except for the lasers with a magic wavelength [25,26]. Therefore, in experiments, one can realize either the same or different trapping potentials for the atom in the g state and e state.

(ii) There exists a spin-exchange interaction between two homonuclear fermionic alkali-earth-metal-like atoms in the e and g states. As a result, these two atoms can exchange their

nuclear-spin states during collision, i.e., the process

$$|e, \uparrow\rangle|g, \downarrow\rangle \rightleftharpoons |e, \downarrow\rangle|g, \uparrow\rangle \quad (1)$$

can occur.

Benefiting from these two properties, one can simulate the Kondo effect with ultracold alkali-earth-metal-like atoms in an optical lattice which is very deep for the atoms in the e state (e atoms) and very shallow for the atoms in the g state (g atoms). In that system the e atoms are localized as impurities and the g atoms remain itinerant [Fig. 1(b)].

Nevertheless, to perform this quantum simulation the enhancement of the intensity of the spin-exchange interaction between the g and e atoms is required so that the Kondo temperature can be high enough and thus attainable by current cooling capabilities. In previous works [22,23], we and our co-workers proposed to solve this problem by confinement-induced resonance (CIR). As shown in Fig. 1(c), in this scheme both the g atoms and the e atoms are confined in a quasi-one-dimensional (quasi-1D) confinement with the same characteristic length $a_\perp$, which is generated by laser beams with a magic wavelength. In addition, there is also a confinement along the axial direction of the quasi-1D tube, which can only be experienced by the e atoms and has characteristic length a_z . One can tune $a_\perp$ and a_z by changing the intensities

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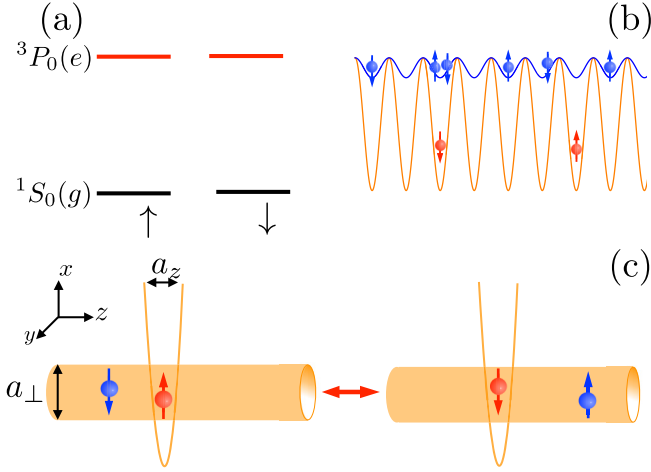


FIG. 1. (a) Energy levels of a fermionic alkali-earth-metal-like atom. Whether the atom is in the electronic orbit 1S_0 state (g state) or the 3P_0 state (e state), the nuclear spin could always be either $\uparrow$ or $\downarrow$. (b) Deep lattice for the e -atom weak lattice for g atoms created by the same standing-wave laser. (c) Quasi-(1+0)-dimensional system for the quantum simulation of the Kondo effect. The spin-exchange process occurs during the scattering between g atoms (itinerant fermions) and localized e atom (impurity). In (b) and (c) the blue and red balls denote the g atom and e atom, respectively, and the arrows denote the nuclear spins.

of the optical lattices. As a result, the g atoms are freely moving in the quasi-1D tube, while the e atoms are localized as quasi-0D impurities. Here we assume that in each axial confinement there is one e atom. In this system the g atom and e atom experience an effective 1D spin-exchange interaction. The strength Ω of this effective interaction is determined by the scattering amplitude between these two atoms, which is a function of $a_\perp$ and a_z . Confinement-induced resonance is the scattering resonance which occurs when $a_\perp$ and a_z are tuned to some specific values. At a CIR point Ω can be resonantly enhanced. In addition, when the system is near a CIR, one can efficiently control Ω by tuning a_z and $a_\perp$.

Our results in Refs. [22,23] are qualitatively consistent with the recent experiment by Riegger *et al.* [24], where the quasi-(1+0)-dimensional system is realized with ultracold ^{173}Yb atoms and the resonant control of the spin-exchange strength via the CIRs is demonstrated. In contrast, for simplicity, some approximations are implemented in our works in Refs. [22,23]. In Ref. [22] we investigate the control of the effective-1D spin-exchange interaction strength by tuning the quasi-1D confinement (transverse confinement). Thus, we approximate this strength as the one for the systems where all atoms are freely moving in the quasi-1D tube, i.e., the axial trap is ignored in our two-body calculations. In Ref. [23] we focus on the effect induced by the axial trap for the e atom. Accordingly, we ignore the transverse degree of freedom and use a pure-1D model which only describes the axial motion. These approximations are reasonable for the cases where the characteristic lengths of the transverse and axial confinements, i.e., $a_\perp$ and a_z , are very different from each other.

However, in realistic systems $a_\perp$ and a_z are generally of the same order. In this case, the cross effect of the transverse and axial confinements can be important. Thus, we should go beyond the above approximations and explicitly take into account both of the two confinements in the theoretical calculation.

In this paper we perform such a complete calculation. In our model the transverse and axial confinements are described by harmonic potentials with finite characteristic lengths $a_\perp$ and a_z , respectively. We exactly solved the quasi-1D–quasi-0D scattering problem between a freely moving g atom and a trapped e atom. In this system, there are two partial-wave scattering channels, i.e., the even-wave channel and the odd-wave channels. We derive the scattering amplitude for both of these partial waves, as well as the effective 1D interaction between these two atoms. Using these results, we investigate the even- and odd-wave CIRs in our systems. It is found that when the ratio $a_z/a_\perp$ is larger, the CIRs can occur in weaker confinements (i.e., the confinements with lower trapping frequencies), which are easier to realize in experiments.

We further compare our results with the recent experimental observations shown in Ref. [24]. The authors of Ref. [24] have done a theoretical calculation based on a two-site model, where the coupling between the center-of-mass motion and relative motion of two atoms in the same site is ignored. Here we use our model to explore the location and widths of the CIRs in this experiment. As shown above, in our model the optical-lattice-induced confinement potentials are approximated as harmonic potentials. In addition, in the experiments, the g atoms also experience a shallow lattice potential in the axial direction, and in our current calculation, we ignore this shallow lattice. Nevertheless, even with these two simplifications, our results are still consistent with the experiment. In particular, our results reveal that one resonance branch of the spin-exchange effect observed in the experiment is due to an even-wave CIR, while the other one is due to an odd-wave CIR. Thus, the effective spin-exchange interactions for these two resonance branches are quite different from each other.

Our results are valuable for the quantum simulation of Kondo effect with alkali-earth-metal-like atoms. Furthermore, our exact solution for the quasi-1D–quasi-0D scattering problem can also be applied to other problems of quasi-1D ultracold gases with localized impurities, e.g., the realization of a high-precision magnetometer with such a system [27,28].

The remainder of this paper is organized as follows. In Sec. II we describe the detail of our model and the approach of our calculation. In Sec. III we illustrate our results and investigate the CIR effects for our system; we also compare our results with the experimental results. A summary and discussion are given in Sec. IV. In the Appendixes we present details of our calculations.

II. EFFECTIVE 1D INTERACTION

A. Model

As shown above, we consider two ultracold fermionic alkali-earth-metal-like atoms of the same species, with one atom in the electronic g state and the other one in the

e state. In our problem the electronic states e and g can be used as the labels of the two atoms, i.e., the e atom and g atom behave as two distinguishable particles. Accordingly, the nuclear-spin states of the g (e) atom can be denoted by $|\uparrow\rangle_{g(e)}$ and $|\downarrow\rangle_{g(e)}$. Here we consider the case with zero magnetic fields, i.e., $B = 0$. We assume that the two atoms are tightly confined in a two-dimensional isotropic harmonic trap in the x - y plane [Fig. 1(c)], which is formed by laser beams with a magic wavelength and thus has the same intensity for both atoms. In addition, there is also an axial harmonic trap in the z direction, which is only experienced by the e atom. We further define $\mathbf{r} \equiv (x_r, y_r, z_r)$ as the relative coordinate of these two atoms and $z_{g(e)}$ as the z coordinate of the g (e) atom, which satisfy $z_r = z_g - z_e$.

