Survey on the Bell nonlocality of a pair of entangled qudits

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(Received 23 May 2018; published 3 October 2018)

The question of how Bell nonlocality behaves in bipartite systems of higher dimensions is addressed. By employing the probability of violation of local realism under random measurements as the figure of merit, we investigate the nonlocality of entangled qudits with dimensions ranging from $d = 2$ up to $d = 10$. We proceed in two complementary directions. First, we study the specific Bell scenario defined by the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality. Second, we consider the nonlocality of the same states under a more general perspective by directly addressing the space of joint probabilities (computing the frequencies of behaviours outside the local polytope). In both approaches we find that the nonlocality decreases as the dimension d grows, but in quite distinct ways. While the drop in the probability of violation is exponential in the CGLMP scenario, it presents, at most, a linear decay in the space of behaviors. Furthermore, in the latter approach the states that produce maximal numeric violations in the CGLMP inequality present low probabilities of violation in comparison to maximally entangled states, so no anomaly is observed. Finally, the nonlocality of states with nonmaximal Schmidt rank is investigated.

DOI: 10.1103/PhysRevA.98.042105

I. INTRODUCTION

The violation of Bell inequalities [1], recently confirmed by experiments not afflicted by detection and locality loopholes [2–5], constitutes one of the most impressive confirmations of the nonlocal character of quantum theory. Presently, the majority of the state-of-the-art experiments in the field involve two qubits in the context of the Clauser-Horne-Shimony-Holt (CHSH) inequality. However, it became clear that the use of systems of higher dimensionality, or *qudits*, may lead to new, interesting phenomena and improvements in the efficiency of some practical tasks [6–10]. In particular, it may be easier in the future to carry out loophole-free Bell tests if qudits are employed [11]. The nonlocality of pairs of entangled qudits have been used to certify high dimensional entanglement and in the study of robustness against noise, imperfect state preparation, and measurements [12–15]. Apart from its foundational relevance, Bell nonlocality is a primary resource within the field of quantum information [16,17].

A more specific, but important question refers to the macroscopic limit. Pioneering works, addressing two spin-s particles, revealed a tendency toward local, classical behaviors as $s \to \infty$ [18,19], in the sense that the range of parameters for which nonclassicality arises vanishes as $1/s$ (however, the considered inequalities are not tight). Complementarily, Gisin and Peres [20] showed that, for particular choices of measurement parameters in the context of the CHSH inequality, it is always possible to obtain violations, but not above the Tsirelson bound.

The authors of [21] employed the resistance to noise as a nonlocality quantifier, and numerically calculated it for maximally entangled states of two qudits up to $d = 9$, each subject to one out of two local measurements characterized by multiport beam splitters and phase shifters (MBSPS) [22]. Rather surprisingly, the authors found that the resistance to white noise increases with the dimension d. Presently, it is acknowledged that, although physically relevant, resistance to noise is not a good measure of nonlocality. Also in this context, a surprising result is that the nonlocality of a system of n qubits tends to increase with n, provided that the ability to individually address each qubit is preserved [23].

Further results indicated that the states that maximally violate the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [24] do not correspond to maximally entangled states for $d > 2$ [25] (this is also valid for optimal Bell tests [26,27]). This unexpected finding has been considered as an "anomaly" of nonlocality. In this context, the probability of violation under random measurements [28,29] has been proposed as a measure of nonlocality [30], and, contrary to these previous works, led to the conclusion that maximally entangled qutrits are maximally nonlocal. This indicates that the anomaly [31] in the nonlocality of entangled qudits may be an artifact of the previously employed measures (see, however, [32]). Recently, other promising quantifiers have been proposed, as, for example, a trace distance measure (within the context of a resource theory for nonlocality) [33], and a nonanomalous realism-based measure [34,35].

