Nonlinear mixing of Bogoliubov modes in a bosonic Josephson junction

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We revisit the dynamics of a Bose-Einstein condensate in a double-well potential, from the regime of Josephson plasma oscillations to the self-trapping regime, by means of the Bogoliubov quasiparticle projection method. For a very small imbalance between the left and right wells only the lowest Bogoliubov mode is significantly occupied. In this regime the system performs plasma oscillations at the corresponding frequency, and the evolution of the condensate is characterized by a periodic transfer of population between the ground and the first excited state. As the initial imbalance is increased, more excited modes—though initially not macroscopically occupied—get coupled during the evolution of the system. Since their population also varies with time, the frequency spectrum of the imbalance turns out to be still peaked around a single frequency, which is continuously shifted towards lower values. The nonlinear mixing between Bogoliubov modes eventually drives the system into the the self-trapping regime, when the population of the ground state can be transferred completely to the excited states at some time during the evolution. For simplicity, here we consider a one-dimensional setup, but the results are expected to hold also in higher dimensions.

I. INTRODUCTION

Two weakly coupled Bose-Einstein condensates (BECs) in a double-well potential constitute a paradigmatic system for investigating the physics of bosonic Josephson junctions [1–4]. Owing to the nonlinear character of the interactions, this system exhibits different dynamical behaviors, ranging from Josephson plasma oscillations (in the limit of a very small imbalance between the population of the two wells) [5], to macroscopic self-trapping where—above a critical value of the imbalance—the population of the two wells is almost locked to the initial value [3,4,6]. Due to the conceptual importance of these phenomena, BECs in double-well potentials and arrays of coupled boson Josephson junctions have been extensively investigated in the last two decades both theoretically [2,3,7–30] and experimentally [4,6,31–39], as well as their counterparts with fermionic superfluid atomic samples [40–43].

The physics of these systems is well captured by a two-mode approximation of the Gross-Pitaevskii (GP) equation, each mode being localized in one of the two wells, which allows for an effective description in terms of only two parameters, namely, the population imbalance $z(t)$ and the phase difference $\phi(t)$ between the left and right components. Here, we provide a complementary description by means of

the quasiparticle projection method of Ref. [44], extending the Bogoliubov treatment of Ref. [30] to the case of arbitrary initial imbalance. For the sake of simplicity, we shall restrict the analysis to the case of a (quasi-) one-dimensional condensate [45].

We find that in the regime of a small initial imbalance, where only one Bogoliubov mode is significantly occupied and the system performs plasma oscillations at the corresponding frequency [30], the evolution of the condensate is characterized by a periodic transfer of population between the ground state and the first excited state. As the initial imbalance is increased, more Bogoliubov modes get coupled during the evolution of the system, and their population also varies with time, contrarily to what happens in a linear system. As a consequence, the frequency spectrum of the imbalance turns out to be still peaked around a single frequency which is shifted towards lower values, rather than getting relevant contributions at higher frequencies, where Bogoliubov modes are located. By further increasing the initial imbalance, the population of the ground state can be completely transferred to the excited states at some time during the evolution, driving the system into the macroscopic self-trapping regime.

The paper is organized as follows. In Sec. II we introduce the formalism, reviewing the definition of the two-mode approach (Sec. II A) and of the quasiparticle Bogoliubov expansion (Sec. II B). Then, in Sec. III we present the results by discussing the behavior of the system in the regime of Josephson plasma oscillations (Sec. III A), the self-trapping regime (Sec. III C), and that intermediate between the former two (Sec. III B), highlighting the role of nonlinear mixing (Sec. III D). Final considerations are drawn in the conclusions.
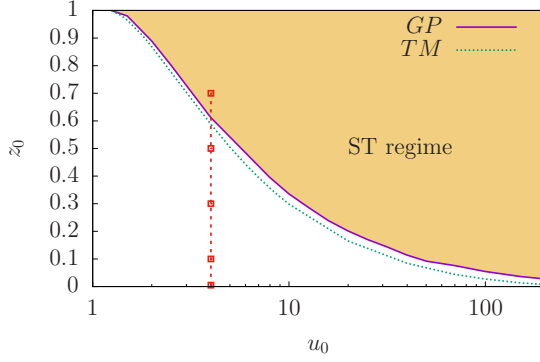
FIG. 1. Plot of the critical initial imbalance z_{0c} vs u_0, as obtained from the solution of the GP equation (solid line). The shaded area indicates the self-trapping (ST) regime, for $\mu/V_0 = 0.25$. Open squares on the vertical dashed line represent the values of z_0 and u_0 considered in this paper. The prediction of the TM model [3] in Eq. (9) is also shown as a reference (dotted line).

