Determination of weak values of quantum operators using only strong measurements

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Weak values have been shown to be helpful especially when considering them as the outcomes of weak measurements. In this paper we show that, in principle, the real and imaginary parts of the weak value of any operator may be elucidated from expectation values of suitably defined density, flux, and Hermitian commutator operators. Expectation values are the outcomes of strong (projective) measurements, implying that weak values are general properties of operators in association with pre- and postselection and they need not be preferentially associated with weak measurements. They should be considered as an important measurable property which provides added information compared with the "standard" diagonal expectation value of an operator. As the first specific example we consider the determination of the real and imaginary parts of the weak value of the momentum operator employing projective time-of-flight experiments. Then the results are analyzed from the point of view of Bohmian mechanics. Finally, we consider recent neutron interferometry experiments used to determine the weak values of the neutron spin.

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I. INTRODUCTION

Weak values naturally appear as a result of weak measurement when one considers pre- and postselected systems [1]. For an initial preselected state $|\Psi\rangle$ and postselected state $|\Phi\rangle$, the weak value of the operator $\hat{A}$ at the time t is defined as [1,2]

$$\langle \hat{A}\rangle_w(t) = \frac{\langle \Phi(t)|\hat{A}|\Psi(t)\rangle}{\langle \Phi(t)|\Psi(t)\rangle}. \tag{1}$$

The relation between weak measurement and weak values was derived by using a linear approximation to unitary time evolution when the coupling of the measurement apparatus to the pre- and postselected system is weak enough [1,3,4]. This is the source of the nomenclature of "weak values." It is therefore not surprising that, subsequently, weak values have commonly been measured using weak measurements (see, e.g., [5–7]).

The introduction of the weak value concept has had a profound impact on our understanding of quantum mechanics. It led to the development of new phenomena such as quantum random walks [8] and superoscillations [9,10]. It has influenced recent theoretical [11–16] and experimental [17–20] studies of quantum foundations. The weak value has been an important tool in the development of precision measurements [21–25], as well as state [26,27] and process [28,29] tomography.

Yet the concepts of a weak value and the related weak measurement are controversial to this very day [30–32]. It has been claimed that the definition of a weak value is a mere generalization of the notion of an expectation value to the case of differing pre- and postselected states but that it does not provide much insight into physical reality, e.g., [33–35]. Others note that weak values and weak measurements have provided and continue to provide interesting physical insights [3,36–40], going beyond the notion of a generalized expectation value [41]. Yet it is still claimed that since the weak value is inevitably linked to a weak measurement involving a "meter," it depends not only on the measured quantum system but also on the measuring meter [42].

There has been a growing number of works in recent years [43–47] which consider inferring weak values using (strong) projective measurements [48,49], yet not with full generality. In this paper we prove via a new and general protocol that both the real and the imaginary parts of weak values can be obtained in principle through strong projective measurements. We thereby disconnect the concept of weak value from the concept of weak measurement, enhancing the validity and applicability of the former.

The paper is organized as follows. We present in Sec. II the general formalism for inferring the weak value of any operator. Then, in Secs. III–VI we consider in detail the special case of obtaining the weak value of the momentum from projective measurements of the density and flux operators. We utilize in our analysis the concept of the transition path time distribution [50,51], as well as time-of-flight experiments. As the first application of our results we revisit in Sec. VII the role of weak values in Bohmian mechanics. As the second application of our formalism we analyze in Sec. VIII recent experiments employing neutron interferometry [48,49]. We end in Sec. IX with a discussion of the implications of these results for the general weak value formulation of quantum mechanics.

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We stress that the aim of this work is not to dismiss the physical origin of weak values as being associated with a shift of a pointer weakly coupled to a pre- and postselected system. We find this traditional understanding interesting and profound. We rather wish to broaden the meaning of weak values and extend the measurement techniques commonly used for inferring them.

II. INFERRING WEAK VALUES FROM STRONG MEASUREMENTS

Consider the operator $\hat{A}$ and its weak value as defined in Eq. (1) for the preselected state $|\Psi\rangle$ at time t_i and a postselected state $|\Phi\rangle$ at time t. The Hermitian density operator related to the postselected state is by definition

$$\hat{D}(\Phi) = |\Phi\rangle\langle\Phi|. \tag{2}$$

We then define a generalized Hermitian "flux" operator associated with the postselected state and the operator $\hat{A}$ as the (Hermitian) anticommutator of the operator $\hat{A}$ and the density operator

$$\hat{F}(\Phi) = \tfrac{1}{2}\{\hat{A}, \hat{D}(\Phi)\} \equiv \tfrac{1}{2}(\hat{A}\hat{D}(\Phi) + \hat{D}(\Phi)\hat{A}^\dagger). \tag{3}$$

We also define the Hermitian commutator operator

$$\hat{C}(\Phi) = \frac{1}{2}[i\hat{A}, \hat{D}(\Phi)] \equiv \frac{i}{2}(\hat{A}\hat{D}(\Phi) - \hat{D}(\Phi)\hat{A}^\dagger). \tag{4}$$

