Collisionless dynamics in two-dimensional bosonic gases

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We study the dynamics of dilute and ultracold bosonic gases in a quasi-two-dimensional (quasi-2D) configuration and in the collisionless regime. We adopt the 2D Landau-Vlasov equation to describe a three-dimensional gas under very strong harmonic confinement along one direction. We use this effective equation to investigate the speed of sound in quasi-2D bosonic gases, i.e., the sound propagation around a Bose-Einstein distribution in collisionless 2D gases. We derive coupled algebraic equations for the real and imaginary parts of the sound velocity, which are then solved also taking into account the equation of state of the 2D bosonic system. Above the Berezinskii-Kosterlitz-Thouless critical temperature we find that there is rapid growth of the imaginary component of the sound velocity, which implies a strong Landau damping. Quite remarkably, our theoretical results are in good agreement with very recent experimental data obtained with a uniform 2D Bose gas of ^{87}Rb atoms.

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I. INTRODUCTION

The Boltzmann-Vlasov equation is the most relevant tool to investigate the kinetics of three-dimensional (3D) quantum gases made of out-of-condensate atoms [1–6]. In the collisionless regime this equation reduces to the Landau-Vlasov equation, where the collisional integral is neglected but the mean-field interaction potential is still present and supports collective modes [7–9]. In the case of fermionic gases the speed of sound in this collisionless regime is the well-known zero-sound velocity of fermions around the Fermi-Dirac distribution [9,10].

In two-dimensional (2D) uniform systems the Mermin-Wagner-Hohenberg theorem [11,12] precludes Bose-Einstein condensation at finite temperature, but quasicondensation and superfluidity are possible below the Berezinskii-Kosterlitz-Thouless critical temperature T_c [13,14]. Very recently, the speed of sound in a uniform quasi-2D Bose gas made of ^{87}Rb atoms was measured [15,16]. These experimental results are in agreement with theoretical predictions [17] based on the two-fluid hydrodynamics of Landau-Khalatnikov only well below T_c.

The authors of [15,16] explain the discrepancy above T_c by suggesting that the experimental conditions are such that in this case collisions are not efficient enough to ensure the local thermodynamic equilibrium required by hydrodynamics and therefore the dynamics is collisionless.

In this paper we suppose that also below T_c, where the superfluid component is present, the dynamics of the normal component is collisionless and therefore the dynamics of the whole fluid is not collisional. To substantiate this hypothesis we investigate the collisionless regime by using an effective 2D Landau-Vlasov equation. We study the speed of sound around a spatially uniform Bose-Einstein distribution. We derive algebraic formulas for the real and imaginary parts of the speed of sound as a function of both temperature and interaction strength. Quite remarkably, our theoretical results for the real part of the sound velocity are in good agreement with the experimental data of Refs. [15,16]. Moreover, we find that the imaginary part of the sound velocity is negligible below the critical temperature T_c, while it becomes sizable close to and above T_c, again in agreement with the recent experiment [16].

II. KINETIC APPROACH FOR THE 2D BOSE GAS

Let us begin by considering a dilute and ultracold three-dimensional (3D) gas made of N identical bosonic atoms of mass m, whose mutual interaction is modeled through a zero-range pseudopotential where $g = 4\pi\hbar^2 a_s/m$ is the 3D interaction strength and a_s is the 3D s-wave scattering length. We assume that the bosonic system is under external confinement given by the trapping potential

$$U_{\rm ext}(\mathbf{r}, z) = \mathcal{U}(\mathbf{r}) + \tfrac{1}{2} m\omega_z^2 z^2, \tag{1}$$

which is the sum of a generic potential $\mathcal{U}(\mathbf{r})$ in the x-y plane, with $\mathbf{r} = (x, y)$ being the 2D position, and a harmonic confinement along the z axis.