II. MODEL

Let us consider the following (dimensionless) Gross-Pitaevskii equation [30],

$$i\partial_t \psi(x,t) = \left[-\tfrac{1}{2}\nabla_x^2 + V(x) + u_0|\psi(x,t)|^2\right]\psi(x,t), \quad (1)$$

with

$$V(x) = \tfrac{1}{2}(x + \delta x)^2 + V_0 e^{-2x^2/w^2}, \quad (2)$$

and $\int dx|\psi(x)|^2 = 1$, describing a (quasi-) one-dimensional condensate trapped in a double-well potential. The latter is composed by a harmonic potential term, plus a barrier of intensity V_0 and width w, with δx providing a relative shift between the two (the distance between the barrier center and the minimum of the potential). Here, we are interested in describing the dynamics triggered by an initial population imbalance between the two wells. This can be obtained by preparing the system in the ground state $\psi_g(x) = \psi(x,0)$ of the above potential with $\delta x \neq 0$, and then suddenly switching $\delta x = 0$ at $t = 0$. Notice that only the harmonic potential is shifted (the barrier does not move), so that the dynamics of the system takes place in a parity symmetric potential. The ground state $\psi_g(x)$ is obtained from

$$\left[-\tfrac{1}{2}\nabla_x^2 + V(x) + u_0|\psi_g(x)|^2\right]\psi_g(x) = \mu\psi_g(x), \quad (3)$$

with μ being the condensate chemical potential. As for the parameters, here we choose $w = 0.3$ and $V_0 = 50$, that correspond to a double-well configuration within reach of current experiments (see, e.g., Ref. [43]), whereas the interaction strength u_0 and the initial shift δx are taken as free parameters, and will be varied for exploring different regimes (see below). In particular, δx is chosen in order to produce the desired initial imbalance z_0.

As it is known, in the limit of a very small initial imbalance the system performs Josephson plasma oscillations [1], and eventually enters a self-trapping (ST) regime at a critical imbalance [3] whose specific value depends on the strength u_0 of the nonlinear term (see Fig. 1). The dynamics of the system will be analyzed by means of an expansion over the

Bogoliubov modes, by comparing with the exact evolution and the two-mode (TM) approach.

A. Two-mode model

Usually, the dynamics of a condensate in a double-well potential is treated by means of the two-mode approach, which consists in writing the condensate wave function as (see, e.g., Refs. [30,46] and references therein)

$$\psi(x,t) = c_L(t)\psi_L(x) + c_R(t)\psi_R(x), \quad (4)$$

where the functions $\psi_{L,R}(x)$ are localized in the left and right well, have unit norm, and are orthogonal to each other, $\langle\psi_L|\psi_R\rangle = 0$. Though somewhat approximate—and not entirely justified from the formal point of view [30]—the two-mode model provides an effective description of the double-well system in several respects, and will be used in the rest of the discussion as a reference. Here, we construct the two modes $\psi_{L,R}(x)$ from the ground state $\psi_g(x)$ (symmetric) and the first excited solution $\psi_1(x)$ (antisymmetric) of the stationary GP equation [47]. Namely, we take the following linear combination, $\psi_{L/R} \equiv (\psi_g \pm \psi_1)/\sqrt{2}$ [46], corresponding to the most common approach in the literature [9,13,18,34,46,48–50]. Then, by defining ($\alpha = L, R$)

$$c_\alpha(t) = \sqrt{N_\alpha(t)}e^{i\phi_\alpha(t)}, \quad (5)$$

and

$$K \equiv -\int dx\,\psi_\alpha(x)\hat{H}_0\psi_\beta(x),$$

$$U_{\alpha m n \beta} \equiv u_0 \int dx\,\psi_\alpha(x)\psi_m(x)\psi_n(x)\psi_\beta(x), \quad (6)$$

one gets the following equations for the phase difference $\phi \equiv \phi_\alpha - \phi_\beta$ and the imbalance $z \equiv N_\beta - N_\alpha$ [13],

$$\frac{\dot{z}}{2K} = (2\Lambda_1 - 1)\sqrt{1 - z^2}\sin\phi + (1 - z^2)\Lambda_2\sin 2\phi, \quad (7)$$

$$\frac{\dot{\phi}}{2K} = (\Lambda - 2\Lambda_2)z + \frac{(1 - 2\Lambda_1)z}{\sqrt{1 - z^2}}\cos\phi - z\Lambda_2\cos 2\phi, \quad (8)$$

with $\Lambda \equiv U_{\alpha\alpha\alpha\alpha}/2K$, $\Lambda_1 \equiv U_{\alpha\alpha\alpha\beta}/2K$, $\Lambda_2 \equiv U_{\alpha\alpha\beta\beta}/2K$. In the following, this set of equations will be referred to as the full two-mode (FTM) model. When the terms Λ_1 and Λ_2 can be neglected, it reduces to the well-known two-mode (TM) model by Smerzi *et al.* [3].

We recall that the TM model predicts that the system enters the ST regime when the parameter Λ exceeds a critical value Λ_c. For $\phi_0 = 0$, it takes the following value: $\Lambda_c(z_0) = 2(\sqrt{1 - z_0^2} + 1/z_0^2)$ [3]. Here, we shall use as the independent parameter u_0 rather than Λ (which will depend on u_0). Then, the previous equation can be easily inverted, yielding

$$z_{0c} = [2\sqrt{\Lambda(u_0) - 1}]/\Lambda(u_0). \quad (9)$$

As shown in Fig. 1, this formula provides a good estimate for the actual critical imbalance extracted from the GP equation, in the whole range considered ($u_0 \in [1, 200]$).