The Hamiltonian for the two-body problem is given by

$$H = H_0 + V, \quad (2)$$

with H_0 and V the free Hamiltonian and the interatomic interaction in 3D space, respectively. Furthermore, in the x - y plane the relative motion of the two atoms can be decoupled from the center-of-mass motion. Therefore, the free Hamiltonian H_0 can be expressed as ($m = \hbar = 1$, with m the single-atom mass)

$$H_0 = -\frac{1}{2} \frac{\partial^2}{\partial z_g^2} + H_\perp + H_e, \quad (3)$$

with

$$H_\perp = -\frac{\partial^2}{\partial x_r^2} - \frac{\partial^2}{\partial y_r^2} + \frac{\omega_\perp^2}{4} (x_r^2 + y_r^2), \quad (4)$$

$$H_e = -\frac{1}{2} \frac{\partial^2}{\partial z_e^2} + \frac{\omega_z^2}{2} z_e^2, \quad (5)$$

where $\omega_\perp$ and ω_z are the frequency of the transverse and axial confinements, respectively. They are related to the characteristic lengths $a_\perp$ and a_z via

$$a_\perp = \sqrt{\frac{2}{\omega_\perp}}, \quad a_z = \sqrt{\frac{1}{\omega_z}}. \quad (6)$$

In addition, for our system the interatomic interaction V is diagonal in the basis of nuclear-spin singlet and triplet states

$$|+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_g |\downarrow\rangle_e - |\downarrow\rangle_g |\uparrow\rangle_e), \quad (7)$$

$$|-, 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_g |\downarrow\rangle_e + |\downarrow\rangle_g |\uparrow\rangle_e), \quad (8)$$

$$|-, +1\rangle = |\uparrow\rangle_g |\uparrow\rangle_e, \quad (9)$$

$$|-, -1\rangle = |\downarrow\rangle_g |\downarrow\rangle_e \quad (10)$$

and can be expressed as

$$V = V_+ \mathcal{P}_+ + V_- \mathcal{P}_-, \quad (11)$$

where

$$\mathcal{P}_+ = |+\rangle\langle+|, \quad \mathcal{P}_- = \sum_{q=0,\pm 1} |-, q\rangle\langle-, q|. \quad (12)$$

Here V_+ and V_- are the interaction potential in the channels of nuclear-spin singlet and triplet states, respectively. They can

be modeled by the Huang-Yang pseudopotential

$$V_\pm = 4\pi a_\pm \delta(\mathbf{r}) \frac{\partial}{\partial r} (r \cdot), \quad (13)$$

where $r = |\mathbf{r}|$ and $a_\pm$ are the corresponding s -wave scattering lengths. For a certain type of alkali-earth-metal-like atom, the two scattering lengths a_+ and a_- are usually different. For instance, for ^{173}Yb atoms we have $a_+ \approx 1878a_0$ and $a_- \approx 216a_0$, with a_0 the Bohr radius [13]. On the other hand, Eq. (11) directly yields that ${}_g\langle\downarrow|{}_e\langle\uparrow|V|\uparrow\rangle_g|\downarrow\rangle_e \propto a_+ - a_-$. Thus, the strength of the spin-exchange interaction in 3D space is proportional to $a_+ - a_-$.

In this work we consider the cases in which the temperature is much lower than $\omega_\perp/k_B$ and ω_z/k_B , with k_B the Boltzmann constant. In these cases, when the two atoms are far away from each other, the relative motion in the x - y plane and the axial motion of the e atom in the z direction are frozen in the ground states of the corresponding harmonic confinements. As a result, our system can be effectively described by a simple model where the g atom and e atom are spin- $\frac{1}{2}$ particles, which are freely moving in the pure-1D space and fixed at $z = 0$, respectively. The effective Hamiltonian of this pure 1D model can be expressed as

$$H_{\text{eff}} = -\frac{1}{2} \frac{\partial^2}{\partial z_g^2} + V_+^{(\text{eff})} \mathcal{P}_+ + V_-^{(\text{eff})} \mathcal{P}_-, \quad (14)$$

where $V_{+(-)}^{(\text{eff})}$ is the effective potential for the nuclear-spin singlet or triplet state. In the 1D scattering problem between the freely moving g atom and the fixed e atom, there are two partial-wave scattering channels, i.e., the even wave and the odd wave. As a result, $V_\xi^{(\text{eff})}$ ($\xi = +, -$) can be expressed as the summation of the 1D zero-range pseudopotentials for these two partial waves. Explicitly, we have [29]

$$V_\xi^{(\text{eff})} = g_\xi^{(\text{even})} \delta(z_g) \hat{d}_e + g_\xi^{(\text{odd})} \delta'(z_g) \hat{d}_o \quad \text{for } \xi = +, -. \quad (15)$$

Here $\delta(z_g)$ is the Dirac delta function, $\delta'(z_g) = \frac{d\delta(z_g)}{dz_g}$, and the operators $\hat{d}_e$ and $\hat{d}_o$ are defined as

$$\hat{d}_e \psi(z_g) \equiv \frac{1}{2} [\psi(z_g)|_{z_g=0^+} + \psi(z_g)|_{z_g=0^-}], \quad (16)$$

$$\hat{d}_o \psi(z_g) \equiv \frac{1}{2} \left[\frac{d}{dz_g} \psi(z_g) \Big|_{z_g=0^+} + \frac{d}{dz_g} \psi(z_g) \Big|_{z_g=0^-} \right]. \quad (17)$$

The operators $\hat{d}_e$ and $\hat{d}_o$ are essentially the projection operators to the even and odd partial-wave channels, respectively, and $g_\xi^{(\text{even})} \delta(z_g) \hat{d}_e$ and $g_\xi^{(\text{odd})} \delta'(z_g) \hat{d}_o$ are the 1D even- and odd-wave pseudopotentials, respectively [29].

Furthermore, the complete effective potential $V_+^{(\text{eff})} \mathcal{P}_+ + V_-^{(\text{eff})} \mathcal{P}_-$ is required to reproduce the correct low-energy scattering amplitude between the freely moving g atom and the e atom in the ground state of the axial trap. Comparing Eqs. (14) and (15) with Eqs. (2) and (11), one can find that this means that the low-energy scattering amplitude for the Hamiltonian $-\frac{1}{2} \frac{\partial^2}{\partial z_g^2} + g_\xi^{(\text{even})} \delta(z_g) \hat{d}_e + g_\xi^{(\text{odd})} \delta'(z_g) \hat{d}_o$ should approximately equal the one for $H_0 + V_\xi$ ($\xi = +, -$). This requirement determines the value of the intensities $g_\pm^{(\text{even})}$ and $g_\pm^{(\text{odd})}$.

On the other hand, the complete effective interaction $V_+^{(\text{eff})}\mathcal{P}_+ + V_-^{(\text{eff})}\mathcal{P}_-$ can be rewritten as

$$V_+^{(\text{eff})}\mathcal{P}_+ + V_-^{(\text{eff})}\mathcal{P}_- = \Lambda(z_g) + \Omega(z_g)\left[\frac{1}{2}\sigma_z^{(g)}\sigma_z^{(e)} + \sigma_+^{(g)}\sigma_-^{(e)} + \sigma_-^{(g)}\sigma_+^{(e)}\right], \quad (18)$$

where $\sigma_z^{(j)} = |\uparrow\rangle_j\langle\uparrow| - |\downarrow\rangle_j\langle\downarrow|$, $\sigma_+^{(j)} = |\uparrow\rangle_j\langle\downarrow|$, and $\sigma_-^{(j)} = \sigma_+^{(j)\dagger}$ ($j = e, g$) are the Pauli operators for the j atom and $\Lambda(z_g)$ and $\Omega(z_g)$ are defined as

$$\Lambda(z_g) = \Lambda^{(\text{even})}\delta(z_g)\hat{d}_e + \Lambda^{(\text{odd})}\delta'(z_g)\hat{d}_0, \quad (19)$$

$$\Omega(z_g) = \Omega^{(\text{even})}\delta(z_g)\hat{d}_e + \Omega^{(\text{odd})}\delta'(z_g)\hat{d}_0, \quad (20)$$

with

$$\Lambda^{[\text{even}(\text{odd})]} = \frac{3}{4}g_-^{[\text{even}(\text{odd})]} + \frac{1}{4}g_+^{[\text{even}(\text{odd})]}, \quad (21)$$

$$\Omega^{[\text{even}(\text{odd})]} = \frac{1}{2}[g_-^{[\text{even}(\text{odd})]} - g_+^{[\text{even}(\text{odd})]}]. \quad (22)$$

Thus, $\Omega^{(\text{even})}$ and $\Omega^{(\text{odd})}$ indicate the strength of the effective 1D spin-exchange interaction.

Since the 1D effective interaction $V_+^{(\text{eff})}\mathcal{P}_+ + V_-^{(\text{eff})}\mathcal{P}_-$ is determined by the four parameters $g_{\pm}^{[\text{even}(\text{odd})]}$, in the next section we calculate $g_{\pm}^{[\text{even}(\text{odd})]}$ via solving the two-atom scattering problem.

B. Quasi-1D–quasi-0D scattering problem

As shown above, the value of $g_{\xi}^{[\text{even}(\text{odd})]}$ ($\xi = +, -$) is determined by the scattering amplitude between a g atom moving in the quasi-1D confinement and an e atom localized by the axial trap, with the two-atom Hamiltonian $H_0 + V_{\xi}$. Thus, to calculate $g_{\xi}^{[\text{even}(\text{odd})]}$ we first solve this quasi-1D–quasi-0D scattering problem. Our approach is similar to that of Massignan and Castin [30], who calculated the scattering amplitude between one atom freely moving in 3D free space and another atom localized in a 3D harmonic trap.