It is then a matter of straightforward calculation to prove that

$$\frac{\langle\Psi|\hat{F}(\Phi)|\Psi\rangle}{\langle\Psi|\hat{D}(\Phi)|\Psi\rangle} = \mathrm{Re}\langle\hat{A}(\Phi;\Psi)\rangle_w \tag{5}$$

and

$$\frac{\langle\Psi|\hat{C}(\Phi)|\Psi\rangle}{\langle\Psi|\hat{D}(\Phi)|\Psi\rangle} = \mathrm{Im}\langle\hat{A}(\Phi;\Psi)\rangle_w. \tag{6}$$

We have thus demonstrated in very general terms that the real and imaginary parts of the weak value of an operator can be obtained through at most three strong projective measurements. The practical question of how one implements them for the relevant operators depends on the identity of the operator $\hat{A}$, as well as the pre- and postselected states, and is not necessarily trivial. However, any weak value associated with the operator $\hat{A}$ may be inferred *in principle* from strong measurements. We now consider the specific example of the weak value of the momentum operator; this example also explains why we relate to the anticommutator [Eq. (3)] as a generalized "flux" operator.

III. MOMENTUM WEAK VALUES THROUGH STRONG MEASUREMENTS

We limit ourselves to a one-dimensional particle, with mass M, whose time evolution is determined by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{q}), \tag{7}$$

where $\hat{q}$ and $\hat{p}$ are the coordinate and momentum operators, respectively. The density and flux Hermitian operators at point x are defined, as usual, as

$$\hat{D}(x) = \delta(\hat{q} - x), \tag{8}$$

$$\hat{F}(x) = \frac{1}{2M}[\hat{p}\delta(\hat{q} - x) + \delta(\hat{q} - x)\hat{p}]. \tag{9}$$

Note the parallelism between these standard definitions and their generalizations as expressed in Eqs. (2) and (3).

We are interested in the weak value of the momentum at a postselected point x using the preselected (normalized) state $|\Psi\rangle$:

$$\langle\hat{p}(x;\Psi)\rangle_w = \frac{\langle x|\hat{p}|\Psi\rangle}{\langle x|\Psi\rangle}. \tag{10}$$

It is a matter of straightforward calculation, using Eqs. (2)–(6) to derive the following three identities:

$$\langle\Psi|\hat{D}(x)|\Psi\rangle = |\langle x|\Psi\rangle|^2, \tag{11}$$

$$\frac{\langle\Psi|\hat{F}(x)|\Psi\rangle}{\langle\Psi|\hat{D}(x)|\Psi\rangle} = \frac{\mathrm{Re}\langle\hat{p}(x;\Psi)\rangle_w}{M}, \tag{12}$$

$$\frac{1}{2}\frac{\langle\Psi|[i\hat{p}, \hat{D}(x)]|\Psi\rangle}{\langle\Psi|\hat{D}(x)|\Psi\rangle} = \mathrm{Im}\langle\hat{p}(x;\Psi)\rangle_w. \tag{13}$$

This shows explicitly that the real and imaginary parts of the weak value of the momentum may be determined with only strong measurements. We now demonstrate, using a transition path time distribution approach, how one may in principle measure the flux and Hermitian commutator operators using strong measurements.

IV. A TRANSITION PATH TIME DISTRIBUTION

We consider a scattering experiment, such that the potential goes to constant values as $x \to \pm\infty$. The particle is prepared initially at time $t = 0$ to be in state $|\Psi_0\rangle$ localized around an initial position y and (positive) momentum p_y, say to the left of the potential. The preselected state $|\Psi_0\rangle$ may, for example, be the coherent state:

$$\langle q|\Psi_0\rangle = \left(\frac{\Gamma}{\pi}\right)^{1/4} \exp\left[-\frac{\Gamma}{2}(q-y)^2 + i\frac{p_y}{\hbar}(q-y)\right]. \tag{14}$$

We then postselect a position x to the right of the potential and measure the time t at which the particle reaches this position. In this scenario the weak value of the operator $\hat{A}$ at the time t is defined as

$$\hat{A} = \frac{\langle\Phi|\hat{A}|\Psi(t)\rangle}{\langle\Phi|\Psi(t)\rangle}. \tag{15}$$