B. Bogoliubov approach

As a complementary description, here we employ the quasiparticle projection method introduced by Morgan *et al.* in Ref. [44]. It amounts to a Bogoliubov expansion [51,52] where the condensate and quasiparticle populations are allowed to vary with time, namely,

$$\psi(x,t) = e^{-i\mu t/\hbar}\{\psi_g(x)[1 + b_g(t)] + \delta\psi(x,t)\}, \quad (10)$$

with

$$\delta\psi(x,t) = \sum_{i>0} b_i(t)\tilde{u}_i(x) + b_i^*(t)\tilde{v}_i^*(x). \quad (11)$$

The functions $\tilde{u}_i(x)$ and $\tilde{v}_i(x)$ are the Bogoliubov eigenmodes, with the tilde indicating that they are chosen to be orthogonal to $\psi_g(x)$ [53]. They are solutions of (from now on we fix $\psi_g^* = \psi_g$ without loss of generality) [54]

$$\begin{pmatrix} \mathcal{L} & u_0\psi_g^2 \\ -u_0\psi_g^2 & -\mathcal{L} \end{pmatrix} \begin{pmatrix} \tilde{u}_i \\ \tilde{v}_i \end{pmatrix} = \omega_i \begin{pmatrix} \tilde{u}_i \\ \tilde{v}_i \end{pmatrix}, \quad (12)$$

with

$$\mathcal{L} \equiv -\tfrac{1}{2}\nabla_x^2 + V(x) + 2g\psi_g^2 - \mu. \quad (13)$$

The solutions of Eq. (12) satisfy the following orthogonality relations [55],

$$\int dx[\tilde{u}_i^*(x)\tilde{u}_j(x) - \tilde{v}_i^*(x)\tilde{v}_j(x)] = \delta_{ij}, \quad (14)$$

$$\int dx[\tilde{u}_i(x)\tilde{v}_j(x) - \tilde{v}_i(x)\tilde{u}_j(x)] = 0. \quad (15)$$

The coefficients $b_g(t)$ and $b_i(t)$ are given by [44]

$$b_g(t) = \int dx[\psi_g(x)\psi(x,t)e^{i\mu t}] - 1, \quad (16)$$

$$b_i(t) = \int dx[\tilde{u}_i^*(x)\psi(x,t)e^{i\mu t} - \tilde{v}_i^*(x)\psi^*(x,t)e^{-i\mu t}]. \quad (17)$$

When the modes remain decoupled during the whole evolution, as it generally assumed in the standard Bogoliubov approach (see, e.g., Refs. [44,51]), the coefficients $b_i(t)$ are solutions of $i\dot{b}_i(t) = \omega_i b_i(t)$, namely,

$$b_i(t) = b_{i0}e^{-i\omega_i t}, \quad (18)$$

where the coefficients $b_{i0} \equiv b_i(0)$, which do not depend on time, are fixed by the initial conditions [see Eq. (17)]

$$b_{i0} = \int dx[\tilde{u}_i^*(x) - \tilde{v}_i^*(x)]\psi_g(x). \quad (19)$$

In the following we shall refer to this regime as the *linear* regime. This has to be contrasted with the general situation, discussed in this paper, in which the quasiparticles interact with the condensate, and both the condensate and quasiparticle populations depend on time [44], characterizing the *nonlinear mixing* regime.

Imbalance. To construct the population imbalance between the right and left wells we start by integrating the particle density

$$n(x,t) \equiv |\psi(x,t)|^2 \simeq |1 + b_g(t)|^2|\psi_g(x)|^2 + 2\,\mathrm{Re}\left[\psi_g(x)[1 + b_g^*(t)]\sum_i [b_i(t)\tilde{u}_i(x) + b_i^*(t)\tilde{v}_i^*(x)]\right] \quad (20)$$

over the positive and negative x semiaxis. By taking into account the symmetries of the problem we have

$$N_{R,L}(t) = A(t) \pm B(t), \quad (21)$$

with

$$A(t) = |1 + b_g|^2 \int_0^{+\infty} dx|\psi_g|^2 + 2\,\mathrm{Re}\left[(1 + b_g^*)\sum_{i\in\mathrm{even}}\left(b_i\int_0^{+\infty} dx\,\psi_g\tilde{u}_i + b_i^*\int_0^{+\infty} dx\,\psi_g\tilde{v}_i^*\right)\right],$$

$$B(t) \simeq 2\,\mathrm{Re}\left[(1 + b_g^*)\sum_{i\in\mathrm{odd}}\left(b_i\int_0^{+\infty} dx\,\psi_g\tilde{u}_i + b_i^*\int_0^{+\infty} dx\,\psi_g\tilde{v}_i^*\right)\right]. \quad (22)$$

Then, the *imbalance* $z(t) \equiv N_R(t) - N_L(t)$ is

$$z(t) = 2B(t). \quad (23)$$

Remarkably, only the Bogoliubov excitations with odd i contribute to the imbalance, owing to the symmetries of the system. In the *linear regime* we have (using also the fact that in our case $\tilde{u}_i$, $\tilde{v}_i$ can be chosen real without loss of generality)

$$B(t) \simeq 2\sum_{i\in\mathrm{odd}}\left[b_{i0}\int_0^{+\infty} dx\,\psi_g(\tilde{u}_i + \tilde{v}_i)\right]\cos(\omega_i t)$$