In this work we employ the probability of violation to quantify the nonlocality of two entangled qudits, in two distinct, complementary perspectives. First, we address a specific experimental situation, i.e., a fixed Bell scenario (CGLMP) and the set of observables which are accessible in a particular experimental realization, namely, multiport beam splitters and phase shifters (MBSPS). Second, we investigate the same set of states in a more fundamental perspective, by calculating the probability of violation directly in the full space of joint probabilities (the space of behaviors). While the first approach corresponds to a situation that can be exhaustively investigated within a single experimental preparation, it also inherits the bias associated with the choice of a particular facet of the local polytope. In the first approach we consider dimensions d with $2 \leqslant d \leqslant 7$, while in the second case we have $2 \leqslant d \leqslant 10$. The second approach is conceptually more powerful since it takes into account all possible Bell inequalities (with a certain number of observables per party), however, the probabilities of violation calculated in the space of behaviors cannot possibly be determined by a single experimental setup. For instance, the MBSPS setup is not able to generate all possible quantum behaviors for $d \geqslant 3$. We discuss both the common points and the differences between the two approaches.

II. NONLOCALITY OF TWO ENTANGLED QUDITS IN THE CGLMP SCENARIO

We start by relating the volume of violation, defined as a quantifier of Bell nonlocality in [30], with the probability of violation under random, directionally unbiased measurements. Here, the nonlocality extent of a quantum state ρ within the scenario of a particular Bell function I will be associated with

$$V_I(\rho) \equiv \frac{1}{\mathcal{N}} \int_{\Gamma_\rho} d^n x, \tag{1}$$

where $\mathcal{X} = \{x_i\}$ is the set of all parameters that characterizes the measurements, $\Gamma_\rho \subset \mathcal{X}$ is the subset of parameters that lead to violations in the Bell inequality $I \leqslant 2$, and $\mathcal{N}$ is a normalization constant. To obtain the probability of violation $p_v(\rho)$ we must write

$$\frac{1}{\mathcal{N}} = \frac{\nu}{V_{\mathcal{X}}},$$

where $V_{\mathcal{X}}$ gives the total volume of the set of measurement parameters

$$V_{\mathcal{X}} \equiv \int_{\mathcal{X}} d^n x,$$

and ν is the number of ways one can relabel Alice's and Bob's observables (the symmetry between Alice and Bob themselves is already considered). Since we have two observables per party in both the CHSH and CGLMP scenarios, say $\{A_1, A_2\}$ for Alice and $\{B_1, B_2\}$ for Bob, there are four inequalities: the original one plus three inequalities corresponding to $A_1 \leftrightarrow A_2$ (Bob's observables unchanged), $B_1 \leftrightarrow B_2$ (Alices's observables unchanged), and $A_1 \leftrightarrow A_2$ and $B_1 \leftrightarrow B_2$. Consider, for instance, the CHSH inequality and the singlet state. If the singlet violates one of the four inequalities it does not violate the other three. However, there are three local operations on the singlet (leading to the other maximally entangled Bell states) that will lead to violations in each of the other inequalities. To take these local equivalences into account we must set $\nu = 4$. We verified that the same reasoning is valid for higher dimensions ($d > 2$).

In this way, $V_I(\rho) \rightarrow p_v(\rho)$ becomes a probability, which is the quantity that we will consider hereafter. A complementary approach was recently used by Atkin and Zohren [36], in which the measurement settings are fixed and the number of outcomes of the measurements is varied for several ensembles of random pure states.

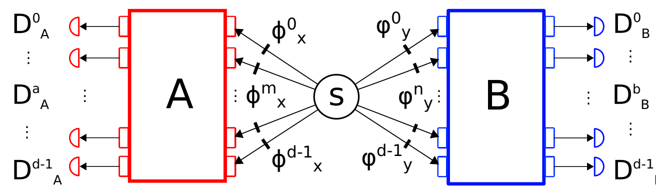

FIG. 1. Schematic illustration of the multiport-beam-splitters-and-phase-shifters (MBSPS) realization of the CGLMP inequality.

