State-independent uncertainty relations

Hubert de Guise,[1] Lorenzo Maccone,[2] Barry C. Sanders,[3,4,*] and Namrata Shukla[3]
[1] *Department of Physics, Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1*
[2] *Dip. Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy*
[3] *Institute for Quantum Science and Technology, University of Calgary, Calgary, Alberta, Canada T2N 1N4*
[4] *Program in Quantum Information Science, Canadian Institute for Advanced Research, Toronto, Ontario, Canada M5G 1M1*

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The standard state-dependent Heisenberg-Robertson uncertainty-relation lower bound fails to capture the quintessential incompatibility of observables as the bound can be zero for some states. To remedy this problem, we establish a class of tight (i.e., inequalities are saturated) variance-based sum-uncertainty relations derived from the Lie algebraic properties of observables and show that our lower bounds depend only on the irreducible representation assumed carried by the Hilbert space of state of the system. We illustrate our result for the cases of the Weyl-Heisenberg algebra, special unitary algebras up to rank 4, and any semisimple compact algebra. We also prove the usefulness of our results by extending a known variance-based entanglement detection criterion.

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For Δw^2 signifying the variance of measurement outcomes for the observable w, Heisenberg's uncertainty relation for position x and momentum p is

$$\Delta x^2 \Delta p^2 \geqslant 1/4, \tag{1}$$

where $[x,p]=i\mathbb{1}$, and $\mathbb{1}$ is the identity operator. Equation (1) fortuitously has a constant lower bound due to the appealing algebraic properties of the commutator of x and p. Robertson's generalization to $\Delta A^2 \Delta B^2 \geqslant |\langle [A,B]\rangle|^2/4$ for arbitrary observables A and B more generally has a state-dependent lower bound [1], and so fails to capture the *intrinsic* incompatibility of noncommuting observables [2,3]. This cannot be amended as the underlying product of uncertainties is null whenever one of the uncertainties is null, an observation that provided impetus for the emergence of uncertainty relations [4–9] that eschew variance in favor of entropy.

Properly assessing uncertainty is important for foundational quantum mechanics [10–12] and for quantum information and communication [13–16]; variance is closer than entropy for practical quantum mechanics, a driving motivation behind research into sum-uncertainty relations (SURs), which deliver state-independent lower bounds [17–25]. Here we discuss SURs by showing connections with the algebras of observables, with examples of the Weyl-Heisenberg $\mathfrak{wh}$, special unitary $\mathfrak{su}(n)$ and $\mathfrak{su}(1,1)$, and generally semisimple compact algebras Extending SURs to general $\mathfrak{su}(n)$ has implications for nuclear physics [26] and quantum information [28]. thereby extending the range to applications of SURs in areas such as nuclear physics [26,27] and quantum information [28], where $\mathfrak{u}(n)$ or $\mathfrak{su}(n)$ symmetries [29,30] are prevalent.

Indeed, single-photon multipath quantum optical interferometry provides a convenient way to realize SU(n) symmetry [31–34] with the experimental signature obtained via sampling photodetection of the photon emerging from each of the n output ports, both by direct detection and by adding special postprocessing interferometers at the output followed by photodetection. Photodetection sampling statistics obtained in these ways yield uncertainties from estimates second-order cumulants for the distributions and, through this process, our uncertainty relations can be empirically tested. Such uncertainty relations are important for assessing the ultimate limits of quantum interferometry [35]. Other applications include Bose-Einstein condensates, where SU(3) symmetries play a role, e.g. [30]. Our approach lays a path towards general uncertainty relations with state-independent lower bounds for arbitrary algebras.

We strongly emphasize that our results refer to the "preparation uncertainty" [11,12] and not to the "measurement uncertainty." The former refers to the variance of the outcomes of measurements of different observables performed on different systems prepared in identical states. The latter, which has been the subject of a lively debate recently [20,36–40], refers to the relation between errors and postmeasurement disturbance in an apparatus. We underline these are very different notions [20,36,38–41], even if this difference is often obscured in the literature.

As our relations are based on the sum of variances, they easily relate more than two observables and possess a simple physical interpretation as the diagonal of the uncertainty volume, depicted in Fig. 1. The uncertainty relations that we derive from algebraic properties are stated below, with the explanation and derivation to follow. Some of our relations are prior knowledge and some are new. Each algebra is defined by its commutator relation, given by Eq. (1) for $\mathfrak{wh}$ and by

$$\begin{gathered}[J_x,J_y]=iJ_z, \quad [J_y,J_z]=iJ_x, \quad [J_z,J_x]=iJ_y,\\ [K_x,K_y]=-iK_z, \quad [K_y,K_z]=iK_x, \quad [K_z,K_x]=iK_y\end{gathered} \tag{2}$$