1. Scattering amplitudes

In the incident state of our problem, the relative transverse motion of the two atoms and the axial motion of the e atom are in the ground states of the corresponding confinements. Therefore, the incident wave function $\Psi^{(0)}(\boldsymbol{\rho}, z_e, z_g)$ can be expressed as

$$\Psi^{(0)}(\boldsymbol{\rho}, z_e, z_g) = \frac{e^{ikz_g}}{\sqrt{2\pi}}\chi_{n_{\perp}=0, m_z=0}(\boldsymbol{\rho})\phi_{n_z=0}(z_e), \quad (23)$$

where k is the incident momentum of the g atom and $\boldsymbol{\rho} = x_r\mathbf{e}_x + y_r\mathbf{e}_y$ is the transverse relative position vector, with $\mathbf{e}_{x(y)}$ the unit vector along the x (y) direction. Here $\chi_{n_{\perp}, m_z}(\boldsymbol{\rho})$

is the eigenstate of the transverse relative Hamiltonian $H_{\perp}$ defined in Eq. (4), with $n_{\perp}$ and m_z the principle quantum number and the quantum number of the angular momentum along the z direction, respectively. It satisfies

$$H_{\perp}\chi_{n_{\perp}, m_z}(\boldsymbol{\rho}) = (n_{\perp} + 1)\omega_{\perp}\chi_{n_{\perp}, m_z}(\boldsymbol{\rho}), \quad (24)$$

with $m_z = 0, \pm 1, \pm 2, \dots$ and $n_{\perp} = |m_z|, |m_z| + 2, |m_z| + 4, \dots$. In addition, in Eq. (23) the function $\phi_{n_z}(z_e)$ ($n_z = 0, 1, 2, \dots$) is the eigenstate of the axial Hamiltonian H_e of the e atom and satisfies

$$H_e\phi_{n_z}(z_e) = (n_z + \frac{1}{2})\omega_z\phi_{n_z}(z_e). \quad (25)$$

Actually this function can be expressed as $\phi_{n_z}(z_e) = \sqrt{1/a_z}\sqrt{\pi}2^{n_z}n_z!e^{-z_e^2/2a_z^2}H_{n_z}(z_e/a_z)$, with H_{n_z} the Hermite polynomial. It is clear that $\Psi^{(0)}$ is an eigenstate of the total free Hamiltonian H_0 with eigenvalue

$$E = \frac{k^2}{2} + \omega_{\perp} + \frac{\omega_z}{2}. \quad (26)$$

Here we assume that the incident kinetic energy $k^2/2$ is smaller than the energy gap between the ground and first excited state of H_e or $H_{\perp}$ with $m_z = 0$, i.e.,

$$0 \leq \frac{k^2}{2} < 2\omega_{\perp}, \quad 0 \leq \frac{k^2}{2} < \omega_z. \quad (27)$$

The scattering wave function $\Psi_{\xi}(\boldsymbol{\rho}, z_e, z_g)$ corresponding to the incident state $\Psi^{(0)}$ is determined by the Schrödinger equation $(H_0 + V_{\xi})\Psi_{\xi} = E\Psi_{\xi}$, with V_{ξ} ($\xi = +, -$) given by Eq. (13), as well as the outgoing boundary condition in the limit $|z_g| \rightarrow \infty$. These requirements can be equivalently reformulated as the integral equation [30]

$$\begin{aligned} \Psi_{\xi}(\boldsymbol{\rho}, z_e, z_g) &= \Psi^{(0)}(\boldsymbol{\rho}, z_e, z_g) + 4\pi a_{\xi} \int dz' G_E \\ &\times (\boldsymbol{\rho}, z_e, z_g; \mathbf{0}, z', z')\eta_{\xi}(z'), \end{aligned} \quad (28)$$

where the function $\eta_{\xi}(z')$ is the regularized scattering wave function, defined as

$$\eta_{\xi}(z') = \frac{\partial}{\partial z_r} \left[z_r \Psi_{\xi} \left(\mathbf{0}, z' - \frac{z_r}{2}, z' + \frac{z_r}{2} \right) \right] \Big|_{z_r \rightarrow 0^+}, \quad (29)$$

and G_E is the retarded Green's function for the free Hamiltonian H_0 . Using the Dirac bracket, we can express G_E as

$$\begin{aligned} G_E(\boldsymbol{\rho}, z_e, z_g; \boldsymbol{\rho}', z'_e, z'_g) \\ = \langle \boldsymbol{\rho}, z_e, z_g | \frac{1}{E + i0^+ - H_0} | \boldsymbol{\rho}', z'_e, z'_g \rangle, \end{aligned} \quad (30)$$

where $|\boldsymbol{\rho}, z_e, z_g\rangle$ and $|\boldsymbol{\rho}', z'_e, z'_g\rangle$ are the eigenstates of the transverse relative position and the axial coordinates of the g and e atoms.

We can extract the scattering amplitude from the behavior of $\Psi_{\xi}(\boldsymbol{\rho}, z_e, z_g)$ in the long-range limit $|z_g| \rightarrow \infty$. To this end, we reexpress the Green's function $G_E(\boldsymbol{\rho}, z_e, z_g; \mathbf{0}, z', z')$ as

$$\begin{aligned} G_E(\boldsymbol{\rho}, z_e, z_g; \mathbf{0}, z', z') &= -i \frac{e^{ik|z_g - z'|}}{k} \chi_{n_{\perp}=0, m_z=0}(\boldsymbol{\rho}) \chi_{n_{\perp}=0, m_z=0}^*(\mathbf{0}) \phi_{n_z=0}(z_e) \phi_{n_z=0}^*(z') \\ &- \sum_{\substack{n_z=0, 1, 2, \dots; \\ n_{\perp}=0, 2, 4, \dots; \\ (n_z, n_{\perp}) \neq (0, 0)}} \frac{e^{-\kappa_{n_{\perp}, n_z} |z_g - z'|}}{\kappa_{n_{\perp}, n_z}} \chi_{n_{\perp}, m_z=0}(\boldsymbol{\rho}) \chi_{n_{\perp}, m_z=0}^*(\mathbf{0}) \phi_{n_z}(z_e) \phi_{n_z}^*(z'), \end{aligned} \quad (31)$$

with $\kappa_{n_\perp, n_z} = \sqrt{2[(n_\perp + 1)\omega_\perp + (n_z + 1/2)\omega_z] - k^2}$. In the derivation of Eq. (31) we have used the fact that $\chi_{n_\perp, m_z}(\mathbf{0}) = 0$ for $m_z \neq 0$. Furthermore, due to the low-energy assumption (27), in the limit $|z_g| \rightarrow \infty$ all the terms in the summation in Eq. (31) decay to zero. Substituting Eq. (31) into Eq. (28) and using this result, we obtain

$$\begin{aligned} \Psi_\xi(\boldsymbol{\rho}, z_e, |z_g| \rightarrow \infty) \\ = \frac{1}{\sqrt{2\pi}} [e^{ikz_g} + f_\xi^{\text{even}}(k)e^{ik|z_g|} + f_\xi^{\text{odd}}(k)\text{sgn}(z_g)e^{ik|z_g|}] \\ \times \chi_{n_\perp=0, m_z=0}(\boldsymbol{\rho})\phi_{n_z=0}(z_e). \end{aligned} \quad (32)$$

Here the scattering amplitudes $f_\xi^{\text{even}}(k)$ and $f_\xi^{\text{odd}}(k)$ can be expressed as [23]

$$\begin{aligned} f_\xi^{\text{even (odd)}}(k) = -i \frac{2(2\pi)^{3/2} a_\xi}{k} \chi_{n_\perp=0, m_z=0}^*(\mathbf{0}) \int dz' F_{\text{even (odd)}} \\ \times (k, z') \phi_{n_z=0}^*(z') \eta_\xi(z'), \end{aligned} \quad (33)$$

with $F_{\text{even}}(k, z') = \cos(kz')$ and $F_{\text{odd}}(k, z') = -i \sin(kz')$. It is clear that Eq. (32) can be reexpressed in a convenient form ($k > 0$)

$$\begin{aligned} \Psi_\xi(\boldsymbol{\rho}, z_e, |z_g| \rightarrow \infty) \\ = \chi_{n_\perp=0, m_z=0}(\boldsymbol{\rho})\phi_{n_z=0}(z_e) \\ \times \begin{cases} \frac{1}{\sqrt{2\pi}} [e^{ikz_g} + r_\xi(k)e^{-ikz_g}] & \text{for } z_g \rightarrow -\infty \\ \frac{1}{\sqrt{2\pi}} t_\xi(k)e^{ikz_g} & \text{for } z_g \rightarrow +\infty, \end{cases} \end{aligned} \quad (34)$$

where $r_\xi(k)$ and $t_\xi(k)$ are the reflection and transmission amplitudes, respectively, related to $f_\xi^{\text{even (odd)}}(k)$ via

$$r_\xi(k) = f_\xi^{\text{even}}(k) - f_\xi^{\text{odd}}(k), \quad (35)$$

$$t_\xi(k) = f_\xi^{\text{even}}(k) + f_\xi^{\text{odd}}(k) + 1. \quad (36)$$

Actually, $f_\xi^{\text{even (odd)}}(k)$ are nothing but the two partial-wave scattering amplitudes. Explicitly, the complete Hamiltonian H in Eq. (2) is invariable under the total reflection operation

$$T : \{z_g \rightarrow -z_g, z_e \rightarrow -z_e\}. \quad (37)$$

As a result, the parity $\mathbb{P}$ with respect to this reflection operation is conserved. Therefore, there are two partial waves for our quasi-1D-quasi-0D scattering problem, i.e., the even wave (corresponding to $\mathbb{P} = +1$) and the odd wave (corresponding to $\mathbb{P} = -1$). As shown in Appendix A, it can be proved that $f_\xi^{\text{even (odd)}}(k)$ given by Eq. (33) is just the scattering amplitude for the even (odd) partial waves, respectively.