$$\equiv 2\sum_{i\in\mathrm{odd}} B_{0i}\cos(\omega_i t). \quad (24)$$

III. RESULTS AND DISCUSSION

Here, we shall discuss the evolution of the imbalance for different values of its initial value $z_0 \equiv z(0)$ [throughout this work we set $\phi_0 \equiv \phi(0) = 0$], discussing the behavior of the system in terms of the quasiparticle projection method [44] introduced in the previous section. The general behavior of $z(t)$ has already been extensively studied, at least in the framework of the TM model (see, e.g., the seminal Ref. [3]). In the rest of this paper we fix the ratio $\mu/V_0 \equiv 0.25$, a value that characterizes a typical Josephson regime (with the chemical potential much lower than the barrier height [46]). A discussion of the actual shape of the ground and first excited states (which enter explicitly in the calculations) can be found
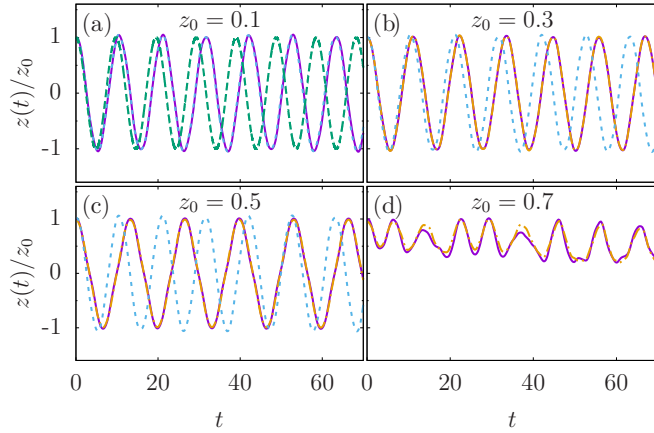
FIG. 2. Evolution of the normalized imbalance, $z(t)/z_0$, for $z_0 =$ 0.1 (a), 0.3 (b), 0.5 (c), 0.7 (d). The different lines correspond to the solution of the GP equation (solid purple line), the TM model [long-dashed green line, in (a)], and the prediction of the quasiparticle projection method in Eq. (23), with $B(t)$ given by Eq. (22) [dotted-dashed orange line, in (b)–(d)] and Eq. (24) [short-dashed cyan line, in (a)–(c)].
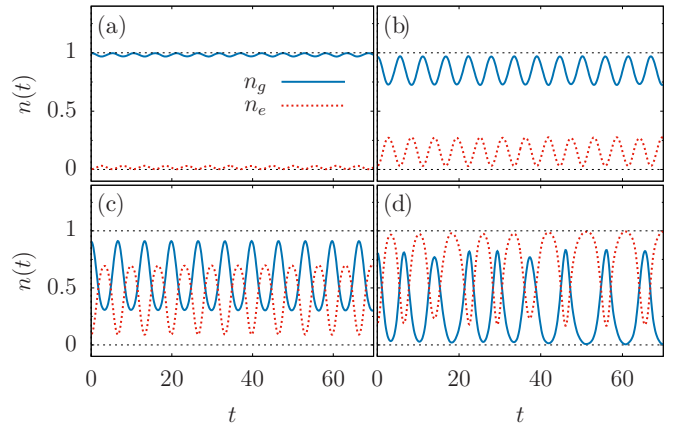

FIG. 3. Evolution of the occupation number of the ground state and of the Bogoliubov modes, n_g and n_e respectively, for $z_0 = 0.1$ (a), 0.3 (b), 0.5 (c), 0.7 (d). Notice that in (a), where only the lowest excited mode is appreciably populated, $n_e \equiv n_{e_1}$, the oscillation period of the population is half of that of the imbalance, namely, $T = \pi/\omega_1$ [see Eq. (28)]. The horizontal lines at $n = 0, 1$ are a guide to the eye.

in Ref. [30]. As an example, we consider here a moderate value of the interactions, $u_0 = 4$. There is nothing special in this value, and we have verified that the general picture discussed in the following is quite general and holds equally for higher values of the interactions (see also Ref. [30]). The explicit behavior of the imbalance evolution is shown in Fig. 2 for $z_0 = 0.1, 0.3, 0.5$, and 0.7 (open squares in Fig. 1), ranging from the regime of Josephson plasma oscillations ($z_0 \lesssim 0.1$), to the ST regime ($z_0 \gtrsim 0.62$). A detailed description of the different dynamical behaviors and of the various lines plotted in the figure is given in the following.

A. Josephson plasma oscillations

In the limit of very small imbalance, the system performs Josephson plasma oscillations characterized by a frequency ω_J. This frequency corresponds to the energy of the lowest Bogoliubov mode [30]. In fact, in this limit only one Bogoliubov mode is occupied, the system is in the *linear regime*, and $z(t)$ is well reproduced by Eqs. (23) and (24) with the only contribution of B_{01}, namely [30],

$$z(t) = 4B_{01}\cos(\omega_1 t). \qquad (25)$$

This is shown in Fig. 2(a), where the GP prediction (solid purple line) is perfectly reproduced by that of Eq. (25) (dotted cyan line). In general, if u_0 is not too large, namely, the interaction term does not exceed significantly the kinetic one, also the frequency obtained from the TM model [3,30,46,56]

$$\omega_J^{\text{TM}} = (2K/\hbar)\sqrt{1 + \Lambda} \qquad (26)$$

can provide a reasonable estimate. In the present case ($\mu/V_0 = 0.25$, $u_0 = 4$), the prediction of the TM model—that here coincides with that of the FTM model—exceeds the exact frequency by approximately a 8% ($\omega_J = 0.595$, $\omega_J^{\text{TM}} = 0.643$) [see the dotted green line in Fig. 2(a)]. This difference may increase further by increasing u_0 [30].