*sandersb@ucalgary.ca; iqst.ca/people/peoplepage.php?id=4

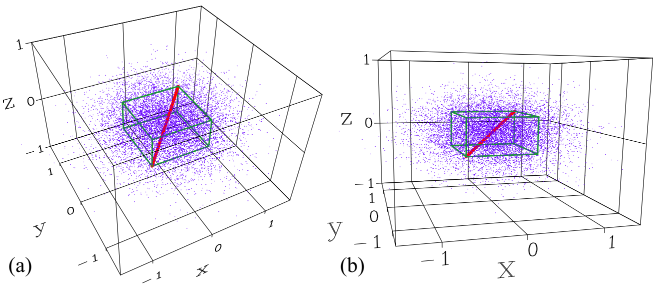

FIG. 1. Sum of variances is a measure of total uncertainty. Given a (green) box with the uncertainties as edges, the sum of variances is the squared length of the (red) diagonal. [Here 30 000 (blue) points with Gaussian distribution corresponding to $\Delta x = \Delta y = 1/2,\ \Delta z = 1/4$ are plotted as an illustration.]

for $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$, respectively. For semisimple compact Lie algebras [42–44], we use $\{e_i\}$ in an operator basis with diagonal Killing form, $\{\lambda_i\}$ as group irrep labels, $|\Lambda\rangle$ as the integral dominant weight, and $|\delta\rangle$ as the Weyl root.

We claim the following tight state-independent SURs:

$$\mathfrak{wh} : \Delta x^2 + \Delta p^2 \geqslant 1 \text{ [45]}, \tag{3}$$

$$\mathfrak{su}(1,1) : \Delta K_x^2 + \Delta K_y^2 - \Delta K_z^2 \geqslant \kappa, \tag{4}$$

κ being Bargmann index and, more generally, for the semisimple compact case,

$$\frac{1}{2}\sum_i \Delta e_i^2 \geqslant 2\langle\Lambda|\delta\rangle, \tag{5}$$

with specialization to

$$\mathfrak{su}(2) : \Delta J_x^2 + \Delta J_y^2 + \Delta J_z^2 \geqslant j, \text{ e.g., in [46–48]}, \tag{6}$$

$$\mathfrak{su}(3) : \frac{1}{2}\sum_i \Delta e_i^2 \geqslant 2(\lambda_1 + \lambda_2), \tag{7}$$

$$\mathfrak{su}(4) : \frac{1}{2}\sum_i \Delta e_i^2 \geqslant 3\lambda_1 + 4\lambda_2 + 3\lambda_3, \tag{8}$$

$$\mathfrak{su}(5) : \frac{1}{2}\sum_i \Delta e_i^2 \geqslant 4\lambda_1 + 6\lambda_2 + 6\lambda_3 + 4\lambda_4. \tag{9}$$

All these state-independent lower bounds are given functions that depend only linearly on the choice of irreducible representation (irrep). As discussed later, there are states for which the equality holds. A geometric intuition for relations (4)–(9) follows from Pythagoras' theorem: the left-hand side is the squared length of the diagonal of a "box" with uncertainties as edges shown in Fig. 1. These edges are a measure of the "total" uncertainty and bounded from below by a positive constant.

We begin by developing our approach to the SUR based on the familiar $\mathfrak{wh}(1)$ algebra (1) treated by Heisenberg [45]. Heisenberg's uncertainty relation (1) [1,45] follows from

$$(\Delta x - \Delta p)^2 \geqslant 0 \Rightarrow \Delta x^2 + \Delta p^2 \geqslant 2\Delta x\Delta p \geqslant 1, \tag{10}$$

which incorporates both the sum (3) and the Heisenberg product $\Delta x \Delta p \geqslant 1/2$ (1) relations in the same expression. Our focus is on the sum relation, and we now rederive this SUR by another approach.

We express $\mathfrak{wh}$ in terms of lowering $a := (x + ip)/\sqrt{2}$ and raising $a^\dagger$ ladder operators so

$$\mathfrak{wh} = \operatorname{span}\{a, a^\dagger, \mathbb{1}\}, \quad [a, a^\dagger] = \mathbb{1}, \tag{11}$$

with the "weight," or number, operator denoted $n = a^\dagger a$. The (Fock) eigenstates $\{|m\rangle\,; m \in \mathbb{N}\}$ satisfy

$$n\,|m\rangle = m\,|m\rangle\,, \quad a^\dagger\,|m\rangle = \sqrt{m+1}\,|m+1\rangle\,. \tag{12}$$