The probability density $\rho(x|t)$ for the particle to reach the position x at the time t is

$$\rho(x|t) = |\langle x|\Psi_t\rangle|^2, \tag{16}$$

where

$$|\Psi_t\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\Psi_0\rangle \tag{17}$$

is the time-evolved preselected state. The distribution ρ is normalized,

$$\int_{-\infty}^{\infty} dx\rho(x|t) = 1. \tag{18}$$

One may also define the probability density $\rho(t|x)$ for the distribution of times at which the particle will reach the postselected point x. It is given by the transition path time distribution [50,51]

$$\rho(t|x) = \frac{|\langle x|\Psi_t\rangle|^2}{\int_0^\infty dt|\langle x|\Psi_t\rangle|^2} \equiv \frac{|\langle x|\Psi_t\rangle|^2}{N(x)}, \tag{19}$$

and by definition,

$$\int_0^\infty dt\rho(t|x) = 1. \tag{20}$$

$\rho(t|x)$ is termed the transition path time probability distribution associated with the preselected state $|\Psi_0\rangle$ and the postselected position x. This time distribution is in principle measurable by sufficient repetition of a single-atom time-of-flight apparatus [52], which measures the time $t = 0$ at which a particle, prepared in state $|\Psi_0\rangle$, exits a source [53] and then the time t at which it reaches the detector located at x.

To measure $\rho(t|x)$ at any point x, one may place a detector at x and divide a reasonably long time interval T into N equal steps $t_n = n\Delta T$, where $n \in \mathbb{N}$, such that $N\Delta T = T$. This will enable one to obtain in a coarse-grained fashion the spatial derivative $\frac{\partial\rho(t|x)}{\partial x}$ of the transition path time distribution using finite differences Δx in space.

Aharonov *et al.* [54] have shown that the time of arrival cannot be measured more accurately than $\Delta t \approx \hbar/E_k$, where E_k is the initial kinetic energy of the particle. Current detectors of massive particles typically have a temporal resolution of picoseconds [55], so for a kinetic energy higher than 10^{-22} J, this temporal resolution can be met. For neutrons, this implies a nonrelativistic velocity of (at least) $v \approx 350$ m/s, which is not extremely high.

V. INFERRING THE IMAGINARY PART OF THE WEAK VALUE OF THE MOMENTUM

Consider, then, a time-of-flight measurement of the distribution, once at $x - \Delta x/2$ and then at $x + \Delta x/2$. Noting that the coordinate representation of the momentum operator is such that

$$\langle x|\hat{p}|\Psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\Psi\rangle, \tag{21}$$

one readily finds that

$$\frac{\rho\left(t|x+\frac{\Delta x}{2}\right) - \rho\left(t|x-\frac{\Delta x}{2}\right)}{\Delta x} \simeq -\frac{\partial \ln N(x)}{\partial x} - \frac{2}{\hbar}\rho(t|x)\mathrm{Im}\left[\frac{\langle x|p|\Psi_t\rangle}{\langle x|\Psi_t\rangle}\right] + O(\Delta x). \tag{22}$$

In most scattering cases, if the postselected position x is sufficiently far out in the asymptotic region, the normalization $N(x)$ becomes independent of x [56] so that measuring the transition path time distribution at the postselected positions $x - \frac{\Delta x}{2}$, x and $x + \frac{\Delta x}{2}$ allows the direct determination (without invoking weak measurements) of the imaginary part of the weak value of the momentum at position x,

$$\mathrm{Im}\left[\frac{\langle x|p|\Psi_t\rangle}{\langle x|\Psi_t\rangle}\right] \simeq -\frac{\hbar}{2}\frac{\partial \ln \rho(t|x)}{\partial x} = -\frac{\hbar}{2}\frac{\partial \ln \rho(x|t)}{\partial x}, \tag{23}$$

and this is identical to the result given in Eq. (13). The time-of-flight measurement therefore provides an experimentally implementable protocol for obtaining the imaginary part of the weak value of the momentum. Even if the normalization is a function of x it is of course time independent so that it just serves as a constant baseline which may be subtracted out.

Further notes regarding the imaginary part of the momentum weak value

One of the challenges posed by weak values is that they are complex, leading to discussion of the significance of the imaginary part. Here, we show how one may relate the imaginary part of the momentum weak value to a physically measurable velocity. For this purpose we consider time averaging, for example, the mean time it takes the particle to reach the postselected position x:

$$\langle t(x)\rangle \equiv \int_0^\infty dtt\rho(t|x). \tag{24}$$

This is an experimentally measurable quantity; it implies placing a "screen" at position x and then measuring the time of flight of particles exiting a source and reaching the screen. The mean time is just $\langle t(x)\rangle$. We can repeat this measurement at two successive values of x which are close to each other and in this way also measure how this mean time changes with the position of the screen. Specifically,

$$\begin{aligned}\frac{\partial\langle t(x)\rangle}{\partial x} &= \int_0^\infty dtt\frac{\partial}{\partial x}\rho(t|x)\\ &= -\frac{1}{N(x)}\frac{\partial N(x)}{\partial x}\langle t(x)\rangle\\ &\quad + \frac{1}{N(x)}\int_0^\infty dtt\frac{\partial|\langle x|\Psi_t\rangle|^2}{\partial x},\end{aligned} \tag{25}$$

where $N(x)$ has been defined in Eq. (19). On the other hand, the imaginary part of the weak value of the momentum as seen from Eq. (22) is