An effective 2D configuration can be realized when the harmonic confinement along the z axis is tight enough. In order to effectively constrain atoms on a plane, the energy $\hbar\omega_z$ of longitudinal confinement must be much larger than the planar average kinetic energy $(p_x^2 + p_y^2)/(2m)$, with $\mathbf{p} = (p_x, p_y)$ being the planar linear momentum, a condition actual experiments can provide quite easily. The 3D system is then forced to occupy the longitudinal ground state along the confining axis, and one finds [18] that the planar distribution $f(\mathbf{r}, \mathbf{p})$ of atoms in the four-dimensional single-particle phase space $[(\mathbf{r}, \mathbf{p}) = (x, y, p_x, p_y)]$ satisfies the effective 2D

Landau-Vlasov equation [9,10]:

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}-\nabla_{\mathbf{r}}(\mathcal{U}+\mathcal{U}_{\rm mf})\cdot\nabla_{\mathbf{p}}\right]f(\mathbf{r},\mathbf{p},t)=0, \tag{2}$$

where

$$\mathcal{U}_{\rm mf}(\mathbf{r},t)=g_{2D}\int\frac{d^2\mathbf{p}}{(2\pi\hbar)^2}\ f(\mathbf{r},\mathbf{p},t) \tag{3}$$

is the self-consistent Hartree-Fock dynamical mean-field term [6,19,20] and the memory of the original 3D character of the system is encoded in the renormalized 2D coupling constant

$$g_{2D}=\frac{\sqrt{8\pi}\hbar^2}{m}\left(\frac{a_s}{a_z}\right), \tag{4}$$

with $a_z=\sqrt{\hbar/(m\omega_z)}$ being the characteristic length of the axial harmonic confinement.

III. COLLECTIVE DYNAMICS IN COLLISIONLESS 2D BOSE GAS

The calculation of transport quantities requires the solution of Eq. (2). In the following we prove that a collisionless dynamical description based on Eq. (2) recovers experimental data obtained in a homogeneous configuration of area L^2, realized by implementing a box potential on the x-y plane [15,16]. Thus, we set $\mathcal{U}(\mathbf{r})=0$ and also

$$f(\mathbf{r},\mathbf{p},t)=f_0(\mathbf{p})+\delta f(\mathbf{r},\mathbf{p},t), \tag{5}$$

where $f_0(\mathbf{p})$ is a stationary and isotropic distribution and $\delta f(\mathbf{r},\mathbf{p},t)$ is a very small perturbation around it. It follows that the linearized Landau-Vlasov equation for $\delta f(\mathbf{r},\mathbf{p},t)$ reads

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}\right]\delta f(\mathbf{r},\mathbf{p},t) = g_{2D}\int\frac{d^2\mathbf{p}'}{(2\pi\hbar)^2}\nabla_{\mathbf{r}}\delta f(\mathbf{r},\mathbf{p}',t)\cdot\nabla_{\mathbf{p}}f_0(\mathbf{p}). \tag{6}$$

Performing the Fourier transform of this equation according to $\widehat{\delta f}(\mathbf{k},\mathbf{p},\omega)=\int dt\int d^2\mathbf{r}\delta f(\mathbf{r},\mathbf{p},t)\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, with $\mathbf{k}$ being a 2D wave vector and ω being the angular frequency, one finds an implicit formula for the dispersion relation [9], given by

$$1-g_{2D}\int\frac{d^2\mathbf{p}}{(2\pi\hbar)^2}\,\frac{\mathbf{k}\cdot\nabla_{\mathbf{p}}f_0(\mathbf{p})}{\mathbf{p}\cdot\mathbf{k}/m-\omega}=0. \tag{7}$$

Note that this equation is nothing other than the condition to find the pole of the dynamic response function of the system within the random-phase approximation [10]. Equation (6) is also called linearized Boltzmann transport equation without the collisional term. In Ref. [21] it was solved numerically by preparing the system at equilibrium in the presence of a weak stationary potential generating a sinusoidal density modulation of a given wavelength. Then the potential was suddenly removed to generate a damped time-dependent oscillation and hence the speed of sound.

On the contrary, here we directly solve Eq. (7) by a fully analytical approach.

In Eq. (7) there is a singularity on the integration path for $\omega=\mathbf{p}\cdot\mathbf{k}/m$. In order to attach a meaning to the integral, we must interpret ω as a complex quantity, i.e., $\omega=\omega_R+i\omega_I$, where $\omega_I>0$ in order to avoid an exponential growth of the perturbation [9].