Henceforth, a general arbitrary state is expressed as a sum $|\psi\rangle = \sum_m \psi_m\,|m\rangle$ over the weights $\{m\}$ determined by diagonal operators for the algebra being studied. If some weights are repeated, the sum extends over the orthogonal states of the same weights. For $\mathfrak{wh}$,

$$\Delta x^2 + \Delta p^2 = 2\langle a^\dagger a\rangle + 1 - \langle x\rangle^2 - \langle p\rangle^2 \tag{13}$$

is bounded by first considering

$$\langle x\rangle + i\langle p\rangle = \sqrt{2}\sum_m \nu_m, \quad \nu_m := \sqrt{m+1}\,\psi_{m+1}\psi_m^*. \tag{14}$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}\langle x\rangle^2 + \langle p\rangle^2 &= 2\left|\sum_{n=0}^{\infty}\nu_n\right|^2\\ &\leqslant 2\sum_{n=0}^{\infty}(n+1)|\psi_{n+1}|^2\sum_{n'=0}^{\infty}|\psi_{n'}|^2 = 2\langle a^\dagger a\rangle,\end{aligned} \tag{15}$$

which proves Eq. (3), and the "lowest-weight state" $|0\rangle$ saturates this bound.

Next we apply this $\mathfrak{wh}$ technique to the ubiquitous $\mathfrak{su}(2)$ algebra, pertinent to spinlike systems with $2j \in \mathbb{N}$. For $J_\pm = J_x \pm iJ_y$ the $\mathfrak{su}(2)$ raising and lowering operators, $\mathfrak{su}(2) = \operatorname{span}\{J_+, J_-, J_z\}$ such that

$$[J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm. \tag{16}$$

The eigenstates $\{|m\rangle\,; 0 \leqslant m \leqslant 2j\}$ of the weight operator J_z, satisfying $J_z\,|m\rangle = (m - j)\,|m\rangle$ and

$$J_+\,|m\rangle = \sqrt{(m+1)(2j-m)}\,|m+1\rangle\,, \tag{17}$$

form a basis for the $(2j+1)$-dimensional irrep of $\mathfrak{su}(2)$ with

$$C_2 = J_z^2 + \tfrac{1}{2}(J_+J_- + J_-J_+) = J_z^2 + J_x^2 + J_y^2 = c_2\mathbb{1} \tag{18}$$

and eigenvalue $\langle c_2\rangle = j(j+1)$. Then

$$\Delta J_x^2 + \Delta J_y^2 + \Delta J_z^2 = c_2 - \sum_{i=x,y,z}\langle J_i\rangle^2. \tag{19}$$

This sum (19) can be bounded, analogous to the $\mathfrak{wh}$ case, by expanding for an arbitrary state

$$|\psi\rangle = \sum \psi_m\,|m\rangle\,. \tag{20}$$

For

$$\mu_m := \psi_{m+1}^*\psi_m\sqrt{(m+1)(2j-m)}, \tag{21}$$

we have

$$\langle J_z\rangle = \sum_{m=0}^{2j} m|\psi_m|^2 - j, \quad \langle J_x\rangle + i\langle J_y\rangle = \sum_{m=0}^{2j-1}\mu_m, \tag{22}$$

which leads to

$$\begin{aligned}\langle J_x\rangle^2 + \langle J_y\rangle^2 &= \left|\sum_{m=0}^{2j-1}\mu_m\right|^2 \leqslant \left(\sum_{m=0}^{2j-1}|\psi_{m+1}|^2(m+1)\right)\\ &\quad\times\left(\sum_{m'=0}^{2j-1}|\psi_{m'}|^2(2j-m')\right)\\ &= 2j\sum_m|\psi_m|^2 m - \left(\sum_m|\psi_m|^2 m\right)^2, \end{aligned}\tag{23}$$

using the Cauchy-Schwarz inequality. As

$$\langle J_z\rangle^2 = j^2 - 2j\sum_m|\psi_m|^2 m + \left(\sum_m|\psi_m|^2 m\right)^2, \tag{24}$$

we obtain the desired $\mathfrak{su}(2)$ uncertainty relation (6). Next we see how this approach robustly extends to the noncompact case.

Closely related to $\mathfrak{su}(2)$ is the noncompact $\mathfrak{su}(1,1) = \mathrm{span}\{K_+, K_-, K_z\}$ with ladder operators

$$K_\pm = K_x \pm iK_y \tag{25}$$

and commutation relations

$$[K_+, K_-] = -2K_z, \quad [K_z, K_\pm] = \pm K_\pm, \tag{26}$$

where the operators $K_{x,y,z}$ are self-adjoint on Hilbert space. K_z eigenstates $\{|m\rangle\}$, such that

$$K_z|m\rangle = (m+\kappa)|m\rangle, \quad m,\kappa \geqslant 0, \tag{27}$$

form a basis for the infinite-dimensional unitary irrep κ. We restrict our discussion to irreps of the positive discrete series, where the representation label common in physics are