$$\mathrm{Im}\langle\hat{p}(x;\Psi_t)\rangle_w = -\frac{\hbar}{2|\langle x|\Psi_t\rangle|^2}\frac{\partial|\langle x|\Psi_t\rangle|^2}{\partial x} \tag{26}$$

so that its time-averaged value is

$$\begin{aligned}\left\langle\mathrm{Im}\langle\hat{p}(x;\Psi)\rangle_w\right\rangle &\equiv \int_0^\infty dt\rho(t|x)\mathrm{Im}\langle\hat{p}(x;\Psi_t)\rangle_w\\ &= -\frac{\hbar}{2}\int_0^\infty dt\left[\frac{\partial\rho(t|x)}{\partial x} + \rho(t|x)\frac{\partial \ln N(x)}{\partial x}\right]\\ &= -\frac{\hbar}{2}\frac{\partial \ln N(x)}{\partial x}.\end{aligned} \tag{27}$$

We thus find that

$$\frac{\partial\langle t(x)\rangle}{\partial x} = \frac{2}{\hbar}\langle \mathrm{Im}\langle \hat{p}(x;\Psi)\rangle_w\rangle\langle t(x)\rangle - \frac{2}{\hbar}\int_0^\infty dt t\rho(t|x)\mathrm{Im}\langle \hat{p}(x;\Psi_t)\rangle_w, \quad (28)$$

which shows how the imaginary part of the weak value of the momentum determines $\frac{\partial\langle t(x)\rangle}{\partial x}$, and this in turn may be considered as the inverse of a mean velocity of the particle at the point x.

VI. INFERRING THE REAL PART OF THE WEAK VALUE OF THE MOMENTUM

Instead of measuring the transition path time distribution as defined above, one may also measure the number of particles per unit time arriving at the postselected point x at the time t. The experiment one has in mind is the following. Initially, one prepares particles described by the initial wave function as before. They will escape from the source. The shutter of the source is opened for a time Dt which is much shorter than the time it takes them to arrive at the postselected point x. During this time Dt we assume that N_i particles came out of the source. This means that, initially, around $t = 0$ the number of particles per unit time exiting the source is N_i/Dt. Now one postselects point x in the asymptotic product region (to the right of the potential) and measures the number of particles per unit time crossing this point at time t. This is the flux of particles at x at time t. Different particles will arrive at x at different times so that one can measure the flux distribution at x at time t. In principle, not all particles will be transmitted. The transmission probability for particles reaching the postselected point x is by definition the ratio of the number of particles reaching the screen located at x (N_f) to the total number of incident particles exiting the source located at x_i (N_i),

$$T = \frac{N_f}{N_i} \equiv \frac{\int_0^\infty dt\langle\Psi_t|\hat{F}(x)|\Psi_t\rangle}{\int_{-Dt/2}^{Dt/2} dt\langle\Psi_t|\hat{F}(x_i)|\Psi_t\rangle}, \quad (29)$$

where $\hat{F}(x)$ is the flux operator defined in Eq. (9).

The analog of the transition path time distribution is then the normalized flux time distribution at the postselected point x,

$$f(t|x) = \frac{\langle\Psi_t|\hat{F}(x)|\Psi_t\rangle}{T\int_{-Dt/2}^{Dt/2} dt\langle\Psi_t|\hat{F}(x_i)|\Psi_t\rangle} = \frac{\langle\Psi_t|\hat{F}(x)|\Psi_t\rangle}{\int_0^\infty dt\langle\Psi_t|\hat{F}(x)|\Psi_t\rangle} \equiv \frac{\langle\Psi_t|\hat{F}(x)|\Psi_t\rangle}{N_f}, \quad (30)$$

and we note that N_f is independent of x due to the conservation of flux.

Using the definition of the flux operator as in Eq. (9) and the momentum operator as in Eq. (21), the normalized flux time distribution may be rewritten as

$$f(t|x) = \frac{N(x)}{MN_f}\rho(t|x)\mathrm{Re}\left[\frac{\langle x|\hat{p}|\Psi_t\rangle}{\langle x|\Psi_t\rangle}\right], \quad (31)$$

and this is identical to the formal result given in Eq. (12). In words, the real part of the weak value of the momentum at the postselected point x is proportional to the ratio of the flux and density time distributions. Hence there is also no need to use weak measurement to obtain the real part of the weak value of the momentum.