In this regime we also have

$$\int dx|\psi(x,t)|^2 \simeq n_g(t) + n_{e_1}(t), \qquad (27)$$

where $n_g(t) \equiv |1 + b_g(t)|^2$ represents the (relative) population of the ground state, and

$$n_{e_1}(t) \equiv |b_{10}|^2 \int dx(|\tilde{u}_1|^2 + |\tilde{v}_1|^2)$$
$$+ 2b_{10}^2 \cos(2\omega_1 t) \int dx\tilde{u}_1\tilde{v}_1 \qquad (28)$$

that of the first Bogoliubov excitation, the other excited modes being essentially irrelevant. The evolution of n_g and n_{e_1} is plotted in Fig. 3(a) for $z_0 = 0.1$ (the other three panels will be discussed later on). A sinusoidal oscillation—with frequency $2\omega_1$ [see Eq. (28)]—is clearly visible in Fig. 3(a). It corresponds to a (small) periodic transfer of population between the ground state and the first excited state, contrarily to what happens in a truly linear system, where the occupation number of each energy level is constant.

B. Intermediate regime

In general, when one increases the initial imbalance z_0, the form of $z(t)$ changes, and its frequency (the inverse of the period) changes as well [3] [see Figs. 2(b) and 2(c)]. In particular, before entering the ST regime, the imbalance is still characterized by periodic oscillations, but with a frequency ω that is shifted with respect to the plasma value ω_J as an effect of the nonlinearity [see Fig. 4(a)]. These changes are reflected in the change of the Fourier spectrum, in Fig. 4(b).

Remarkably, in this regime the spectrum is still peaked around a single frequency, that is shifted continuously towards lower-frequency values with respect to $\omega_J \equiv \omega_1$, contrarily to the naive expectation of having more Bogoliubov modes macroscopically occupied [the first Bogoliubov frequencies is indicated by the solid (red) point on the horizontal axis of
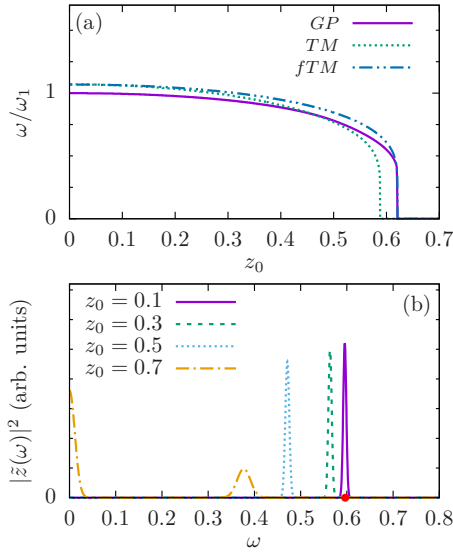
FIG. 4. (a) Frequency $\omega = 2\pi/T$ of the oscillations of $z(t)$ as a function of the initial imbalance z_0 obtained from the solution of the GP equation and from the TM and the fTM models. (b) Fourier spectrum of $z(t)$ (calculated over an interval of size $t = 10^3$), for different values of the initial imbalance z_0. The (red) point on the horizontal axis indicates the value of the first Bogoliubov frequency, $\omega_1 \simeq 0.6$. Higher modes lie far outside the present range (e.g., $\omega_2 \simeq 1.84$, $\omega_3 \simeq 2.19$).

Fig. 4(b), higher modes lie far outside the present range]. In fact, we find that the system exits the linear regime, namely, Eq. (24) fails in reproducing the actual behavior of $z(t)$—see Figs. 2(b) and 2(c)—even if higher Bogoliubov modes have an initial population that is still below 1% that of the lowest mode. In other words, the linear approach fails not because some of the other excited modes are *initially* macroscopically occupied (as it would be the case for a truly linear system), but because of the nonlinear mixing during the evolution of the system.

In this regime, the analog of the decomposition (28) becomes more complicated as the contribution of all the excited modes to the total density is now

$$n_e(t) = \sum_{i,j} \left[b_i(t)b_j^*(t) \int dx (\tilde{u}_i \tilde{u}_j^* + \tilde{v}_j^* \tilde{v}_i) \right.$$
$$\left. + b_i(t)b_j(t) \int dx \tilde{u}_i \tilde{v}_j + b_i^*(t)b_j^*(t) \int dx \tilde{u}_j^* \tilde{v}_i^* \right], \quad (29)$$

meaning that it is not possible to write the total density as the sum of separate contributions of each Bogoliubov mode. The evolution of $n_g(t)$ and $n_e(t)$ is shown in Fig. 3 for increasing values of the initial imbalance z_0. This figure shows that the transfer of population between the ground and the excited states increases by increasing z_0. Initially, when the system exits the linear regime but z_0 is not too large [e.g., $z_0 = 0.3$, Fig. 3(b)], $n_g(t)$ and $n_e(t)$ are still characterized by sinusoidal oscillations. For larger values of z_0, the oscillations in the population deviate from this behavior, as does the corresponding imbalance [see, e.g., Figs. 3(c) and 2(c)]. In any case, the oscillations of $n_g(t)$ are always in phase with those

of $|z(t)|$ (that is, the maximal imbalance is obtained when the population of the ground state is maximal).