$$\kappa = 1/2, 1, 3/2, \ldots. \tag{28}$$

The analysis also applies to the two limits of discrete series with labels $\kappa = 1/4, 3/4$. The eigenvalue m is discrete; continuous m [49] is a topic for future investigation. The $\mathfrak{su}(1,1)$ raising operators satisfies

$$K_+|m\rangle = \sqrt{(m+1)(2\kappa+m)}\,|m+1\rangle, \tag{29}$$

and $K_- = K_+^\dagger$. Evidently the ladder of $|m\rangle$ states is unbounded above, but the K_z eigenstate $|m=0\rangle$ with eigenvalue κ is annihilated by K_-. The quadratic Casimir operator is

$$C_2 = K_z^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) = K_z^2 - K_x^2 - K_y^2 = c_2\mathbb{1},$$

with $c_2 = \kappa(\kappa-1)$. The sum of variances is

$$\begin{aligned}\Delta K_x^2 + \Delta K_y^2 + \Delta K_z^2 &\geqslant \Delta K_x^2 + \Delta K_y^2 - \Delta K_z^2\\ &= \langle K_x\rangle^2 + \langle K_y\rangle^2 - \langle K_z\rangle^2 - c_2.\end{aligned}\tag{30}$$

Assuming all sums converge, we bound this by expanding an arbitrary state $|\psi\rangle = \sum\psi_m|m\rangle$ to obtain

$$\langle K_z\rangle = \sum_{m=0}^{\infty} m|\psi_m|^2 + \kappa, \quad \langle K_x\rangle + i\langle K_y\rangle = \sum_{m=0}^{\infty}\lambda_m, \tag{31}$$

with $\lambda_m := \psi_{m+1}^*\psi_m\sqrt{(m+1)(2\kappa+m)}$, whence

$$\begin{aligned}\langle K_x\rangle^2 + \langle K_y\rangle^2 &= \left|\sum_{m=0}^{\infty}\lambda_m\right|^2 \leqslant \left(\sum_{m=0}^{\infty}|\psi_{m+1}|^2(m+1)\right)\\ &\quad\times\left(\sum_{m'=0}^{\infty}|\psi_{m'}|^2(2\kappa+m')\right)\\ &= \left(\sum_{m=0}^{\infty}|\psi_m|^2 m\right)\left(\sum_{m'=0}^{\infty}|\psi_{m'}|^2(2\kappa+m')\right)\\ &= 2\kappa\sum_m|\psi_m|^2 m + \left(\sum_m|\psi_m|^2 m\right)^2, \end{aligned}\tag{32}$$

using the Cauchy-Schwarz inequality. As

$$\langle K_z\rangle^2 = \kappa^2 + 2\kappa\sum_m|\psi_m|^2 m + \left(\sum_m|\psi_m|^2 m\right)^2, \tag{33}$$

we can recover the right side of Eq. (30) by subtracting Eq. (32) from Eq. (33) to obtain

$$\langle K_z\rangle^2 - \langle K_x\rangle^2 - \langle K_y\rangle^2 \geqslant \kappa^2 \tag{34}$$

and thus the desired SUR (4). Actually we have proven the inequality (30), which is even stronger than desired.

We have so far discussed three cases of SURs, all based on ladder operator relations and Casimir operators, although the $\mathfrak{wh}$ case has a trivial Casimir operator, namely $\mathbb{1}$. We now have the tools to investigate more general cases involving semisimple Lie algebras.

Consider a compact semisimple rank-r Lie algebra

$$\mathfrak{g} = \mathrm{span}\{e_k, e_m; k\in\{1,\ldots,r\}, m\in\{r+1,\ldots,\ell\}\}, \tag{35}$$

with e_k a diagonal Cartan element and e_m a nondiagonal operator. For $\mathfrak{su}(2)$ this would be the Hermitian basis with $e_1 = J_z$ and $e_{2,3} = J_{x,y}$. The $\ell - r$ operators are combinations of raising and lowering operators so, crucially, have null expectation value on any eigenstate of the Cartan elements, i.e., on any state of definite weight.

The Casimir operator C_2 and its state-independent eigenvalue c_2 are

$$C_2 = \frac{1}{2}\sum_{k=1}^{\ell} e_k^2, \quad c_2 = 2\langle\Lambda|\delta\rangle + \langle\Lambda|\Lambda\rangle,$$
$$|\Lambda\rangle := \sum_{i=1}^{r}\lambda_i|w_i\rangle, \tag{36}$$

with $|\Lambda\rangle$ the highest weight for the irrep $\Lambda = (\lambda_1,\ldots,\lambda_r)$, $|w_i\rangle$ the ith fundamental weight, and $|\delta\rangle$ the Weyl root. The Weyl root is half the sum of all positive roots as detailed in [42] or [43]. Scalar products (36) are computed with a metric matrix G [43]: for

$$|\mu\rangle := \sum_i \mu_i\, |w_i\rangle\,, \quad \langle\mu|\tau\rangle = \mu\cdot G\cdot\tau. \tag{37}$$