Here we would like to emphasize that, even though our 3D bare interaction $V_\pm$ defined in Eq. (13) only includes the s -wave component, both the even- and odd-wave scattering amplitudes $f_\xi^{\text{even (odd)}}(k)$ are nonzero. This can be explained as follows. The total parity $\mathbb{P}$ with respect to the reflection T can be expressed as

$$\mathbb{P} = \mathbb{P}_{\text{c.m.}} \times \mathbb{P}_r, \quad (38)$$

where $\mathbb{P}_{\text{c.m.}}$ is the parity corresponding to the reflection of the center-of-mass coordinate [i.e., the transformation $\{Z \rightarrow -Z; z_r \rightarrow z_r\}$, with $Z = (z_e + z_g)/2$ and $z_r = z_g - z_e$ as defined above] and $\mathbb{P}_r$ is the parity corresponding to the

reflection of the relative coordinate (i.e., the transformation $\{Z \rightarrow Z; z_r \rightarrow -z_r\}$). Therefore, in the odd-wave subspace (i.e., the subspace with $\mathbb{P} = -1$), there are some states with $\mathbb{P}_{\text{c.m.}} = -1$ and $\mathbb{P}_r = +1$. Thus, although the s -wave Huang-Yang pseudopotentials $V_\pm$ only operate on the states with $\mathbb{P}_r = 1$, they have nonzero projection for the odd-wave subspace. As a result, the odd-wave scattering amplitude $f_\xi^{\text{odd}}(k)$ is nonzero. Similarly, $f_\xi^{\text{even}}(k)$ is also nonzero. It has been shown that in the scattering problems of two ultracold atoms in a mixed-dimensional system, even if the interatomic interaction is described by an s -wave Huang-Yang pseudopotential, the high partial-wave scattering amplitudes are usually nonzero [31].

2. Calculation of $f_\xi^{\text{even (odd)}}(k)$

Equation (33) shows that the scattering amplitudes $f_\xi^{\text{even (odd)}}(k)$ are functionals of the regularized wave function $\eta_\xi(z')$ defined in Eq. (29). In contrast, substituting Eq. (28) into Eq. (29), we can find that $\eta_\xi(z)$ satisfies another integral equation (Appendix B)

$$\eta_\xi(z) = \Psi^{(0)}(\mathbf{0}, z, z) + \hat{\mathcal{O}}_\xi[\eta_\xi(z)]. \quad (39)$$

Here $\hat{\mathcal{O}}_\xi$ is an integral operator with the explicit form given in Appendix B. In our calculation, we first numerically solve Eq. (39) and obtain $\eta_\xi(z)$, and then substitute our results into Eq. (33) and obtain the scattering amplitudes $f_\xi^{\text{even (odd)}}(k)$.

3. Low-energy behaviors of $f_\xi^{\text{even (odd)}}(k)$

Furthermore, in the low-energy limit $k \rightarrow 0$ the behaviors of the scattering amplitudes $f_\xi^{\text{even (odd)}}(k)$ are given by [32–38]

$$f_\xi^{\text{even}}(k) \approx -\frac{1}{1 + ika_\xi^{(\text{even})}}, \quad (40)$$

$$f_\xi^{\text{odd}}(k) \approx \frac{-ik}{ik + 1/a_\xi^{(\text{odd})}}, \quad (41)$$

where

$$a_\xi^{(\text{even})} \equiv \lim_{k \rightarrow 0} \frac{i}{k} \left[1 + \frac{1}{f_\xi^{\text{even}}(k)} \right], \quad (42)$$

$$a_\xi^{(\text{odd})} \equiv \lim_{k \rightarrow 0} \frac{i}{k} \left[1 + \frac{1}{f_\xi^{\text{odd}}(k)} \right]^{-1} \quad (43)$$

are the effective 1D scattering lengths for the even and odd waves, respectively.

4. Effective 1D interaction

In addition, the low-energy scattering amplitudes in Eqs. (40) and (41) can be reproduced by the effective 1D interaction $g_\xi^{(\text{even})} \delta(z_g)(z_g) \hat{d}_e + g_\xi^{(\text{odd})} \delta'(z_g) \hat{d}_o$, with $\hat{d}_{e,o}$ defined in Eqs. (16) and (17) and the intensities $g_\xi^{[\text{even (odd)}]}$ given by [29]

$$g_\xi^{(\text{even})} = -\frac{1}{a_\xi^{(\text{even})}}, \quad (44)$$

$$g_\xi^{(\text{odd})} = -a_\xi^{(\text{odd})}. \quad (45)$$

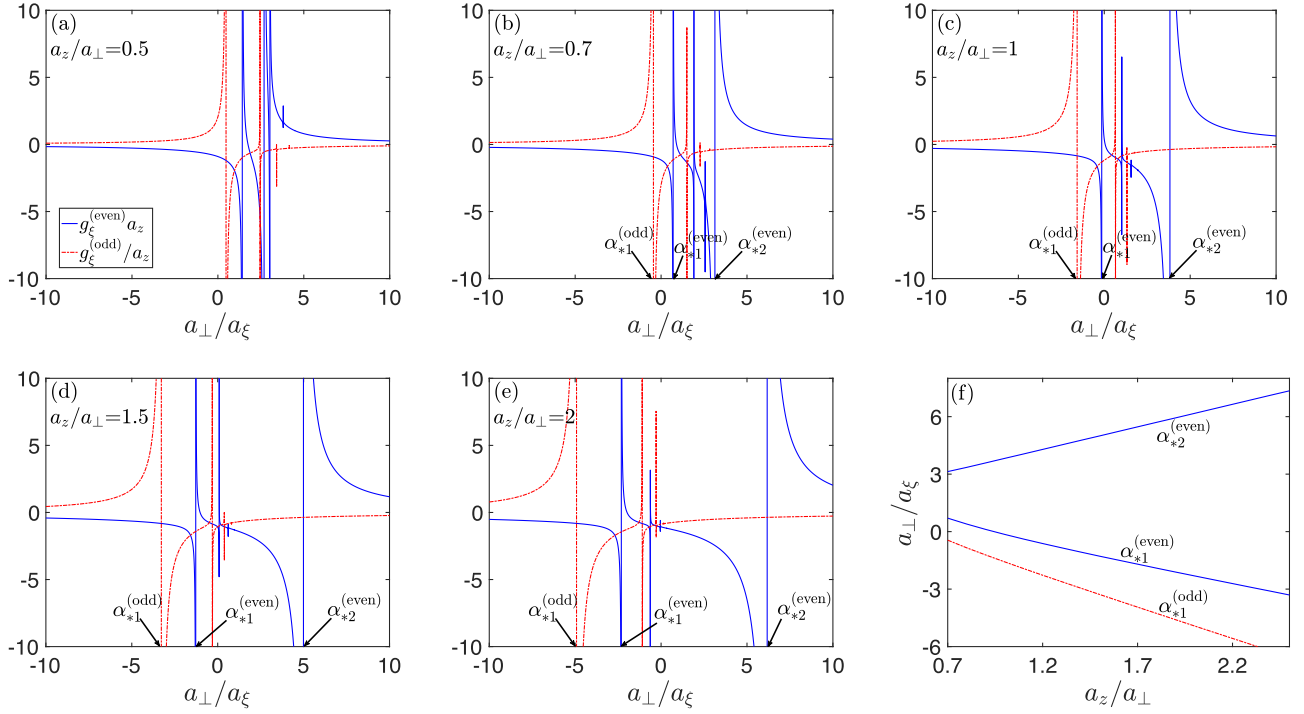


FIG. 2. (a)–(e) Effective 1D interaction strength $g_\xi^{(even)}$ (blue solid line) and $g_\xi^{(odd)}$ (red dash-dotted line) as functions of $a_\perp/a_\xi$ for (a) $a_z/a_\perp = 0.5$, (b) $a_z/a_\perp = 0.7$, (c) $a_z/a_\perp = 1$, (d) $a_z/a_\perp = 1.5$, and (e) $a_z/a_\perp = 2$. (f) Locations $\alpha_{*1}^{(even)}$ and $\alpha_{*2}^{(even)}$ of the two broadest even-wave CIRs and the location $\alpha_{*1}^{(odd)}$ of the broadest odd-wave CIR, for various $a_z/a_\perp$.

Therefore, when the scattering amplitude $f_\xi^{(even/odd)}(k)$ is obtained, we can calculate the effective 1D scattering length $a_\xi^{[even/odd]}$ via Eq. (42) [Eq. (43)] and then derive the effective 1D interaction intensity $g_\xi^{[even/odd]}$ via Eq. (44) [Eq. (45)].

III. RESULTS AND ANALYSIS

In the preceding section we showed our approach for the numerical calculation for the strengths $g_\pm^{[even/odd]}$ of the effective 1D interaction $V_\pm^{(eff)}$. In this section we illustrate our results and study the CIRs for our system and then compare our results with the experimental observations in Ref. [24].