C. Self-trapping regime

By further increasing the initial imbalance z_0, the period of $z(t)$ gets larger and larger (see also Ref. [3]), and eventually diverges at the critical value z_{0c} where $\omega \propto 1/T \to 0$ [see Fig. 4(a)] ($z_{0c} \simeq 0.62$ in the present case). Notice that the value of z_{0c} obtained from the solution of the GP equation is reproduced with great accuracy by the FTM model (we have verified that this holds true even for values of u_0 larger than that considered in the present paper). Remarkably, the *onset of ST* corresponds to a situation in which the population of the ground state can be transferred completely to the excited states, namely, when $n_g(t) = 0$ at some t during the evolution [see Fig. 3(d)]. This feature is indeed a distinctive characteristic of the ST regime. In this regime the imbalance is stuck on the positive side (or the negative one, depending on the initial conditions), still oscillating, but with an irregular pattern [3]. The latter reflects in the shape of the frequency spectrum, that significantly broadens and acquires a relevant contribution from the low-frequency region, $\omega \simeq 0$ [dotted-dashed orange line in Fig. 4(b)].

D. Nonlinear mixing

Owing to the coupling between the different Bogoliubov modes [see Eq. (29)], we argue that $|b_i(t)|^2$ cannot be identified with the occupation number of the ith quasiparticle level, contrarily to the the interpretation given in Ref. [44]. However, since the coefficients $b_i(t)$ represent the quasiparticle amplitudes in the sense of the expansion (11), in the following we shall consider their modulus squared $|b_i(t)|^2$ as a measure of the weight of each mode in the system dynamics. Their evolution (for $i = 1, 2, 3, 4$) is shown in Fig. 5 (left), along with the corresponding time-averaged values

$$\langle |b_i|^2 \rangle(t) \equiv \frac{1}{t} \int_0^t |b_i(t')|^2 dt', \quad (30)$$

for the same values of the initial imbalance as in the previous figures, namely, $z_0 = 0.1, 0.3, 0.5, 0.7$. In Fig. 5 (right) we show the corresponding region of the complex plane spanned by the real and imaginary parts of $b_i(t)$ [here normalized to $b_1(0)$, for ease of visualization] during the evolution of the system. In Fig. 5(e) we also show the trajectory of the lowest Bogoliubov mode ($i = 1$) for $z_0 = 0.005$, indicating that in the limit $z_0 \to 0$ the expected behavior is recovered: In this case the coefficient $b_1(t)$ is constant in modulus as dictated by Eq. (18) for the linear regime, and the contribution of all higher excited modes is negligible [30]. As z_0 is increased, the dynamics in the complex plane becomes chaotic, each mode spanning a larger portion of the plane. Notice also the change in the orbit shape, from circular (in the limit $z_0 \to 0$) to elliptical (for $z_0 \gtrsim 0.1$). Both left and right panels evidence a mixing between different modes, especially those with $i = 2$ and $i = 3$. Remarkably, the mode $i = 2$—which, being even, does not contribute directly to the imbalance [see Eqs. (22) and (23)]—indeed affects it through the mixing with other modes during the evolution of the system.
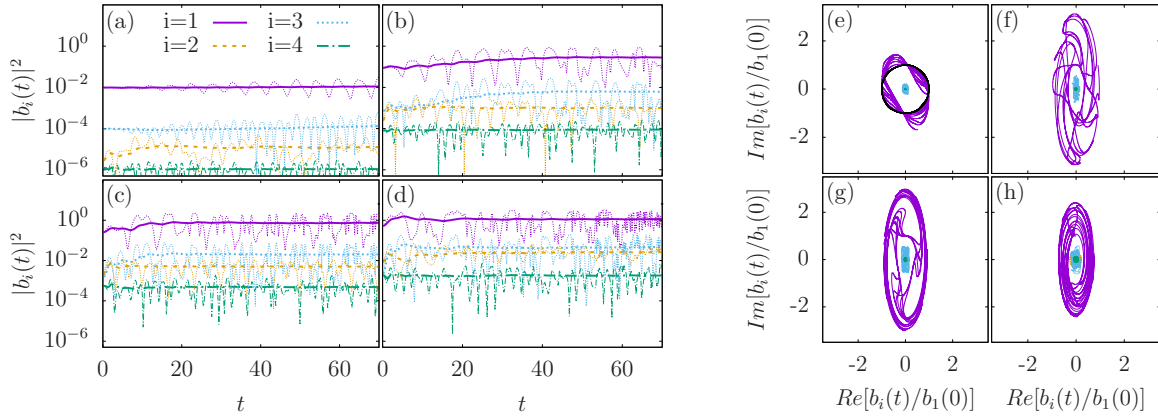
FIG. 5. Left: Evolution of (the modulus square of) the quasiparticle amplitudes $|b_i(t)|^2$ ($i = 1, 2, 3, 4$, dotted lines, from top to bottom) and of their corresponding time-averaged values $\langle |b_i|^2 \rangle(t)$ [solid lines, see Eq. (30)], for $z_0 = 0.1$ (a), 0.3 (b), 0.5 (c), 0.7 (d). Right: Evolution of the real and imaginary part of $b_i(t)$ [normalized to the initial value of $b_1(t)$] in the complex plane, for $z_0 = 0.1$ (e), 0.3 (f), 0.5 (g), 0.7 (h). The legends of the two figures are the same. The (black) circle in (e) represents the trajectory for mode 1 in the limit $z_0 \to 0$ (here, $z_0 = 0.005$).