VII. BOHMIAN TRAJECTORIES AND WEAK MOMENTUM VALUE TIME EVOLUTION

We now revisit the role of weak values within Bohmian mechanics in the context of their determination via strong measurements. We consider a particle with mass M moving under the influence of a potential energy $V(x)$. In Bohmian mechanics the time-dependent wave function of the particle is represented as

$$\langle x|\varphi_t\rangle = \sqrt{r(x,t)}\exp\left[i\frac{S(x,t)}{\hbar}\right], \quad (32)$$

where $r(x,t)$ is a positive function, the density, and $S(x,t)$ is a real-valued phase. It is well known that the time-dependent Schrödinger equation may be written in terms of the time-dependent density and phase as

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2M}\left[\frac{\partial S(x,t)}{\partial x}\right]^2 + V_{\mathrm{eff}}(x,t) = 0, \quad (33)$$

$$\frac{\partial r(x,t)}{\partial t} + \frac{1}{M}\frac{\partial}{\partial x}\left[r(x,t)\frac{\partial S(x,t)}{\partial x}\right] = 0, \quad (34)$$

where the effective potential is

$$V_{\mathrm{eff}}(x,t) = V(x) - \frac{\hbar^2}{2M\sqrt{r(x,t)}}\left(\frac{d^2}{dx^2}\sqrt{r(x,t)}\right). \quad (35)$$

In Bohmian mechanics the time-dependent momentum is identified as the spatial derivative of the phase

$$p_B(x,t) \equiv \frac{\partial S(x,t)}{\partial x} = \mathrm{Re}\left[\frac{\langle \mathrm{x}|\mathrm{p}|\varphi_{\mathrm{t}}\rangle}{\langle \mathrm{x}|\varphi_{\mathrm{t}}\rangle}\right] \quad (36)$$

and this connects the real part of the weak value of the momentum with the Bohmian momentum. Note, though, that with this formulation the coordinate x does not vary with time; it is our postselected point.

One may, however, "measure" the real part of the momentum at different values of the coordinate. Bohmian trajectories are defined by allowing the coordinate to change with time by using the classical equation of motion for its time derivative. One then has the coupled set of equations

$$M\frac{dx}{dt} = p_B, \quad (37)$$

$$\frac{dp_B}{dt} = -\frac{dV_{\mathrm{eff}}(x)}{dx}, \quad (38)$$

and these define the Bohmian trajectory $x(t)$, $p_B(t)$.

If, however, one keeps the postselected coordinate x fixed in time, one finds that

$$\frac{dp_B(x,t)}{dt} = \frac{d}{dt}\mathrm{Re}\left[\frac{\langle \mathrm{x}|\mathrm{p}|\varphi_{\mathrm{t}}\rangle}{\langle \mathrm{x}|\varphi_{\mathrm{t}}\rangle}\right] \quad (39)$$

$$= -\frac{\partial}{\partial x}\left(V_{\mathrm{eff}}(x,t) + \frac{p_B^2(x,t)}{2M}\right), \quad (40)$$

and this differs from the time evolution of the Bohmian momentum. The time evolution of the real part of the weak value of the momentum at the fixed postselected state $|x\rangle$ is not identical to the time evolution of the momentum of the Bohmian trajectory.

Suppose, though, that we allow the coordinate to be a function of time, such that indeed $M\frac{dx}{dt} = p_B(x,t)$. Then we have that

$$\frac{dp_B(x,t)}{dt} = \frac{\partial}{\partial t}\mathrm{Re}\left[\frac{\langle \mathrm{x}|\mathrm{p}|\varphi_\mathrm{t}\rangle}{\langle \mathrm{x}|\varphi_\mathrm{t}\rangle}\right] + \frac{\mathrm{p_B(x,t)}}{\mathrm{M}}\frac{\partial}{\partial \mathrm{x}}\mathrm{Re}\left[\frac{\langle \mathrm{x}|\mathrm{p}|\varphi_\mathrm{t}\rangle}{\langle \mathrm{x}|\varphi_\mathrm{t}\rangle}\right] = -\frac{\partial}{\partial x}V_{\mathrm{eff}}(x,t) \tag{41}$$

and we have regained the Bohmian trajectory equation. In this case, the evolution of the coordinate is not through the propagator. If we define the time dependence of the momentum using the Heisenberg time evolution operator so that

$$P(x,t) = \mathrm{Re}\left[\frac{\langle \mathrm{x}|\mathrm{p_t}|\varphi\rangle}{\langle \mathrm{x}|\varphi\rangle}\right] = \mathrm{Re}\left[\frac{\langle \mathrm{x}|\exp\left(\frac{\mathrm{i}}{\hbar}\mathrm{Ht}\right)\mathrm{p}\exp\left(-\frac{\mathrm{i}}{\hbar}\mathrm{Ht}\right)|\varphi\rangle}{\langle \mathrm{x}|\varphi\rangle}\right], \tag{42}$$

then

$$\frac{dP(x,t)}{dt} = \mathrm{Re}\left[\frac{\mathrm{i}}{\hbar}\frac{\langle \mathrm{x}|\exp\left(\frac{\mathrm{i}}{\hbar}\mathrm{Ht}\right)[\mathrm{H},\mathrm{p}]\exp\left(-\frac{\mathrm{i}}{\hbar}\mathrm{Ht}\right)|\varphi\rangle}{\langle \mathrm{x}|\varphi\rangle}\right] = -\mathrm{Re}\left[\frac{\langle \mathrm{x}|\exp\left(\frac{\mathrm{i}}{\hbar}\mathrm{Ht}\right)\frac{\mathrm{dV(q)}}{\mathrm{dq}}\exp\left(-\frac{\mathrm{i}}{\hbar}\mathrm{Ht}\right)|\varphi\rangle}{\langle \mathrm{x}|\varphi\rangle}\right], \tag{43}$$