A. Multiport beam splitters and phase shifters

We will be concerned with bipartite systems with Alice and Bob sharing a pure entangled state $|\Psi\rangle$ of two d-level systems. The state of such a system can always be written as a Schmidt decomposition

$$|\Psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle_A \otimes |j\rangle_B. \tag{2}$$

Each of the parties can execute one out of two d-outcome projective measurements ($x, y = 1, 2$) limited to a MBSPS scheme, which consists in diagonal phase-shift unitary operations: $U_{mm} = e^{i\phi_x^m}$ (Alice) and $U_{nn} = e^{i\varphi_y^n}$ (Bob), followed by discrete Fourier transforms U_{FT} and U_{FT}^* on Alice's and Bob's subsystems, respectively, and then a projection onto the original basis [21,22,24,26,27,37] (see Fig. 1). It is important to note that this does not exhaust the CGLMP scenario, however, we obtain a great simplification by remaining within MBSPS realizations, which are often employed in CGLMP tests. In addition, this was exactly the considered situation when the anomaly in the nonlocality of two qutrits was first reported. It has also been conjectured that the optimal settings are contained in the MBSPS scenario [37], which has been proved in the two-qutrit case in [38].

The joint probability associated with the kth and lth outputs for Alice and Bob, respectively, given that their choices of observable were x and y reads

$$P_{xy}(k,l) = \frac{1}{d^2} + \frac{2}{d^2} \sum_{m>n=0}^{d-1} \mathrm{Re}(\alpha_m \alpha_n^*) \cos \Delta_{xy}^{mn}(k,l), \tag{3}$$

with

$$\Delta_{xy}^{mn}(k,l) = \phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n + \frac{2\pi}{d}(m-n)[k \oplus (-l)],$$

where $\oplus$ denotes sum modulo d.

The corresponding CGLMP inequality is a facet of the associated local polytope [39] and reads

$$I_d = \sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) \{\mathcal{B}_k - \mathcal{B}_{-(k+1)}\} \leqslant 2, \tag{4}$$

here $[\zeta]$ indicates the integer part of ζ and $\mathcal{B}_k = P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)$, where $P(A_x = B_y + k)$ is the probability

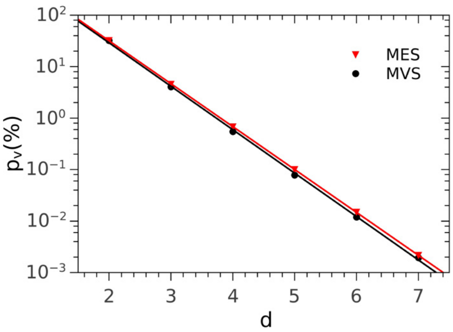

FIG. 2. Monolog plot of the probabilities of violation (in percents) of the maximally entangled state (MES) and maximally violating state (MVS) as a function of the dimension d (CGLMP inequality and MBSPS measurements). The nonlocality decreases exponentially with the dimension.

that the outcomes corresponding to the observables A_x and B_y differ by k, modulo d.

Introducing the joint probabilities (3) into (4) the CGLMP-Bell function can be rewritten in a simpler form, compatible with the MBSPS constraints (see the Appendix)

$$I_d = \sum_{x,y=1}^{2} \sum_{m>n=0}^{d-1} C_{xy}^{mn} \cos\left(\phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n + \Psi_{xy}^{mn}\right),$$

with coefficients C_{xy}^{mn} and Ψ_{xy}^{mn} given by (A1) and (A2).

The volume element of the set of measurement parameters is simply given by $d\Phi = \prod_{x,y=1}^{2} \prod_{j,k=0}^{d-1} d\phi_x^j d\varphi_y^k$. This "trivial" measure is due to the fact that all involved parameters are in-plane angles (in the MPBSPS scheme). The total volume is $V_{\mathcal{X}} = (2\pi)^{4d}$, then the probability of violation may be calculated as

$$p_v(\rho) = \frac{4}{(2\pi)^{4d}} \int_{\Gamma_\rho} d\Phi, \tag{5}$$

where Γ_ρ corresponds to the subset of $\mathcal{X}$ for which the measurement parameters lead to violation of the inequality $I_d \leqslant 2$ for a given state ρ.

The results presented in this section have been obtained via Monte Carlo integrations, corresponding to several runs of a Bell experiment using uniform random measurement configurations on a definite quantum state.