IV. CONCLUSIONS

We have analyzed the dynamics of a (quasi-) one-dimensional Bose-Einstein condensate in a double-well potential [57], from the regime of Josephson plasma oscillations to the self-trapping regime, by means of the Bogoliubov quasiparticle projection method [44]. In the limit of a very small initial imbalance, the system performs Josephson plasma oscillations characterized by the frequency of the lowest Bogoliubov mode (the only Bogoliubov mode being significantly occupied) [30]. In this regime, the evolution of the system is characterized by a periodic transfer of population between the ground state and the first excited state. As the initial imbalance is increased, the system still performs periodic oscillations between the left and right wells, but with a frequency that is continuously shifted towards values lower than the plasma frequency. This occurs because of the nonlinear mixing of the Bogoliubov modes during the evolution of the system, and not because some of the excited modes (besides the lowest one) are initially macroscopically occupied, contrarily to what

happens in a linear system. The frequency spectrum of the imbalance is therefore still peaked around a single frequency, and the corresponding period diverges when the system enters the self-trapping regime. This corresponds to a situation in which the population of the ground state can be transferred completely to the excited states at some time during the evolution. This feature is indeed a distinctive characteristic of the ST regime. The present picture is expected to hold also in higher dimensions.

[1] J. Javanainen, Phys. Rev. Lett. **57**, 3164 (1986).

[2] G. J. Milburn, J. Corney, E. M. Wright, and D. F. Walls, Phys. Rev. A **55**, 4318 (1997).

[3] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).

[4] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. **95**, 010402 (2005).

[5] B. D. Josephson, Phys. Lett. **1**, 251 (1962).

[6] R. Gati, M. Albiez, J. Fölling, B. Hemmerling, and M. Oberthaler, Appl. Phys. B **82**, 207 (2006).

[7] F. Dalfovo, L. Pitaevskii, and S. Stringari, Phys. Rev. A **54**, 4213 (1996).

[8] I. Zapata, F. Sols, and A. J. Leggett, Phys. Rev. A **57**, R28 (1998).

[9] S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy, Phys. Rev. A **59**, 620 (1999).

[10] S. Giovanazzi, A. Smerzi, and S. Fantoni, Phys. Rev. Lett. **84**, 4521 (2000).

[11] Y. Zhang and H. J. W. Müller-Kirsten, Phys. Rev. A **64**, 023608 (2001).

[12] E. Sakellari, M. Leadbeater, N. J. Kylstra, and C. S. Adams, Phys. Rev. A **66**, 033612 (2002).

[13] D. Ananikian and T. Bergeman, Phys. Rev. A **73**, 013604 (2006).

[14] M. Rosenkranz, D. Jaksch, F. Y. Lim, and W. Bao, Phys. Rev. A **77**, 063607 (2008).

[15] S. Giovanazzi, J. Esteve, and M. Oberthaler, New J. Phys. **10**, 045009 (2008).

[16] R. Ichihara, I. Danshita, and T. Nikuni, Phys. Rev. A **78**, 063604 (2008).

[17] B. Juliá-Díaz, M. Melé-Messeguer, M. Guilleumas, and A. Polls, Phys. Rev. A **80**, 043622 (2009).

[18] B. Juliá-Díaz, J. Martorell, M. Melé-Messeguer, and A. Polls, Phys. Rev. A **82**, 063626 (2010).

[19] C. Ottaviani, V. Ahufinger, R. Corbalán, and J. Mompart, Phys. Rev. A **81**, 043621 (2010).

[20] B. Juliá-Díaz, J. Martorell, and A. Polls, Phys. Rev. A **81**, 063625 (2010).

[21] B. Juliá-Díaz, D. Dagnino, M. Lewenstein, J. Martorell, and A. Polls, Phys. Rev. A **81**, 023615 (2010).

[22] M. Melé-Messeguer, B. Juliá-Díaz, M. Guilleumas, A. Polls, and A. Sanpera, New J. Phys. **13**, 033012 (2011).

[23] S. Wüster, B. J. Dabrowska-Wüster, and M. J. Davis, Phys. Rev. Lett. **109**, 080401 (2012).

[24] B. Juliá-Díaz, E. Torrontegui, J. Martorell, J. G. Muga, and A. Polls, Phys. Rev. A **86**, 063623 (2012).

[25] B. Juliá-Díaz, T. Zibold, M. K. Oberthaler, M. Melé-Messeguer, J. Martorell, and A. Polls, Phys. Rev. A **86**, 023615 (2012).

[26] B. Juliá-Díaz, A. D. Gottlieb, J. Martorell, and A. Polls, Phys. Rev. A **88**, 033601 (2013).

[27] D. M. Jezek, P. Capuzzi, and H. M. Cataldo, Phys. Rev. A **87**, 053625 (2013).