The sum of the variances of all $\{e_i\}$ is

$$\frac{1}{2}\sum_{k=1}^{\ell}\Delta e_k^2 = \langle C_2\rangle - \frac{1}{2}\sum_{k=1}^{\ell}\langle e_k\rangle^2 = c_2 - \frac{1}{2}\sum_{k=1}^{r}\langle e_k\rangle^2, \tag{38}$$

where, in the last equality, we assume the system state $|\lambda\rangle$ is an eigenstate of the r Cartan operators so that

$$\langle e_m\rangle = \langle\lambda|\, e_m\, |\lambda\rangle = 0 \tag{39}$$

for $m > r$ due to the action of the raising and lowering operators. For the weight $|\lambda\rangle$,

$$\frac{1}{2}\sum_{k=1}^{r}\langle e_k\rangle^2 = \langle\lambda\,|\lambda\rangle \leqslant \langle\Lambda\,|\Lambda\rangle\,, \tag{40}$$

where the upper bound is attained for the highest-weight state. Combining Eqs. (38) and (40) yields

$$\frac{1}{2}\sum_{k=1}^{\ell}\Delta e_k^2 \geqslant c_2 - \langle\Lambda\,|\Lambda\rangle = 2\langle\Lambda\,|\delta\rangle\,, \tag{41}$$

which is the desired SUR (5) for semisimple compact Lie algebras. Moreover, the uncertainty-sum relation is tight as the inequality is saturated by the highest-weight state $|\Lambda\rangle$, its Weyl-reflected images, and any state in the group orbit of $|\Lambda\rangle$, i.e., any coherent state [50].

We now demonstrate the value of Eq. (41) through its application to examples of compact unitary algebras, namely $\mathfrak{su}(3)$, $\mathfrak{su}(4)$, and $\mathfrak{su}(5)$. For $\mathfrak{su}(3)$, Eq. (7) follows immediately from Eq. (41) using

$$G_{\mathrm{SU}(3)} = \frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}. \tag{42}$$

The Hermitian basis for the defining, i.e., three-dimensional $(\lambda_1,\lambda_2) = (1,0)$, irrep of $\mathfrak{su}(3)$ is

$$A_- = \begin{pmatrix}0 & -i & 0\\ +i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad A_+ = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$B_- = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad B_+ = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

$$C_- = \begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}, \quad C_+ = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},$$

$$h_1 = \mathrm{diag}(1,-1,0), \quad h_2 = \mathrm{diag}(1,1,-2). \tag{43}$$

The Killing form is $2\times\mathbb{1}$, whereas the quadratic Casimir operator

$$C_2 = \tfrac{1}{2}\left(A_+^2 + A_-^2 + B_+^2 + B_-^2 + C_+^2 + C_-^2 + h_1^2 + h_2^2\right)$$

has eigenvalue

$$c_2(\lambda_1,\lambda_2) = \tfrac{2}{3}\left(\lambda_1^2 + \lambda_2^2 + 3[\lambda_1+\lambda_2] + \lambda_1\lambda_2\right) \tag{44}$$

for irrep (λ_1,λ_2). For the (1,0) irrep (43), $C_2 = \frac{8}{3}\mathbb{1}$.

Now we verify inequality (7) for a different $\mathfrak{su}(3)$ irrep, namely the (8-dimensional) adjoint irrep (1,1). In this case With this, we easily verify the lower uncertainty bound

$$\begin{aligned}&\frac{1}{2}[\Delta(A_+)^2 + \Delta(A_-)^2 + \Delta(B_+)^2 + \Delta(B_-)^2\\ &+ \Delta(C_+)^2 + \Delta(C_-)^2 + \Delta(h_1)^2 + \Delta(h_2)^2]\\ &:= \frac{1}{2}\sum_i(\Delta\tilde{e}_i)^2 \geqslant 2(\lambda_1+\lambda_2).\end{aligned} \tag{45}$$

which confirms that the general formula (41) gives the correct inequality (7).

We generalize this procedure to $\mathfrak{su}(4)$ and $\mathfrak{su}(5)$, yielding (8) and (9), respectively, and confirm our procedure for irreps $(\lambda_1,\lambda_2,\lambda_3)$ and $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, respectively. For $\mathfrak{su}(4)$, we obtain the Gell-Mann matrices Λ_{1-15} in Appendix A following Stover's procedure [51] to obtain

$$\begin{aligned}c_2(\lambda_1,\lambda_2,\lambda_3) = \tfrac{1}{4}\big(&3\lambda_1^2 + 2[2\lambda_2+\lambda_3+6]\lambda_1 + 4\lambda_2^2\\ &+ 4\lambda_2[\lambda_3+4] + 3\lambda_3[\lambda_3+4]\big).\end{aligned} \tag{46}$$

The lower bound (8) is successfully obtained with each e_i replaced by Λ_i so our expression is confirmed for this $\mathfrak{su}(4)$ irrep.