which is just the Ehrenfest equation. When considering the transition path time distribution we are "measuring" the weak momentum value at a fixed postselected coordinate x and a fixed time t. From the Bohmian point of view the transition path time distribution will then involve contributions from different Bohmian trajectories. However, one does not need to determine them to obtain the distribution.

Osmotic velocity and Bohmian potential

In analogy with the Bohmian momentum associated with the real part of the weak momentum value we may define an "osmotic" momentum associated with its imaginary part,

$$p_O(x,t) \equiv -\frac{\hbar}{2r(x,t)}\frac{\partial r(x,t)}{\partial x} = \mathrm{Im}\left[\frac{\langle \mathrm{x}|\mathrm{p}|\varphi_\mathrm{t}\rangle}{\langle \mathrm{x}|\varphi_\mathrm{t}\rangle}\right]. \tag{44}$$

The velocity $v_O \equiv p_O/m$ is often called the "osmotic" [60,61] or "diffusive" [62] velocity, as it is related to changes in the density rather than the phase. Furthermore, the resulting prefactor $D \equiv i\hbar/2m$ is often interpreted as an imaginary diffusion coefficient within stochastic quantum mechanics [60].

The kinetic term of the total energy may be defined as $T_B = p_B^2/2M$. Similarly, one may define a nonnegative internal energy $I_O = p_O^2/2M$ [63]. This definition is meaningful because one finds that the mean of the total energy is

$$\langle H\rangle = \int \Psi^*(x,t)\left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial^2 x} + V(x)\right)\Psi(x,t)dx = \langle T_B + I_O + V\rangle = \text{const.}, \tag{45}$$

which is a conserved quantity. By its definition, the internal energy I_O is related to the quantum potential $Q = -\frac{\hbar^2}{2mr}\frac{\partial^2 r}{\partial x^2}$, since $\langle Q\rangle = \langle I\rangle$. The quantum potential in turn affects the dynamics of the Bohmian momentum p_B:

$$\left(\frac{\partial}{\partial t} + p_B\frac{\partial}{\partial x}\right)p_B = -\frac{\partial}{\partial x}(Q+V). \tag{46}$$

The imaginary part of the weak momentum value thus reveals the dynamics underlying the Bohmian trajectories which is expressed by the real part. Therefore, both real and imaginary parts of the momentum weak value play important roles in Bohmian mechanics, and as shown, both can be strongly inferred.

VIII. NEUTRON INTERFEROMETRY EXPERIMENTS

The purpose of this section is to show the connection between our formal results and the recent neutron interferometry experiments reported in Refs. [48] and [49], which also demonstrate how a strong measurement may be used to infer weak values. The experiments employ a combined system and measuring device. The interferometer creates the neutron "paths" whose two possible "states" are denoted P. The neutron spin, denoted S, is used as a probe or meter. A preselected state is prepared as

$$|\Psi_i\rangle = |P_i\rangle|S_i\rangle, \tag{47}$$

where $|P_i\rangle$ are the initial path spin states and $|S_i\rangle$ the spin states. In the experiments the initial spin state was chosen to be positive in the x direction:

$$|S_i\rangle = |S_x;+\rangle. \tag{48}$$

The magnetic field is applied in the z direction with a field strength given by α. After the scattering event is over, considering only the interaction Hamiltonian, which is linear in the path and spin operators, they show that the initial preselected state changes to

$$|\Psi_i(\alpha)\rangle = \cos\left(\frac{\alpha}{2}\right)|P_i\rangle|S_x;+\rangle - i\hat{\sigma}_z^P\sin\left(\frac{\alpha}{2}\right)|P_i\rangle|S_x;-\rangle, \tag{49}$$

where $\hat{\sigma}_z^P$ is the path spin operator in the z direction. The postselected state in the path direction is denoted $|P_f\rangle$ and the weak value of interest is

$$\langle\hat{\sigma}_z^P\rangle_w = \frac{\langle P_f|\hat{\sigma}_z^P|P_i\rangle}{\langle P_f|P_i\rangle}. \tag{50}$$

The postselected state of the path and probe can take six forms,

$$|\Psi_f(j;\pm)\rangle = |P_f\rangle|S_j;\pm\rangle, \quad j = x,y,z; \tag{51}$$

that is, the probe may be strongly measured in any of the x, y, z directions and may point either up or down.