A. Locations and widths of the CIRs

As shown above, $g_+^{[even/odd]}$ and $g_-^{[even/odd]}$ are given by the scattering amplitudes for the same scattering problem with different 3D scattering lengths a_+ and a_- , respectively. As a result, an immediate dimensional analysis yields that

$$g_\xi^{(l)} = a_\xi^{\lambda_l} S_l\left(\frac{a_\perp}{a_\xi}, \frac{a_z}{a_\perp}\right) \quad (46)$$

for $l = (\text{even}, \text{odd})$ and $\xi = (+, -)$, with $\lambda_{\text{even}} = -1$, $\lambda_{\text{odd}} = +1$, and $S_{\text{even}}(\alpha, \beta)$ and $S_{\text{odd}}(\alpha, \beta)$ ξ -independent universal functions which can be obtained via the numerical calculation shown in Sec. II. This result shows that for a system with fixed 3D scattering length a_ξ , the control effect of the parameters a_z and $a_\perp$ for $g_\xi^{[even/odd]}$ can be described by the two dimensionless parameters $a_\perp/a_\xi$ and $a_z/a_\perp$ with the following clear physical meanings. The absolute value of $a_\perp/a_\xi$ describes the intensity of the transverse confinement. Explicitly, $|a_\perp/a_\xi|$

is larger for weaker transverse confinement. Similarly, the ratio $a_z/a_\perp$ describes the relative intensity of the axial and transverse confinement.

In addition, Eq. (46) shows that the conditions for the CIR in the even- and odd-wave channels can be expressed as

$$\left(\frac{a_z}{a_\xi}, \frac{a_z}{a_\perp}\right) = (\alpha_*^{(even)}, \beta_*^{(even)}) \quad (47)$$

and

$$\left(\frac{a_z}{a_\xi}, \frac{a_z}{a_\perp}\right) = (\alpha_*^{(odd)}, \beta_*^{(odd)}), \quad (48)$$

respectively, with $(\alpha_*^{[even/odd]}, \beta_*^{[even/odd]})$ being any singularity of the function $S_{\text{even/odd}}(\alpha, \beta)$. It is clear that the values of $(\alpha_*^{[even/odd]}, \beta_*^{[even/odd]})$ are ξ independent. When the condition in Eq. (47) or (48) is satisfied for a specific ξ , we have $g_\xi^{(even)} = \infty$ or $g_\xi^{(odd)} = \infty$. According to Eqs. (18), (20), and (22), in this case the strength of the effective 1D spin-exchange interaction, i.e., $\Omega^{(even)}$ or $\Omega^{(odd)}$, also diverges.

Now we investigate the locations and widths of the CIRs. To this end, in Figs. 2(a)–2(e) we illustrate the dependence of $g_\xi^{(even)}$ (in units of $1/a_z$) and $g_\xi^{(odd)}$ (in units of a_z) on $a_\perp/a_\xi$ for given values of $a_z/a_\perp$. In addition, in Fig. 2(f) we plot the locations $\alpha_{*1}^{(even)}$ and $\alpha_{*2}^{(even)}$ of the two broadest even-wave CIRs, as well as the location $\alpha_{*1}^{(odd)}$ of the broadest odd-wave CIR, as functions of $a_z/a_\perp$. The results in these figures can be summarized and understood as follows.

(A) Multiple CIRs can appear for both even- and odd-wave channels. This result is qualitatively consistent with our previous work with the pure-1D model [23]. As stated in [23], it can be explained as the result of the coupling between

the center-of-mass motion and the relative motion of the two atoms in the z direction. Similar multiresonance phenomena were also found in other scattering problems between two ultracold atoms, where the center-of-mass motion is coupled to the relative motion [30,31,39–45].

(B) For the odd partial wave, the broadest CIR is the one located at the lower end of $a_{\perp}/a_{\xi}$. The location is defined as $a_{\perp}/a_{\xi} = \alpha_{*1}^{(\text{odd})}$. Other odd-wave CIRs become more and more narrow when $a_{\perp}/a_{\xi}$ increases. In addition, as shown in Figs. 2(c)–2(e), for $a_z/a_{\perp} \gtrsim 0.7$, we have $\alpha_{*1}^{(\text{odd})} < 0$, i.e., the broadest odd-wave CIR can appear only when the s -wave scattering length a_{ξ} is negative.

(C) For the even partial wave, when $a_z/a_{\perp}$ is small [e.g., $a_z/a_{\perp} = 0.5$, as shown in Fig. 2(a)], the situation is similar to the odd partial wave, i.e., the broadest CIR is the one located at the lower end of $a_{\perp}/a_{\xi}$. When the value of $a_z/a_{\perp}$ becomes large [Fig. 2(b)], some narrow CIRs, which appear for relatively large $a_{\perp}/a_{\xi}$, gradually merge with each other and form another broad CIR. Furthermore, in the parameter region with $a_z/a_{\perp} \gtrsim 1$ [Figs. 2(c)–2(e)], there are always two relatively broad CIRs which are located at $a_{\perp}/a_{\xi} = \alpha_{*1}^{(\text{even})}$ and $\alpha_{*2}^{(\text{even})}$, with $\alpha_{*1}^{(\text{even})} < 0$ and $\alpha_{*2}^{(\text{even})} > 0$. In addition, many relatively narrow CIRs can occur for $\alpha_{*1}^{(\text{even})} < a_{\perp}/a_{\xi} < \alpha_{*2}^{(\text{even})}$. Thus, when $a_z/a_{\perp} \gtrsim 1$, a broad CIR, which is usually very advantageous for the control of interatomic interaction, can always be realized for the systems with either positive or negative 3D scattering length a_{ξ} .

(D) As shown in Fig. 2(f), for $a_z/a_{\perp} \gtrsim 0.7$ the absolute values of $\alpha_{*1}^{(\text{odd})}$ and $\alpha_{*1,2}^{(\text{even})}$, i.e., the locations of the broadest odd-wave CIRs and the two broadest even-wave CIRs, almost linearly increase with $a_z/a_{\perp}$. Thus, for either positive or negative a_{ξ} , by increasing the ratio $a_z/a_{\perp}$ one can always realize a broad even-wave CIR via the confinements with larger characteristic lengths of $a_{\perp}$ and a_z . These confinements can be created via weaker laser beams and thus are more feasibly prepared in the experiments. Similarly, when a_{ξ} is negative one can also realize a broad odd-wave CIR in these confinements. Nevertheless, in realistic systems $a_z/a_{\perp}$ cannot be infinitely increased. That is because, when $a_z/a_{\perp} \rightarrow \infty$, we have either $\omega_z \rightarrow \infty$ or $a_{\perp} \rightarrow 0$. In the former case, we also have $\omega_z \rightarrow 0$ and thus the low-temperature condition $T \ll \omega_z/k_B$ would be violated. In the latter case, the transverse confinement has to be realized via very strong laser beams, which is formidable in experiments.

B. Comparison of theory and experiment

Now we compare our theoretical results with the recent experiment of ultracold ^{173}Yb atoms [24]. In this experiment the transverse confinement for both atoms and the axial confinement for the e atom are realized via a 2D optical lattice with magic wavelengths $\lambda_{\perp} = 759$ nm and a 1D optical lattice with wavelength $\lambda_z = 680$ nm, respectively. The explicit potentials of these two lattices are given by

$$V_{\text{lattice}}^{(\perp)} = U_{\perp} \sum_{j=e,g} [\cos^2(2\pi x_j/\lambda_{\perp}) + \cos^2(2\pi y_j/\lambda_{\perp})], \quad (49)$$

$$V_{\text{lattice}}^{(z)} = U_z \cos^2(2\pi z_e/\lambda_z), \quad (50)$$

where x_j and y_j ($j = e, g$) are the x and y coordinates of the j atom, respectively, and U_z and $U_{\perp}$ are the intensities of the lattices. By expanding these two potentials around the minimum points we can obtain the harmonic trapping potentials shown in Eqs. (4) and (5). The characteristic lengths a_z and $a_{\perp}$ are given by

$$a_{\perp} = \frac{\sqrt{\lambda_{\perp}}}{2^{1/4} \sqrt{\pi} U_{\perp}^{1/4}}, \quad a_z = \frac{\sqrt{\lambda_z}}{2^{3/4} \sqrt{\pi} U_z^{1/4}} \quad (51)$$

and can be tuned via the intensities U_z and $U_{\perp}$. In the experiments of Ref. [24], the e atoms and g atoms are initially prepared in the states $|\downarrow\rangle_e$ and $|\uparrow\rangle_g$, respectively. After a finite holding time, the spin of some e atoms is flipped to the state $|\uparrow\rangle_e$ by the effective spin-exchange interaction. The number $N_{e\uparrow}$ of spin-flipped e atoms is measured. This number is supposed to be positively correlated with the absolute value of the effective 1D spin-exchange intensity $\Omega^{[\text{even}(\text{odd})]}$, which is given by $\Omega^{[\text{even}(\text{odd})]} = (g_{-}^{[\text{even}(\text{odd})]} - g_{+}^{[\text{even}(\text{odd})]})/2$ in our theory, as shown in Eqs. (18) and (22). Explicitly, when the system is around a CIR of either $g_{+}^{[\text{even}(\text{odd})]}$ or $g_{-}^{[\text{even}(\text{odd})]}$, the atom number $N_{e\uparrow}$ should be resonantly enhanced.