For $\mathfrak{su}(5)$ we have the 5×5 Gell-Mann matrices Λ'_{1-24} given in Appendix A and we obtain

$$\begin{aligned}c_2(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \tfrac{2}{5}\big[&2\lambda_1^2 + 3\lambda_2\lambda_1 + 2\lambda_3\lambda_1 + \lambda_4\lambda_1 + 10\lambda_1\\ &+ 3\lambda_2^2 + 3\lambda_3^2 + 2\lambda_4^2 + 15\lambda_2 + 4\lambda_2\lambda_3\\ &+ 15\lambda_3 + 2\lambda_2\lambda_4 + 3\lambda_3\lambda_4 + 10\lambda_4\big].\end{aligned} \tag{47}$$

Replacing each e_i in inequality (9) by the Gell-Mann matrix Λ_i given in Appendix A confirms that the SUR (9) holds for this $\mathfrak{su}(5)$ irrep. We note that the bound is the same for conjugate irreps, v.g., the $\mathfrak{su}(3)$ irreps (λ_1,λ_2) and (λ_2,λ_1) have the same bound, with similar symmetry for conjugate representations holding for $\mathfrak{su}(4)$ and $\mathfrak{su}(5)$ irreps.

Whereas Eq. (41) is an equality on the sums of all variances, one can also obtain various inequalities involving sums over a restricted set of variances. There are also some special cases of our inequalities that appear in [52,53]. Tóth *et al.* [54] have given inequalities involving variances and expectation values of $\mathfrak{su}(2)$ for detecting bound entanglement in spin systems. One can reproduce their proofs for $\mathfrak{su}(n)$ (Appendix B) and find that a violation of

$$(N-1)\sum_{k=1}^{n-1}(\Delta e_k)^2 \geqslant \sum_m\left\langle e_m^2\right\rangle - 2(n-1)N \tag{48}$$

implies entanglement for N particles, each in the fundamental representation $(1,\ldots,0)$ of $\mathfrak{su}(n)$. In (B1), the sum over k is a sum of the variances of the $r = n-1$ elements in the Cartan subalgebra of $\mathfrak{su}(n)$, whereas the sum over m is over the remaining elements not in the Cartan subalgebra.

As a simple example of application of Eq. (B1), fix n and consider the n-fold coupling of the fundamental of $\mathfrak{su}(n)$, i.e., the n-fold coupling of $(1,0,\ldots,0)$. The scalar irrep $(0,0,\ldots,0)$ occurs once in this decomposition. The states of the scalar irrep in this n-fold coupling are determinants in the n states. In $\mathfrak{su}(2)$, this would be the coupling of two spin-1/2 particles to the entangled $s = 0$ singlet state. For $\mathfrak{su}(3)$, with basis states $|100\rangle$, $|010\rangle$, $|001\rangle$, the scalar that appears as the three-particle coupling is the (entangled) determinant state

$$|\psi\rangle = \mathcal{N}\begin{vmatrix} |100\rangle_1 & |010\rangle_1 & |001\rangle_1 \\ |100\rangle_2 & |010\rangle_2 & |001\rangle_2 \\ |100\rangle_3 & |010\rangle_3 & |001\rangle_3 \end{vmatrix},$$

where $\mathcal{N}$ is a normalization. Clearly since this state is in (0,0) of $\mathfrak{su}(3)$, $\Delta e_k = 0$ and $\langle e_m^2\rangle = 0$ for all k and m for this state. Thus our inequality (B1) becomes

$$0 \geqslant 0 - 2\times 2\times 3 = -12$$

and so is clearly violated, correctly implying that $|\psi\rangle$ is entangled.

Finally, Tóth *et al.* also obtain an inequality containing the sums of all the variances: this is nothing but our Eq. (41) for states in the irrep $(N, 0, \dots)$, which are not the highest weight state, its reflection, or a coherent state for this irrep. This generalizes the entanglement detection results of [54] to $\mathfrak{su}(n)$.