Equation (7) can be further simplified by assuming, without loss of generality, that $\mathbf{k}\parallel\hat{e}_x$, i.e., $\mathbf{k}=(k,0)$. In this way one finds

$$1-g_{2D}\int\frac{dp_x}{(2\pi\hbar)}\frac{\partial\tilde{f}_0(p_x)}{\partial p_x}\frac{1}{\frac{p_x}{m}-c}=0, \tag{8}$$

where $c=\omega/k$ and $\tilde{f}_0(p_x)=\int f_0(p_x,p_y)dp_y/(2\pi\hbar)$. Clearly, from Eq. (8) one can extract the speed c of sound in our collisionless regime. This velocity is, in general, a complex number such that $c=\omega/k=c_R+ic_I$, with $c_R=\omega_R/k$ and $c_I=\omega_I/k$.

In the limit of a weakly damped wave, i.e., $c_I\ll c_R$, an elegant formulation is provided for the real and imaginary parts of c [18]. In particular, one finds two coupled equations for the real part c_R and the imaginary part c_I of the speed of sound. The equation derived from the real part of Eq. (8) reads

$$1-g_{2D}\mathcal{P}\int\frac{dp_x}{(2\pi\hbar)}\left[\frac{\partial\tilde{f}_0(p_x)/\partial p_x}{p_x/m-c_R}\right]-\pi c_I\frac{\partial\phi(c)}{\partial c}\bigg|_{c_R}=0, \tag{9}$$

where we denote $\phi(c)=\frac{mg_{2D}}{(2\pi\hbar)}\frac{\partial\tilde{f}_0}{\partial p_x}|_{p_x=mc}$ and $\mathcal{P}$ means the principal value.

The equation derived from the imaginary part of Eq. (8) is instead given by

$$c_I=\frac{\pi\frac{\partial\tilde{f}_0(p_x)}{\partial p_x}\Big|_{p_x=mc_R}}{\frac{\partial}{\partial c_R}\left\{\mathcal{P}\int\frac{dp_x}{(2\pi\hbar)}\left[\frac{\partial\tilde{f}_0(p_x)/\partial p_x}{p_x/m-c_R}\right]\right\}}. \tag{10}$$

IV. SOUND VELOCITY FOR THE 2D BOSE GAS

In order to describe the behavior of the quasi-2D uniform Bose gas below or just above the critical temperature, we choose the Bose-Einstein distribution function

$$f_0(\mathbf{p})=\frac{1}{L^2}\frac{1}{e^{\beta\left(\frac{p^2}{2m}+g_{2D}n-\mu\right)}-1} \tag{11}$$

as the thermal equilibrium distribution of 2D weakly interacting bosonic atoms with uniform 2D number density $n=N/L^2$, where $\beta\equiv(k_BT)^{-1}$, k_B is the Boltzmann constant, and T is the absolute temperature. Here μ is the 2D chemical potential of the interacting system. Clearly, the Hartree interaction term $g_{2D}n$ can be formally removed by introducing a shifted chemical potential $\tilde{\mu}=\mu-g_{2D}n$.

The equation of state, relating the shifted chemical potential $\tilde{\mu}$ to the number density $n=N/L^2$, is simply derived from the normalization condition

$$N=\int\frac{d^2\mathbf{r}d^2\mathbf{p}}{(2\pi\hbar)^2}f_0(\mathbf{p}), \tag{12}$$

resulting in

$$\tilde{\mu}=k_BT\ \ln(1-e^{-T_B/T}), \tag{13}$$

where $k_BT_B=2\pi\hbar^2n/m$ is the temperature of the Bose degeneracy and, clearly, $\tilde{\mu}<0$.