In Fig. 3(a) we compare the experimentally observed atom number $N_{e\uparrow}$ and the effective spin-exchange-interaction intensities $\Omega^{(\text{even})}$ and $\Omega^{(\text{odd})}$ obtained from our theory, for the cases with $U_{\perp} = 35.6 E_R^{(\perp)}$, where $E_R^{(z)} = 2\pi^2/\lambda_z^2$ and $E_R^{(\perp)} = 2\pi^2/\lambda_{\perp}^2$ are the recoil energies of the lattices. To be consistent with Ref. [24], in the figure we chose the horizontal ordinate to be $V_z \equiv U_z/3.3$. As shown in this figure, a even-wave CIR (C_A) and an odd-wave CIR (C_B) are found by our calculations, with the positions close to the experimentally observed peaks p_A and p_B of $N_{e\uparrow}$, respectively. According to this result, the peaks p_A and p_B are due to CIRs in different partial-wave channels. In addition, there is some difference between the position of the CIR $C_{A,B}$ and the observed peak $p_{A,B}$. The difference between the positions of C_A and p_A may due to the fact that in our calculation the shallow lattice experienced by the g atom is ignored. On the other hand, the difference between the positions of C_B and p_B may be due to the fact that, in the region of C_B and p_B ($V_z \lesssim 3E_R^{(z)}$, i.e., $U_z \lesssim 9.9E_R^{(z)}$) the depth of the trapping potential $V_{\text{lattice}}^{(z)}$ experienced by the e atom is so weak that the harmonic approximation for this potential does not work very well.

In Figs. 3(b) and 3(c) we further illustrate $\Omega^{(\text{even})}$ and $\Omega^{(\text{odd})}$ given by our calculation, respectively, for the parameter regions $U_{\perp}/E_R^{(\perp)} \in [15, 50]$ and $V_z/E_R^{(z)} \in [0.3, 8]$ (i.e., $U_z/E_R^{(\perp)} \in [1, 26.4]$). In these figures, the locations of the CIRs are the places where the color suddenly changes from deep blue to yellow, where $|\Omega^{[\text{even}(\text{odd})]}|$ is very large and the sign of $\Omega^{[\text{even}(\text{odd})]}$ suddenly changes. Two even-wave CIR branches (A) and (C) and one odd-wave CIR branch (B) are illustrated. Our calculations show that all of them are caused by the CIRs of $g_{+}^{[\text{even}(\text{odd})]}$. The experimental results for the parameter region of Figs. 3(b) and 3(c) are given in Fig. 3 of Ref. [24], where the lattice depth $U_{\perp}$ is denoted by $V_{\perp}$. It is shown that the even-wave CIR branch (A) and odd-wave CIR branch (B) are clearly observed at locations which are

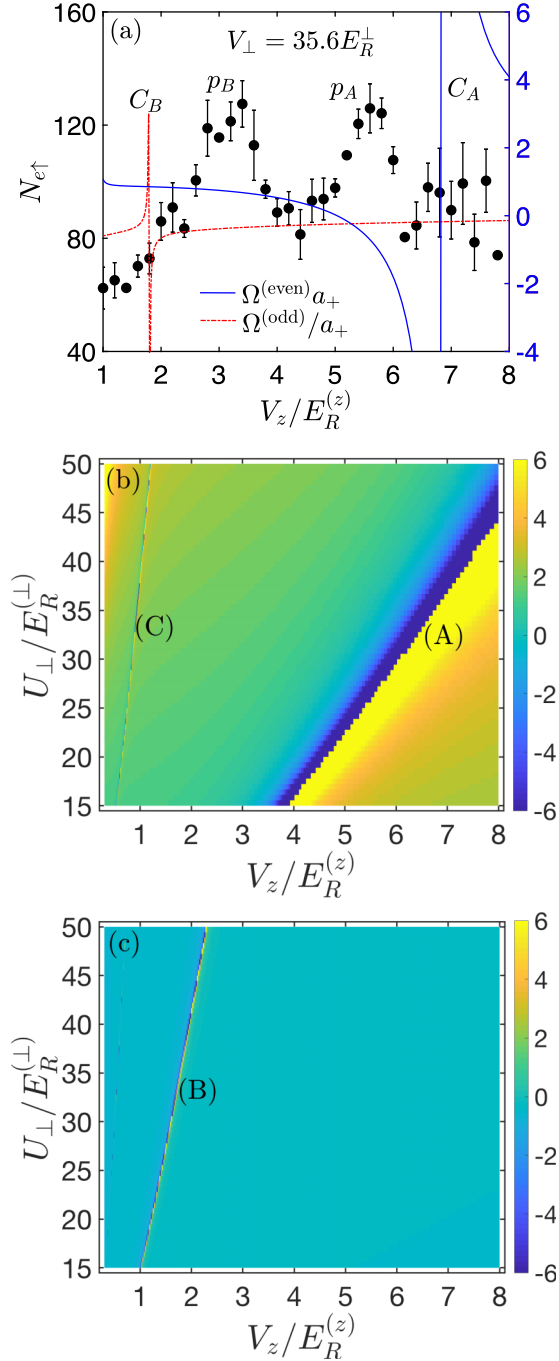


FIG. 3. Comparison between our theory with the experimental results of Ref. [24]. In our calculations we take $a_+ = 1878a_0$ and $a_- = 216a_0$ [13]. (a) Black circles with error bar denote the number $N_{e\uparrow}$ of spin-flipped e atoms observed in the experiment, for the cases with $U_\perp = 35.6E_R^\perp$. The data are taken from Fig. 3(b) of Ref. [24]. The blue solid line shows the effective 1D even-wave spin-exchange intensity $\Omega^{(\text{even})}$ (in units of $1/a_+$) given by our calculation. The red dash-dotted line shows the effective 1D odd-wave spin-exchange intensity $\Omega^{(\text{odd})}$ (in units of a_+) given by our theoretical calculation. As shown in the text, the horizontal ordinate V_z is defined as $V_z \equiv U_z/3.3$, with U_z defined in Eq. (50). (b) Theoretically calculated $\Omega^{(\text{even})}$ (in units of $1/a_+$) as a function of V_z and $U_\perp$. (c) Theoretically calculated $\Omega^{(\text{odd})}$ (in units of a_+) as a function of V_z and $U_\perp$.

close to our theoretical results. In addition, similar to Fig. 3(a), the quantitative shifts between the theoretical prediction and the experimental observation for these two CIR branches are due to the ignorance of the shallow lattice for the g atom as well as the weakness of the trapping lattice potential $V_{\text{lattice}}^{(z)}$ for the e atom in the region of CIR (B). Furthermore, the even-wave CIR branch (C) is not experimentally detected. That is very possibly because this branch is too narrow, as shown in Fig. 3(b).

IV. SUMMARY AND OUTLOOK

We exactly solved the scattering problem between two alkali-earth-metal-like atoms, i.e., one g atom freely moving in a quasi-1D tube and another e atom localized by a 3D harmonic trap. Our solutions show that with the help of the even- or odd-wave CIRs in this system, the effective 1D spin-exchange interaction can be resonantly controlled via the characteristic lengths a_z and $a_\perp$ of the confinements. When $a_z/a_\perp$ is larger, the relatively broad CIRs can be realized in weaker confinements. Our results reveal that the two CIR branches which were observed in the recent experiment in Ref. [24] are due to even-wave and odd-wave CIRs, respectively. In previous studies for the Kondo effect, most of the attention was paid to systems with only an even-wave 1D interaction. The system with the resonant odd-wave spin-exchange interaction was not studied much. Our results show that these systems can be experimentally realized in the ultracold gases of alkali-earth-metal-like atoms.

As shown above, in the experiments a shallow axial lattice potential for the g atom can also be induced by a laser beam which is used to confine the axial motion of the e atom. Further study of the effect induced by this shallow lattice potential is left for future work.

ACKNOWLEDGMENTS

This work was supported in part by the National Key R&D Program of China through Grants No. 2018YFA0307601 (R.Z.) and No. 2018YFA0306502 (P.Z.); NSFC Grants No. 11804268 (R.Z.), No. 11434011 (P.Z.), and No. 11674393 (P.Z.); and the Research Funds of Renmin University of China under Grant No. 16XNLQ03 (P.Z.).

APPENDIX A: PARTIAL-WAVE ANALYSIS

In this appendix we prove that $f_\xi^{\text{even(odd)}}(k)$ given by Eq. (33) is just the scattering amplitude for the even (odd) partial wave.