In conclusion, we have presented a class of state-independent tight SURs based on algebraic properties, and our scheme shows how to generalize to other algebras. Thanks to the concavity of the variance, the results presented here are valid also for mixed states, since $(\Delta e_k)^2_{(\rho)} \geqslant \sum_i p_i(\Delta e_k)^2_{(|\psi_i\rangle)}$ for $\rho = \sum_i p_i\,|\psi\rangle_i\,\langle\psi|$. Inequalities (3) and (6) were known previously [19,46–48,55–57] as is (implicitly) inequality (4) [50], but bounds were not explicitly stated nor was their common algebraic origin from a similar derivation. Furthermore, the state-independent nature of the tight lower bound was not investigated. Instead previous analyses focused on their connection with algebraic coherent states [50]. Different relations for state-independent variance-based uncertainty relations were known explicitly only for qubits [20,37,58]. Whereas state-independent uncertainty relations were traditionally connected to entropic uncertainty relations, our results show how one can obtain them also for variance-based ones.

The result of Eq. (41) exploits the relation between the SUR and the quadratic Casimir operator when the Killing form is diagonal, which is an easily generalizable notion including infinite-dimensional irreps, but the state saturating this lower bound might not be normalizable. Some work needs to be done, such as dealing with continuous irreps, verifying SURs for various irreps, and generalizing to other algebras. Some aspects of our work were known before but not in a unified, explicit, purely algebraic approach as done here.

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APPENDIX A: GELL-MANN MATRICES FOR $\mathfrak{su}(4)$ AND $\mathfrak{su}(5)$

The 4×4 Gell-Mann matrices are

$$\Lambda_1 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\quad \Lambda_2 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix},\quad \Lambda_3 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix},\quad \Lambda_4 = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix},$$

$$\Lambda_5 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{pmatrix},\quad \Lambda_6 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix},\quad \Lambda_7 = \begin{pmatrix} 0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\quad \Lambda_8 = \begin{pmatrix} 0&0&-i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0 \end{pmatrix}$$

and

$$\Lambda_9 = \begin{pmatrix} 0&0&0&-i\\0&0&0&0\\0&0&0&0\\i&0&0&0 \end{pmatrix},\quad \Lambda_{10} = \begin{pmatrix} 0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0 \end{pmatrix},\quad \Lambda_{11} = \begin{pmatrix} 0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&i&0&0 \end{pmatrix},\quad \Lambda_{12} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0 \end{pmatrix},$$

$$\Lambda_{13} = \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},\quad \Lambda_{14} = \begin{pmatrix} \frac{1}{\sqrt{3}}&0&0&0\\0&\frac{1}{\sqrt{3}}&0&0\\0&0&-\frac{2}{\sqrt{3}}&0\\0&0&0&0 \end{pmatrix},\quad \Lambda_{15} = \begin{pmatrix} \frac{1}{\sqrt{6}}&0&0&0\\0&\frac{1}{\sqrt{6}}&0&0\\0&0&\frac{1}{\sqrt{6}}&0\\0&0&0&-\sqrt{\frac{3}{2}} \end{pmatrix}.$$

The 5×5 Gell-Mann matrices are

$$\Lambda'_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda'_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Lambda'_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Lambda'_{11} = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda'_{12} = \begin{pmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{13} = \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda'_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Lambda'_{17} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{19} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix},$$

$$\Lambda'_{20} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, \quad \Lambda'_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{22} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda'_{23} = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda'_{24} = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{10}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & 0 & 0 & -2\sqrt{\frac{2}{5}} \end{pmatrix}.$$

APPENDIX B: DERIVATION OF THE INEQUALITY (48) IN THE MAIN TEXT

All separable states of N systems, with properties described by operators in an $\mathfrak{su}(n)$ algebra, satisfy the following inequality. Namely a violation of the inequality implies that the systems are entangled:

$$(N-1)\sum_{k=1}^{r}(\Delta e_k)^2 \geqslant \sum_{m=r+1}^{\ell}\langle e_m^2\rangle - 2(n-1)N, \tag{B1}$$

where the sum over k is a sum over the $r = n-1$ commuting elements in the Cartan subalgebra of $\mathfrak{su}(n)$ and the sum of m is a sum over the remaining nondiagonal operators in $\mathfrak{su}(n)$. The operators in $\mathfrak{su}(n)$ are normalized so that $\mathrm{Tr}(e_a^\dagger e_b) = 2\delta_{ab}$, as per Eq. (36) for $\mathfrak{su}(3)$. We derive Eq. (B1) explicitly for $\mathfrak{su}(3)$ and discuss the specific parts of the derivation that will generalize to $\mathfrak{su}(n)$.