[28] S. Li, S. R. Manmana, A. M. Rey, R. Hipolito, A. Reinhard, J.-F. Riou, L. A. Zundel, and D. S. Weiss, Phys. Rev. A **88**, 023419 (2013).

[29] K. Sakmann, A. I. Streltsov, O. E. Alon, and L. S. Cederbaum, Phys. Rev. A **89**, 023602 (2014).

[30] A. Burchianti, C. Fort, and M. Modugno, Phys. Rev. A **95**, 023627 (2017).

[31] F. Cataliotti, S. Burger, C. Fort, and P. Maddaloni, Science **293**, 843 (2001).

[32] Th. Anker, M. Albiez, R. Gati, S. Hunsmann, B. Eiermann, A. Trombettoni, and M. K. Oberthaler, Phys. Rev. Lett. **94**, 020403 (2005).

[33] R. Gati, B. Hemmerling, J. Fölling, M. Albiez, and M. K. Oberthaler, Phys. Rev. Lett. **96**, 130404 (2006).

[34] R. Gati and M. Oberthaler, J. Phys. B: At., Mol. Opt. Phys. **40**, R61 (2007).

[35] S. Levy, E. Lahoud, I. Shomroni, and J. Steinhauer, Nature (London) **449**, 579 (2007).

[36] L. J. LeBlanc, A. B. Bardon, J. McKeever, M. H. T. Extavour, D. Jervis, J. H. Thywissen, F. Piazza, and A. Smerzi, Phys. Rev. Lett. **106**, 025302 (2011).

[37] A. Trenkwalder, G. Spagnolli, G. Semeghini, S. Coop, M. Landini, P. Castilho, L. Pezze, G. Modugno, M. Inguscio, A. Smerzi, and M. Fattori, Nat. Phys. **12**, 826 (2016).

[38] R. Labouvie, B. Santra, S. Heun, and H. Ott, Phys. Rev. Lett. **116**, 235302 (2016).

[39] G. Spagnolli, G. Semeghini, L. Masi, G. Ferioli, A. Trenkwalder, S. Coop, M. Landini, L. Pezzè, G. Modugno, M. Inguscio, A. Smerzi, and M. Fattori, Phys. Rev. Lett. **118**, 230403 (2017).

[40] A. Spuntarelli, P. Pieri, and G. C. Strinati, Phys. Rev. Lett. **99**, 040401 (2007).

[41] F. Ancilotto, L. Salasnich, and F. Toigo, Phys. Rev. A **79**, 033627 (2009).

[42] P. Zou and F. Dalfovo, J. Low Temp. Phys. **177**, 240 (2014).

[43] G. Valtolina, A. Burchianti, A. Amico, E. Neri, K. Xhani, J. A. Seman, A. Trombettoni, A. Smerzi, M. Zaccanti, M. inguscio, and G. Roati, Science **350**, 1505 (2015).

[44] S. Morgan, S. Choi, K. Burnett, and M. Edwards, Phys. Rev. A **57**, 3818 (1998).

[45] For the definition of a (quasi-) one-dimensional condensate, see, e.g., Ref. [58] and references therein.

[46] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity*, International Series of Monographs on Physics (Oxford University Press, Oxford, UK, 2016).

[47] The stationary GP equation is solved by imaginary time evolution [51]. Once the ground state ψ_g has been obtained, the first excited solution is found by the same approach, requiring it to be orthogonal to $\psi_g(x)$.

[48] E. Sakellari, N. P. Proukakis, M. Leadbeater, and C. S. Adams, New J. Phys. **6**, 42 (2004).

[49] I. Danshita, K. Egawa, N. Yokoshi, and S. Kurihara, J. Phys. Soc. Jpn. **74**, 3179 (2005).

[50] S. K. Adhikari, H. Lu, and H. Pu, Phys. Rev. A **80**, 063607 (2009).

[51] F. Dalfovo, S. Giorgini, L. P. Pitaevsii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[52] Y. Castin, in *Coherent Atomic Matter Waves*, edited by R. Kaiser, C. Westbrook, and F. David, Lecture Notes of Les Houches Summer School, LXXII, 1999 (Springer, Berlin, 2001), pp. 1–136.

[53] In general, the solutions of Eq. (12) are not orthogonal to ground-state solution ψ_g. However, one has the freedom to impose that condition, posing [44] $u_i = \tilde{u}_i - a_i \psi_g$, $\tilde{v}_i^* = \tilde{v}_i^* + a_i^* \psi_g$, with $a_i = \int \psi_g^* u_i = -\int \psi_g v_i$.

[54] The Bogoliubov equations are transformed into a system of linear algebraic equations equations by grid discretization.

[55] In general, there are three different classes of solutions of Eq. (12), with "norm" $\int \tilde{u}_i^* \tilde{u}_j - \tilde{v}_i^* \tilde{v}_j = \pm \delta_{ij}$, 0; only one of the two families with norm ± 1 has to be considered in the expansion (11) [52].

[56] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).

[57] The present analysis, performed within the framework of the GP mean-field theory, is aimed at describing the short-time behavior of the system. For longer times, the mean-field approach may break down due to the fragmentation of the condensate [29].

[58] M. Modugno, G. Pagnini, and M. A. Valle-Basagoiti, Phys. Rev. A **97**, 043604 (2018).