Following the notation as in Eqs. (4)–(6), the density operator associated with the postselected state is

$$\hat{D}(j;\pm) = |\Psi_f(j;\pm)\rangle\langle\Psi_f(j;\pm)|, \quad j = x, y, z. \tag{52}$$

The "flux" operator associated with the density and with the operator whose weak value is to be determined is

$$\hat{F}(j;\pm) = \frac{1}{2}\left[\hat{\sigma}_z^P \hat{D}(j;\pm) + \hat{D}(j;\pm)\hat{\sigma}_z^P\right], \tag{53}$$

and finally, the "Hermitian commutator" operator takes the form

$$\hat{C}(j;\pm) = \frac{i}{2}\left[\hat{\sigma}_z^P \hat{D}(j;\pm) - \hat{D}(j;\pm)\hat{\sigma}_z^P\right]. \tag{54}$$

The strong value of the density in the x direction with positive spin, using the "time"-evolved preselected state $|\Psi_i(\alpha)\rangle$ is found, after a bit of algebra, to be

$$\langle\Psi_i(\alpha)|\hat{D}(x;+)|\Psi_i(\alpha)\rangle = \cos^2\left(\frac{\alpha}{2}\right)|\langle P_i|P_f\rangle|^2 \equiv I_{x+}, \tag{55}$$

and this is precisely Eq. (10a) in the paper by Sponar *et al.* [49]. The strong value of the flux operator with the probe in the x direction with positive spin is, similarly, found to be

$$\langle\Psi_i(\alpha)|\hat{F}(x;+)|\Psi_i(\alpha)\rangle = |\langle P_i|P_f\rangle|^2 \cos^2\left(\frac{\alpha}{2}\right)\mathrm{Re}\langle\hat{\sigma}_z^P\rangle_w, \tag{56}$$

and the Hermitian commutator operator is

$$\langle\Psi_i(\alpha)|\hat{C}(x;+)|\Psi_i(\alpha)\rangle = |\langle P_i|P_f\rangle|^2 \cos^2\left(\frac{\alpha}{2}\right)\mathrm{Im}\langle\hat{\sigma}_z^P\rangle_w. \tag{57}$$

Equations (5) and (6) are thus specified to

$$\frac{\langle\Psi_i(\alpha)|\hat{F}(x;+)|\Psi_i(\alpha)\rangle}{\langle\Psi_i(\alpha)|\hat{D}(x;+)|\Psi_i(\alpha)\rangle} = \mathrm{Re}\langle\hat{\sigma}_z^P\rangle_w \tag{58}$$

and

$$\frac{\langle\Psi_i(\alpha)|\hat{C}(x;+)|\Psi_i(\alpha)\rangle}{\langle\Psi_i(\alpha)|\hat{D}(x;+)|\Psi_i(\alpha)\rangle} = \mathrm{Im}\langle\hat{\sigma}_z^P\rangle_w. \tag{59}$$

The experimental setup made it possible to measure only densities, as given in Eqs. (10a)–(10f) in Ref. [49], not fluxes. They extracted the real and imaginary parts and the absolute value of the weak value from a combination of the six densities as given in their Eqs. (11a)–(11c). Specifically, their Eqs. (10c) and (10d) are (in their notation)

$$I_{y+} - I_{y-} = \sin\alpha|\langle P_i|P_f\rangle|^2\mathrm{Re}\langle\hat{\sigma}_z^P\rangle_w, \tag{60}$$

$$I_{z+} - I_{z-} = \sin\alpha|\langle P_i|P_f\rangle|^2\mathrm{Im}\langle\hat{\sigma}_z^P\rangle_w, \tag{61}$$

from which we extract

$$|\langle P_i|P_f\rangle|^2\mathrm{Re}\langle\hat{\sigma}_z^P\rangle_w = \frac{I_{y+} - I_{y-}}{\sin\alpha} = \frac{\langle\Psi_i(\alpha)|\hat{F}(x;+)|\Psi_i(\alpha)\rangle}{\cos^2\left(\frac{\alpha}{2}\right)} \tag{62}$$

and

$$|\langle P_i|P_f\rangle|^2\mathrm{Im}\langle\hat{\sigma}_z^P\rangle_w = \frac{I_{z+} - I_{z-}}{\sin\alpha} = \frac{\langle\Psi_i(\alpha)|\hat{C}(x;+)|\Psi_i(\alpha)\rangle}{\cos^2\left(\frac{\alpha}{2}\right)}. \tag{63}$$