The analytical computation of the dispersion relation can be simplified for temperatures $T \ll T_B$ and, at the same time, $c_R^2 \ll k_B T/m$. Within this range of parameters one is allowed to write $f_0(\mathbf{p})L^2 \simeq k_B T/[p^2/(2m) - \tilde{\mu}]$, from which $L^2 \tilde{f}_0(p_x) = k_B T/(\hbar\sqrt{(p_x^2/m) - 2\tilde{\mu}})$. Consequently, the coupled equations (9) and (10) for the real and imaginary parts of the zero-sound velocity respectively read

$$1 + \frac{\tilde{g}_{2D} k_B T}{2\pi}\left[\frac{2}{mc_R^2 - 2\tilde{\mu}} + \frac{\sqrt{mc_R^2}}{\left(mc_R^2 - 2\tilde{\mu}\right)^{3/2}} \ln\left(\frac{\sqrt{mc_R^2 - 2\tilde{\mu}} - \sqrt{mc_R^2}}{\sqrt{mc_R^2 - 2\tilde{\mu}} + \sqrt{mc_R^2}}\right)\right] + \tilde{g}_{2D} k_B T c_I \frac{mc_R^2 + \tilde{\mu}}{\sqrt{m}\left(mc_R^2 - 2\tilde{\mu}\right)^{5/2}} = 0, \tag{14}$$

$$c_I = -\frac{\frac{c_R}{\sqrt{m}\left(mc_R^2 - 2\tilde{\mu}\right)^{3/2}}}{\frac{6c_R}{\left(mc_R^2 - 2\tilde{\mu}\right)^2} + \frac{2\left(mc_R^2 + \tilde{\mu}\right)}{\sqrt{m}\left(mc_R^2 - 2\tilde{\mu}\right)^{5/2}} \ln\left(\frac{\sqrt{mc_R^2 - 2\tilde{\mu}} - \sqrt{mc_R^2}}{\sqrt{mc_R^2 - 2\tilde{\mu}} + \sqrt{mc_R^2}}\right)}. \tag{15}$$

By inserting Eq. (15) in Eq. (14) we get an equation for c_R. This equation can be easily solved numerically, and taking into account Eq. (13), one finds the real part c_R of the zero-sound velocity as a function of temperature T and adimensional interaction strength $\tilde{g}_{2D}$.

In Fig. 1 we compare the solution of Eq. (14) with the experimental data reported in Ref. [16]. The agreement between our results and the experimental points is excellent in the low-temperature regime and still good close to the superfluid threshold given by the Berezinskii-Kosterlitz-Thouless critical temperature T_c. The velocity c_R does not display any discontinuity at the critical temperature T_c. This feature marks a crucial difference with respect to first-sound and second-sound velocities calculated within the superfluid Landau-Khalatnikov model, which intrinsically relies upon a collisional dynamics of the normal component [22,23]. Despite the similar behaviors exhibited far below T_c by the second-sound velocity c_2 [17] and our collisionless velocity c_R, the former is related to the superfluid density, and consequently, it jumps to zero at T_c [17].

The dashed line in Fig. 1 is obtained by using Eq. (14) with $c_I = 0$. Comparing the dashed line with the solid line, which is instead derived solving the coupled equations (14) and (15), one clearly sees the increasingly relevant role played by the imaginary part c_I (the so-called Landau damping) above T_c.

In Fig. 2 we report the absolute value of c_I as predicted by Eq. (15), where c_R is simply the solution of Eq. (14). We remark that Eqs. (9) and (10) are derived by assuming a weakly damped perturbation, i.e., $c_I \ll c_R$. From Fig. 2 it appears clear that our approximation scheme is surely reliable for low temperatures, where Landau damping plays a negligible role, but also in the proximity of the transition temperature T_c. Quite remarkably, the rapid growth of c_I/c_R with the temperature T above T_c is in agreement with the large damping of sound oscillations found in Ref. [16]. A large value of c_I/c_R also signals the breaking of our theoretical scheme, also if the theoretical results reproduce the experimental data.