As shown in Sec. II B, our system is invariable under the total reflection operation $T : \{z_g \rightarrow -z_g, z_e \rightarrow -z_e\}$. Thus, the subspaces $\mathcal{H}_{\text{even}} \equiv \{\Psi | T\Psi = +\Psi\}$ (corresponding to the parity $\mathbb{P} = +1$, or the even partial wave) and $\mathcal{H}_{\text{odd}} \equiv \{\Psi | T\Psi = -\Psi\}$ (corresponding to $\mathbb{P} = -1$, or the odd partial wave) are invariant subspaces of the complete Hamiltonian H . Furthermore, the incident state $\Psi^{(0)}(\boldsymbol{\rho}, z_e, z_g)$ defined in Eq. (23) can be expressed as

$$\Psi^{(0)} = \Psi_{\text{even}}^{(0)} + \Psi_{\text{odd}}^{(0)}, \quad (\text{A1})$$

where $\Psi_{\text{even (odd)}}^{(0)} \in \mathcal{H}_{\text{even (odd)}}$ and are defined as

$$\Psi_{\text{even}}^{(0)}(\boldsymbol{\rho}, z_e, z_g) = \frac{\cos(kz_g)}{\sqrt{2\pi}} \chi_{n_\perp=0, m_z=0}(\boldsymbol{\rho}) \phi_{n_z=0}(z_e), \quad (\text{A2})$$

$$\Psi_{\text{odd}}^{(0)}(\boldsymbol{\rho}, z_e, z_g) = i \frac{\sin(kz_g)}{\sqrt{2\pi}} \chi_{n_\perp=0, m_z=0}(\boldsymbol{\rho}) \phi_{n_z=0}(z_e). \quad (\text{A3})$$

Therefore, the scattering wave function $\Psi_\xi(\boldsymbol{\rho}, z_e, z_g)$ corresponding to the incident state $\Psi^{(0)}$, which we studied in Sec. II B, can be expressed as

$$\Psi_\xi(\boldsymbol{\rho}, z_e, z_g) = \Psi_\xi^{\text{even}}(\boldsymbol{\rho}, z_e, z_g) + \Psi_\xi^{\text{odd}}(\boldsymbol{\rho}, z_e, z_g), \quad (\text{A4})$$

with $\Psi_\xi^{\text{even (odd)}} \in \mathcal{H}_{\text{even (odd)}}$ the scattering wave functions corresponding to the incident states $\Psi_{\text{even (odd)}}^{(0)}$, i.e., the scattering states in the even (odd) partial-wave channel. In addition,

using the analysis which is similar to that in Sec. II B, we can find that

$$\begin{aligned} \Psi_\xi^{\text{even (odd)}}(\boldsymbol{\rho}, z_e, |z_g| \rightarrow \infty) \\ = \Psi_{\text{even (odd)}}^{(0)}(\boldsymbol{\rho}, z_e, z_g) + f_\xi^{\text{even (odd)}}(k) \Lambda_{\text{even (odd)}} \\ \times e^{ik|z_g|} \chi_{n_\perp=0, m_z=0}(\boldsymbol{\rho}) \phi_{n_z=0}(z_e), \end{aligned} \quad (\text{A5})$$

where $\Lambda_{\text{even}} = 1$, $\Lambda_{\text{odd}} = \text{sgn}(z_g)$, and $f_\xi^{\text{even (odd)}}(k)$ is given by Eq. (33). In the derivation of this result, we have used $\eta_\xi(z') = \eta_\xi^{\text{even}}(z') + \eta_\xi^{\text{odd}}(z')$, with $\eta_\xi^{\text{even}}(z')$ defined in Eq. (29), $\eta_\xi^{\text{even (odd)}}(z') = \frac{\partial}{\partial z_r} [z_r \Psi_\xi^{\text{even (odd)}}(\mathbf{0}, z' - \frac{z_r}{2}, z' + \frac{z_r}{2})] |_{z_r \rightarrow 0^+}$, and the fact that $\eta_\xi^{\text{even (odd)}}(z')$ is an even (odd) function of z' . Equation (A5) shows that $f_\xi^{\text{even (odd)}}(k)$ is nothing but the scattering amplitude for the even (odd) partial wave.

APPENDIX B: INTEGRAL EQUATION FOR $\eta_\xi(z)$

In this appendix we derive the integral equation for the regularized wave function $\eta_\xi(z)$ defined in Eq. (29).

1. Green's function $G_E(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z')$

The Green's function $G_E(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z')$ is very important for our calculation. Therefore, here we first reexpress this function in convenient forms. Using Eq. (31) and the fact that $|\chi_{n_\perp, m_z=0}(\mathbf{0})|^2 = \omega_\perp/2\pi$, we have

$$G_E(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z') = \frac{\omega_\perp}{2\pi} \sum_{n_\perp=0, 2, 4, 6, \dots} g[E - (n_\perp + 1)\omega_\perp; z_e, z_g, z', z'], \quad (\text{B1})$$

where the function $g(\mathcal{E}; z_e, z_g; z'_e, z'_g)$ is the Green's function for the axial motion of the g atom and the e atom (i.e., the matrix element of $[\mathcal{E} + i0^+ - (H_0 - H_\perp)]^{-1}$) and can be expressed as

$$g(\mathcal{E}; z_e, z_g; z'_e, z'_g) = \sum_{n_z=0, 1, 2, 3, \dots} \frac{e^{\sqrt{2[(n_z + \frac{1}{2})\omega_z - \mathcal{E}]}|z_g - z'|}}{\sqrt{2[(n_z + \frac{1}{2})\omega_z - \mathcal{E}]}} \phi_{n_z}(z_e) \phi_{n_z}^*(z'_g), \quad (\text{B2})$$

where for the function $\sqrt{q}$ is defined as $\sqrt{q} = i|q|^{1/2}$ for $q < 0$. Equation (B1) yields that

$$G_E(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z') = \frac{\omega_\perp}{2\pi} g[E - \omega_\perp; z_e, z_g; z', z'] + G_{E'}(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z'), \quad (\text{B3})$$

with the energy E' defined as

$$E' = E - \omega_\perp. \quad (\text{B4})$$

With the help of Eq. (B3) we convert the calculation of G_E to the calculations of the functions $g[E - \omega_\perp; z_e, z_g; z', z']$ and $G_{E'}$. The former function can be easily calculated numerically. Furthermore, due to the low-energy assumption shown in Eqs. (26) and (27), we know that E' is lower than the threshold of H_0 , i.e., $E' < \omega_\perp + \omega_z/2$. Due to this fact, the integration $-\int_0^\infty d\beta e^{\beta E'} e^{-\beta H_0}$ converges and we have $(E' - H_0)^{-1} = -\int_0^\infty d\beta e^{\beta E'} e^{-\beta H_0}$. Thus, the Green's function $G_{E'}$ can be reexpressed as

$$G_{E'}(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z') = -\int_0^\infty d\beta e^{\beta E'} K_\beta(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z'), \quad (\text{B5})$$

with the function $K_\beta(\boldsymbol{\rho}, z_e, z_g; \boldsymbol{\rho}', z'_e, z'_g)$ the imaginary-time propagator of the free Hamiltonian H_0 , which can be expressed as

$$\begin{aligned} K_\beta(\boldsymbol{\rho}, z_e, z_g; \boldsymbol{\rho}', z'_e, z'_g) &= \langle \boldsymbol{\rho}, z_e, z_g | e^{-\beta H_0} | \boldsymbol{\rho}', z'_e, z'_g \rangle \\ &= K_\beta^{(\perp)}(\boldsymbol{\rho}, \boldsymbol{\rho}') K_\beta^{(e)}(z_e, z'_e) K_\beta^{(g)}(z_g, z'_g), \end{aligned} \quad (\text{B6})$$

where $K_{\beta}^{(\perp)}$, $K_{\beta}^{(e)}$, and $K_{\beta}^{(g)}$ are the propagators of the two-atom transverse relative motion (two-dimensional harmonic oscillator with mass $1/2$ and frequency $\omega_{\perp}$), the axial motion of the e atom (1D harmonic oscillator with mass 1 and frequency ω_z), and the axial motion of the g atom (1D free particle with mass 1), respectively. Explicitly, we have

$$K_{\beta}^{(\perp)}(\boldsymbol{\rho}, \boldsymbol{\rho}') = \left[\frac{\omega_{\perp}}{4\pi \sinh(\omega_{\perp}\beta)} \right] \exp \left\{ -\frac{\omega_{\perp}[(|\boldsymbol{\rho}|^2 + |\boldsymbol{\rho}'|^2) \cosh(\omega_{\perp}\beta) - 2\boldsymbol{\rho} \cdot \boldsymbol{\rho}']}{4 \sinh(\omega_{\perp}\beta)} \right\}, \quad (\text{B7})$$

$$K_{\beta}^{(e)}(z_e, z'_e) = \sqrt{\frac{\omega_z}{2\pi \sinh(\omega_z\beta)}} \exp \left\{ -\frac{\omega_z[(z_e^2 + z_e'^2) \cosh(\omega_z\beta) - 2z_e z'_e]}{2 \sinh(\omega_z\beta)} \right\}, \quad (\text{B8})$$

$$K_{\beta}^{(g)}(z_g, z'_g) = \sqrt{\frac{1}{2\pi\beta}} \exp \left[-\frac{(z_g - z'_g)^2}{2\beta} \right]. \quad (\text{B9})$$

Substituting Eqs. (B7)–(B9) into Eq. (B6), we can obtain the expression for the function $K_{\beta}(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z')$ in Eq. (B5):

$$K_{\beta}(\mathbf{0}, z_e, z_g; \mathbf{0}, z', z') = \frac{\omega_{\perp}}{8\pi^2 \sinh(\omega_{\perp}\beta)} \sqrt{\frac{\omega_z}{\beta \sinh(\omega_z\beta)}} \exp \left\{ -\frac{\omega_z[(z_e^2 + z_e'^2) \cosh(\omega_z\beta) - 2z_e z'_e]}{2 \sinh(\omega_z\beta)} - \frac{(z_g - z')^2}{2\beta} \right\}. \quad (\text{B10})$$

In our following calculations we will use Eqs. (B3), (B5), and (B10).