We then have that

$$\langle\Psi_i(\alpha)|\hat{F}(x;+)|\Psi_i(\alpha)\rangle = \frac{1}{2}\cot\left(\frac{\alpha}{2}\right)(I_{y+} - I_{y-}) \tag{64}$$

and

$$\langle\Psi_i(\alpha)|\hat{C}(x;+)|\Psi_i(\alpha)\rangle = \frac{1}{2}\cot\left(\frac{\alpha}{2}\right)(I_{z+} - I_{z-}), \tag{65}$$

so that

$$\mathrm{Re}\langle\hat{\sigma}_z^P\rangle_w = \frac{\langle\Psi_i(\alpha)|\hat{F}(x;+)|\Psi_i(\alpha)\rangle}{\langle\Psi_i(\alpha)|\hat{D}(x;+)|\Psi_i(\alpha)\rangle} = \frac{1}{2}\cot\left(\frac{\alpha}{2}\right)\frac{I_{y+} - I_{y-}}{I_{x+}}, \tag{66}$$

$$\mathrm{Im}\langle\hat{\sigma}_z^P\rangle_w = \frac{\langle\Psi_i(\alpha)|\hat{C}(x;+)|\Psi_i(\alpha)\rangle}{\langle\Psi_i(\alpha)|\hat{D}(x;+)|\Psi_i(\alpha)\rangle} = \frac{1}{2}\cot\left(\frac{\alpha}{2}\right)\frac{I_{z+} - I_{z-}}{I_{x+}}, \tag{67}$$

and these are Eqs. (11a) and (11b) in Ref. [49]. It thus becomes evident that the real and imaginary components of the weak spin values which they inferred are obtained through a strong measurement of the generalized density, flux, and Hermitian commutator operators.

IX. DISCUSSION

At first, the result that weak measurements are not needed to obtain weak values might seem surprising. Part of the motivation for introducing weak measurements was to reveal information regarding pre- and postselected systems without changing them much during the process. On the other hand, strong measurement, almost by definition, alters the system. However, the strong measurement protocol proposed here allows us to accurately infer the weak value of the unperturbed system because it is executed exactly at the time of postselection. In a given run of an experiment, this strong measurement coincides with the projective measurement used for performing the postselection and hence does not disturb the initial or final states of the system.

Our protocol is not only consistent with recent experiments [48,49] employing neutron interferometry, but in fact generalizes these schemes from discrete operators to any operator. The comparison with neutron interferometry determination of weak spin values demonstrates the experimental feasibility of our protocol. The methods presented in Refs. [46,48,49] indicate also the possible advantage over the weak measurement technique in terms of precision and accuracy. The proposed protocol still bears some similarity to the case of weak measurements, as it does necessitate accumulating enough statistics over a large ensemble of similarly prepared pre- and postselected states.

Although appearing ever more frequently in the physics literature, weak values are still controversial. The question whether or not they can be strongly measured is still under debate [57], reflecting on earlier discussions regarding their conceptual meaning and practical significance. The theorem derived in this paper provides a new approach for strongly inferring the weak value of operators based on time-of-flight measurements. The protocol needs only projective measurements, thus strengthening the status of weak values as profound quantities in the quantum mechanical description of pre- and postselected systems. The fact that the proposed protocol also accords well with neutron interferometry experiments [48,49], which showed that strong measurements of weak values can outperform weak measurements, further demonstrates the generality of the result and its practical relevance.

Previously, it was shown using the von Neumann measurement scheme that the imaginary part of the weak value arises from the disturbance due to coupling with the measuring pointer. This part thus reflects how the initial state is unitarily disturbed by the measured observable [58]. On the one hand, Eq. (6), which depends on the commutator, accords with this view, but on the other hand, it suggests an alternative way to understand the imaginary part in a manner which does not require an auxiliary measuring pointer. Equations (5) and (6) show that both the real and the imaginary parts of the weak value are physically significant and that both are amenable to direct, strong inference. The experimental significance of the imaginary part of the momentum weak value was also discussed.

The importance of the weak value, especially of the momentum operator, cannot be overstressed. The real and imaginary parts of the momentum weak value allow the reconstruction of the wave function since they contain the necessary information regarding the phase and amplitude of the wave function, respectively. Specifically, representing the wave function as $\Psi(x,t)=\sqrt{\rho(x,t)}\exp[iS(x,t)/\hbar]$, the phase may be reconstructed from Eq. (12),

$$S(x,t)=\int \mathrm{Re}\left[\frac{\langle x|\hat{p}|\Psi_t\rangle}{\langle x|\Psi_t\rangle}\right]dx, \tag{68}$$

and the density from Eq. (13),

$$\rho(x,t)=e^{-\frac{2}{\hbar}\int \mathrm{Im}[\frac{\langle x|p|\Psi_t\rangle}{\langle x|\Psi_t\rangle}]dx}. \tag{69}$$

The more general Eqs. (5) and (6) allow us in principle to reconstruct the wave function in any other basis.

Weak values have been also used for reconstructing Bohmian trajectories, since the real part of the weak value of the momentum is identical to the Bohmian momentum [18,59]. The Bohmian approach is also a somewhat different route towards reconstructing the wave function.

To conclude, we have shown in this paper that weak values need not be considered only in the context of weak measurement; they may be inferred directly from a strong measurement protocol. These results will hopefully pave the way to a better understanding of weak values, as well as to feasible strong-measurement-based methods for inferring and using them in practical applications.

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