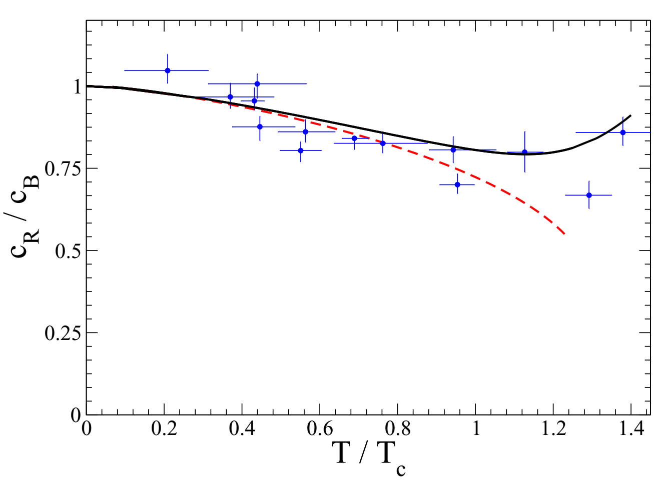

FIG. 1. Sound velocity c_R in units of $c_B = \sqrt{g_{2D} n/m}$ as a function of the scaled temperature T/T_c for $\tilde{g}_{2D} \simeq 0.16$. The solid black line represents our prediction based on Eqs. (14) and (15), while the blue dots are the experimental data of Ref. [16]. The red dashed line is obtained by using Eq. (14) with $c_I = 0$. On the basis of universal relations [20], for $\tilde{g}_{2D}$ the Berezinskii-Kosterlitz-Thouless critical temperature is $T_c = 0.13\ T_B$.

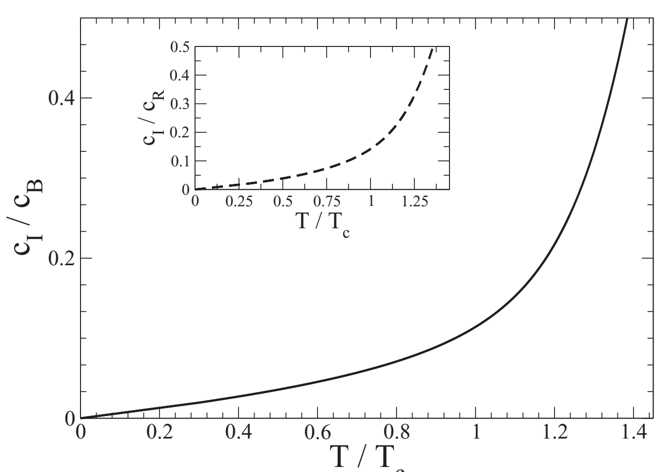

FIG. 2. Imaginary part c_I of the sound velocity in units of $c_B = \sqrt{g_{2D} n/m}$ as a function of the scaled temperature T/T_c for $\tilde{g}_{2D} \simeq 0.16$. The solid black line is obtained from Eq. (15), where c_R was derived by solving Eq. (14). Inset: Ratio between the imaginary and the real parts of c as a function of the temperature.

V. CONCLUSIONS

We have analyzed the sound propagation in collisionless bosonic gases assuming a 2D configuration. By solving the linearized 2D Landau-Vlasov equation in the degenerate regime, where bosonic statistical effects play a relevant role, we have derived an integral equation for the speed of sound as a function of temperature and interaction strength. From this integral equation we have obtained two coupled algebraic equations for the real and imaginary parts of the sound velocity. We have also compared our theoretical results with experimental data of a recent experiment [15,16], where the ^{87}Rb atoms of the bosonic cloud are expected to be in the collisionless regime. This expectation is fully confirmed: the agreement between our theory and the experiment is very encouraging. Our theoretical analysis strongly suggests that the density perturbation used in the experiment of Ref. [16] has excited the "bosonic zero sound," i.e., the sound of a collisionless bosonic fluid. For a superfluid system, a density perturbation can be used to excite the second sound only if the system is weakly interacting and collisional [17]. By increasing the interaction strength g_{2D} the 2D bosonic system enters in the collisional regime where the Landau-Vlasov equation (2) loses its validity. The collisional regime is, in fact, correctly described by the two-fluid model of Landau-Khalatnikov, which reduces to the usual hydrodynamics above the critical temperature T_c.

Note added. Recently, a theoretical preprint on the same topic appeared [21]. The conclusions of Ref. [21], based on the stochastic Gross-Pitaevskii equation and dynamic response function, are similar to ours.

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