Denote $\langle e_\alpha^i\rangle = \lambda_\alpha^i$, where α labels a Gell-Mann matrix and i the particle. The collective operators $e_\alpha = \sum_{i=1}^N e_\alpha^i$. The Cartan elements are e_1 and e_2. Then

$$(N-1)((\Delta e_1)^2 + (\Delta e_2)^2) - \sum_{m=3}^{8}\langle e_m^2\rangle + 4N \geqslant 0. \tag{B2}$$

Using

$$(\Delta e_1)^2 = \langle e_1^2\rangle - \langle\lambda_1\rangle^2, \tag{B3}$$

$$= \left\langle\left(\sum_i e_1^i\right)\left(\sum_j e_1^j\right)\right\rangle - \left\langle\left(\sum_i e_1^i\right)\right\rangle\left\langle\left(\sum_j e_1^j\right)\right\rangle. \tag{B4}$$

If the states are factorizable (i.e., separable), then the average values satisfy

$$(\Delta e_1)^2 = \sum_i \langle (e_1^i)^2\rangle - \sum_i (\lambda_1^i)^2. \tag{B5}$$

Doing the same for e_2 and summing gives

$$((\Delta e_1)^2 + (\Delta e_2)^2) \tag{B6}$$

$$= \sum_i [\langle (e_1^i)^2\rangle + \langle (e_2^i)^2\rangle] - \sum_i [(\lambda_1^i)^2 + (\lambda_2^i)^2]. \tag{B7}$$

From the explicit expression of the Gell-Mann matrices, one finds

$$(e_1^i)^2 + (e_2^i)^2 = \frac{4}{3}\mathbb{1}_{3\times 3}, \tag{B8}$$

so that

$$(N-1)((\Delta e_1)^2 + (\Delta e_2)^2) \tag{B9}$$

$$= \frac{4}{3}N(N-1) - (N-1)\left[\sum_i (\lambda_7^i)^2 + (\lambda_8^i)^2\right]. \tag{B10}$$

Next,

$$\sum_m \langle e_m^2\rangle - 4N = \sum_\beta \left\langle\left(\sum_i e_m^i\right)\left(\sum_j e_m^j\right)\right\rangle, \tag{B11}$$

$$= \sum_m \left[\sum_i \langle (e_m^i)^2\rangle + \sum_{i\neq j}\langle e_m^i e_m^j\rangle\right] - 4N. \tag{B12}$$

One easily verifies that, for fixed i,

$$\sum_m (e_m^i)^2 = 4\times \mathbb{1}_{3\times 3}, \tag{B13}$$

so that we now have

$$\sum_m \langle e_m^2\rangle - 4N = \sum_m\sum_{i\neq j}\lambda_m^i\lambda_m^j \tag{B14}$$

$$\leqslant \sum_m\left(\sum_i \lambda_m^i\right)^2 - \sum_m\sum_i (\lambda_m^i)^2, \tag{B15}$$

$$\leqslant \sum_m N\sum_i (\lambda_m^i)^2 - \sum_m\sum_i (\lambda_m^i)^2, \tag{B16}$$

$$\leqslant (N-1)\sum_m\sum_i (\lambda_m^i)^2. \tag{B17}$$

Subtracting Eqs. (B17) from (B10) yields

$$(N-1)((\Delta e_1)^2 + (\Delta e_2)^2) - \sum_m \langle e_m^2\rangle - 4N$$

$$\geqslant \frac{4}{3}N(N-1) - (N-1)\sum_{\alpha=1}^{8}\sum_i (\lambda_\alpha^i)^2. \tag{B18}$$

Finally, one readily verifies that, for fixed i,

$$\sum_{\alpha=1}^{8}(\lambda_\alpha^i)^2 = \frac{4}{3}, \tag{B19}$$

so that

$$(N-1)\sum_{\alpha=1}^{8}\sum_i (\lambda_\alpha^i)^2 = \frac{4}{3}N(N-1), \tag{B20}$$

from which Eq. (B2) follows.

For $\mathfrak{su}(4)$, Eq. (B7) becomes

$$(e_1^i)^2 + (e_2^i)^2 + (e_3^i)^2 = \frac{3}{2}\mathbb{1}_{4\times 4}, \tag{B21}$$

while Eq. (B19) gives $\sum_{\alpha=1}^{15}(\lambda_\alpha^i)^2 = \frac{3}{2}$. Moreover, the factor $4N$ is replaced by $6N$ so that Eq. (B1) follows.

For $\mathfrak{su}(5)$, Eq. (B7) becomes

$$(e_1^i)^2 + (e_2^i)^2 + (e_3^i)^2 + (e_4^i)^2 = \frac{8}{5}\mathbb{1}_{5\times 5}, \tag{B22}$$

while Eq. (B19) gives $\sum_{\alpha=1}^{24}(\lambda_\alpha^i)^2 = \frac{8}{5}$. This time, the factor $4N$ is replaced by $8N$ and Eq. (B1) follows. Finally, we can conjugate the entire algebra by any (global) $n\times n$ unitary matrix U since, by Eq. (B13), the sum $\sum_m (e_m^i)^2$ of squares of Cartan elements is proportional to the unit matrix and thus invariant under transformation by U.

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