2. Short-range behavior of $\Psi_{\xi}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2})$

Now we derive the equation for $\eta_{\xi}(z)$. According to Eq. (29), $\eta_{\xi}(z)$ is determined by the behavior of the function $\Psi_{\xi}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2})$ in the short-range limit $|z_r| \rightarrow 0$. Thus, we first study this behavior. According to Eqs. (28) and (B3), the function $\Psi_{\xi}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2})$ satisfies the equation

$$\begin{aligned} \Psi_{\xi}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}) &= \Psi^{(0)}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}) + 2\omega_{\perp}a_{\xi} \int dz' g[E - \omega_{\perp}; z - \frac{z_r}{2}, z + \frac{z_r}{2}; z', z'] \eta_{\xi}(z') \\ &\quad + 4\pi a_{\xi} \eta_{\xi}(z) \int dz' G_{E'}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}; \mathbf{0}, z', z') \\ &\quad + 4\pi a_{\xi} \int dz' G_{E'}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}; \mathbf{0}, z', z') [\eta_{\xi}(z') - \eta_{\xi}(z)]. \end{aligned} \quad (\text{B11})$$

In the limit $z_r \rightarrow 0$, the first and second terms on the right-hand side of Eq. (B11) converge. Now we study the behavior of the third term. To this end, we define

$$U \equiv \int dz' G_{E'}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}; \mathbf{0}, z', z'). \quad (\text{B12})$$

Then Eq. (B5) yields that

$$U = \int_0^{\infty} d\beta \mathcal{L}[\beta, z, z_r], \quad (\text{B13})$$

with the function $\mathcal{L}[\beta, z, z_r]$ defined as

$$\mathcal{L}[\beta, z, z_r] = - \int_{-\infty}^{\infty} dz' e^{\beta E'} K_{\beta}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}; \mathbf{0}, z', z'). \quad (\text{B14})$$

Substituting Eq. (B10) into Eq. (B14), we further obtain

$$\begin{aligned} \mathcal{L}[\beta, z, z_r] &= - \frac{\omega_{\perp} \sqrt{\omega_z}}{2(2\pi)^{3/2} \sinh(\omega_{\perp}\beta) \sqrt{\omega_z\beta \cosh(\omega_z\beta) + \sinh(\omega_z\beta)}} \\ &\quad \times \exp \left\{ \beta E' + \frac{\left[\frac{\omega_z(z - \frac{z_r}{2})}{\sinh(\omega_z\beta)} + \frac{(z + \frac{z_r}{2})}{\beta} \right]^2}{2(\frac{\omega_z}{\tanh(\omega_z\beta)} + \frac{1}{\beta})} - \frac{\omega_z(z - \frac{z_r}{2})^2}{2 \tanh(\omega_z\beta)} - \frac{1}{2\beta} \left(z + \frac{z_r}{2} \right)^2 \right\}. \end{aligned} \quad (\text{B15})$$

Furthermore, the integration $\int_0^{\infty} d\beta \mathcal{L}[\beta, z, z_r]$ diverges in the limit $z_r \rightarrow 0$. This divergence is due to the behavior of $\mathcal{L}[\beta, z, z_r]$ in the limit $\beta \rightarrow 0$. Thus, we can obtain the behavior of $\int_0^{\infty} d\beta \mathcal{L}[\beta, z, z_r]$ in this limit by reexpressing $\mathcal{L}[\beta, z, z_r]$ as

$$\mathcal{L}[\beta, z, z_r] = \mathcal{L}_0[\beta, z, z_r] + \mathcal{L}_1[\beta, z, z_r], \quad (\text{B16})$$

where

$$\mathcal{L}_0[\beta, z, z_r] = \mathcal{L}[\beta \rightarrow 0^+, z, z_r] = \frac{1}{8(\pi\beta)^{3/2}} \exp \left(-\frac{z_r^2}{4\beta} \right)$$

and

$$\mathcal{L}_1[\beta, z, z_r] = \mathcal{L}[\beta, z, z_r] - \mathcal{L}_0[\beta, z, z_r].$$

Thus, we have

$$U = \int_0^\infty \mathcal{L}_0[\beta, z, z_r] d\beta + \int_0^\infty \mathcal{L}_1[\beta, z, z_r] d\beta = -\frac{1}{4\pi|z_r|} + F_1(z) + O(z_r), \quad (\text{B17})$$

where the function $F_1(z)$ is given by

$$\begin{aligned} F_1(z) &= \int_0^\infty \mathcal{L}_1[\beta, z, z_r = 0] d\beta = \int_0^\infty \{\mathcal{L}[\beta, z, z_r = 0] - \mathcal{L}_0[\beta, z, z_r = 0]\} d\beta \\ &= -\frac{1}{4\pi^{3/2}} \int_0^\infty d\beta \left[\frac{\omega_\perp \sqrt{\omega_z} \exp\left(\beta E' - \frac{\omega_z[\omega_z \beta + 2 \tanh(\omega_z \beta/2)]}{2[1 + \omega_z \hbar \beta \coth(\omega_z \beta)]} z^2\right)}{\sqrt{2} \sinh(\beta \omega_\perp) \sqrt{\omega_z \beta} \cosh(\omega_z \beta) + \sinh(\omega_z \beta)} - \frac{1}{2\beta^{3/2}} \right]. \end{aligned} \quad (\text{B18})$$

Using the result in Eqs. (B13) and (B17), we can obtain the behavior of the third term on the right-hand side of Eq. (B11) in the limit $|z_r| \rightarrow 0$.

Finally, we can show that last term on the right-hand side of Eq. (B11) is convergent in the limit $|z_r| \rightarrow 0$ with the following analysis, which is quite similar to the analysis around Eq. (E12) of Ref. [30]. This term is proportional to $\int_{-\infty}^{+\infty} dz' G_{E'}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}; \mathbf{0}, z', z') [\eta_\xi(z') - \eta_\xi(z)]$. By defining $u = z' - z$, we can rewrite this integration as $I \equiv \int_0^{+\infty} du [\mathcal{G}(z_r; u) B(u) + \mathcal{G}(z_r; -u) B(-u)]$, with $\mathcal{G}(z_r; u) = G_{E'}(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}; \mathbf{0}, z + u, z + u)$ and $B(u) = \eta_\xi(z + u) - \eta_\xi(z)$. In the limit $|z_r| \rightarrow 0$, the only possible cause for the divergence of I is the fact that $\mathcal{G}(z_r = 0; u)$ diverges as $1/u^2$ for $u \rightarrow 0$. However, when $u \rightarrow 0$ we also have $B(\pm u) = \pm B'u + B''u^2$, with $B' = dB(u)/du|_{u=0}$ and $B'' = d^2B(u)/du^2|_{u=0}$, which leads to $[\mathcal{G}(z_r; u) B(u) + \mathcal{G}(z_r; -u) B(-u)] \propto \frac{1}{u^2} [2B''u^2 + O(u^3)] \propto 2B'' + O(u)$. Notice that the linear terms $\pm B'u$ in $B(u)$ and $B(-u)$ cancel each other. Thus, the divergence of $\mathcal{G}(z_r = 0; u)$ is canceled by the functions $B(\pm u)$, and the integration I and the last term on the right-hand side of Eq. (B11) are thus convergent.

With our above results, especially Eqs. (B13) and (B17), we obtain the behavior of $\Psi_\xi(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2})$ in the short-range limit $|z_r| \rightarrow 0$,

$$\begin{aligned} \lim_{z_r \rightarrow 0} \Psi_\xi\left(\mathbf{0}, z - \frac{z_r}{2}, z + \frac{z_r}{2}\right) &= -\frac{1}{|z_r|} a_\xi \eta_\xi(z) + \Psi^{(0)}(\mathbf{0}, z, z) + 2\omega_\perp a_\xi \int dz' g[E - \omega_\perp; z, z; z', z'] \eta_\xi(z') \\ &\quad + 4\pi a_\xi F_1(z) \eta_\xi(z) + 4\pi a_\xi \int dz' F_2(z, z') [\eta_\xi(z') - \eta_\xi(z)] + O(z_r), \end{aligned} \quad (\text{B19})$$

where the function $F_2(z, z')$ is defined as $F_2(z, z') \equiv G_{E'}(\mathbf{0}, z, z; \mathbf{0}, z', z')$. Using Eqs. (B5) and (B10), we obtain

$$F_2(z, z') = -\int_0^\infty d\beta \frac{\omega_\perp}{8\pi^2 \sinh(\omega_\perp \beta)} \sqrt{\frac{\omega_z}{\beta \sinh(\omega_z \beta)}} \exp\left\{\beta E' - \frac{\omega_z[(z^2 + z'^2) \cosh(\omega_z \beta) - 2zz']}{2 \sinh(\omega_z \beta)} - \frac{(z - z')^2}{2\beta}\right\}. \quad (\text{B20})$$

3. Integral equation for $\eta_\xi(z)$

Substituting Eq. (B20) into Eq. (29), we obtain the integral equation for $\eta_\xi(z)$, which is Eq. (39), where $\hat{\mathcal{O}}_\xi$ is an integral operator which is defined as

$$\hat{\mathcal{O}}_\xi[\eta_\xi(z)] \equiv 2\omega_\perp a_\xi \int dz' g[E - \omega_\perp; z, z; z', z'] \eta_\xi(z') + 4\pi a_\xi F_1(z) \eta_\xi(z) + 4\pi a_\xi \int dz' F_2(z, z') [\eta_\xi(z') - \eta_\xi(z)], \quad (\text{B21})$$

with the function $g[E - \omega_\perp; z, z; z', z']$ defined in Eq. (B2), the function $F_1(z)$ defined in Eq. (B18), and the function $F_2(z, z')$ defined in Eq. (B20).

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