Calculations of the probability of violation of pairs of qudits in maximally entangled states (MES) and maximally violating states (MVS) under the CGLMP inequality and MBSPS measurements were carried out up to $d = 7$. The results are shown in a monolog plot in Fig. 2. Each point in this figure is the average of ten Monte Carlo runs with 10^{10} points each. Error-bar sizes in this plot are smaller than the scatters used to represent the points. In all cases the uncertainties are orders of magnitude smaller than the mean values. As it can be seen, the higher the dimension, the lower the probability of violation. In this way it is possible to conclude that the nonlocal content of a quantum entangled state of two qudits *exponentially* decreases with the dimensionality of the system, which is in agreement with the notion of restoration of classical features in the limit of high quantum numbers. However, we stress that the CGLMP scenario refers to two observables per party, no matter the value of d. We found that the exponential-decay behavior assumes a particularly simple form if we use 2π as the basis (this is a natural basis in MBSPS scenarios). The points are well described by

$$p_v(d) \sim (2\pi)^{-d}, \tag{6}$$

where $p_v(d)$ refers to the maximally entangled state (MES) of two qudits with d levels each. In Fig. 2, these points are represented by (red) triangles, and the upper continuous line corresponds to the best fitting with $p_v(d) \sim (2\pi)^{-1.04d}$. The squares correspond to the states that yield the maximal numeric violation of the CGLMP inequality. Except for $d = 2$ (for which equal probabilities are obtained), the MES present a higher probability in comparison to the maximally violating states. The probability of violation for the MVS's drops off approximately as $(2\pi)^{-1.07d}$. This extends the results of [30], showing that there is no anomaly in the nonlocality of two entangled qudits up to $d = 7$, at least in the CGLMP scenario, when p_v is used as a figure of merit. However, very recently, it has been shown that this conclusion cannot be extended to $d > 7$ [40]. In this reference the authors numerically demonstrate that, for $d = \{8, 9, 10\}$, the volume of violation of the corresponding MES' is again smaller than that of MVS'. We atribute this to the increasing bias (as d grows) introduced by the choice of a single facet of the local polytope rather than to a fundamental anomaly. In fact, in the next section we show that no anomaly shows up for $2 \leqslant d \leqslant 10$ when the whole space of behaviors is considered.

Below, we provide a list of numerically calculated MVS's for $3 \leqslant d \leqslant 10$:

$$\left|\psi_{\mathrm{MVS}}^{\mathrm{rank}=3}\right\rangle = 0.6169|00\rangle + 0.4888|11\rangle + 0.6169|22\rangle, \tag{7}$$

$$\left|\psi_{\mathrm{MVS}}^{\mathrm{rank}=4}\right\rangle = 0.5686|00\rangle + 0.4204|11\rangle + 0.4204|22\rangle + 0.5686|33\rangle, \tag{8}$$

$$\left|\psi_{\mathrm{MVS}}^{\mathrm{rank}=5}\right\rangle = 0.5368|00\rangle + 0.3859|11\rangle + 0.3859|22\rangle + 0.3859|33\rangle + 0.5368|44\rangle, \tag{9}$$

$$\left|\psi_{\mathrm{MVS}}^{\mathrm{rank}=6}\right\rangle = 0.5137|00\rangle + 0.3644|11\rangle + 0.3214|22\rangle + 0.3214|33\rangle + 0.3644|44\rangle + 0.5137|55\rangle, \tag{10}$$

$$\left|\psi_{\mathrm{MVS}}^{\mathrm{rank}=7}\right\rangle = 0.4957|00\rangle + 0.3493|11\rangle + 0.3011|22\rangle + 0.2882|33\rangle + 0.3011|44\rangle + 0.3493|55\rangle + 0.4957|66\rangle, \tag{11}$$

$$\left|\psi_{\mathrm{MVS}}^{\mathrm{rank}=8}\right\rangle = 0.4812|00\rangle + 0.3379|11\rangle + 0.2872|22\rangle + 0.2679|33\rangle + 0.2679|44\rangle + 0.2872|55\rangle + 0.3379|66\rangle + 0.4812|77\rangle, \tag{12}$$

$$\left|\psi_{\rm MVS}^{\rm rank=9}\right\rangle = 0.4690|00\rangle + 0.3288|11\rangle + 0.2770|22\rangle + 0.2541|33\rangle + 0.2474|44\rangle + 0.2541|55\rangle + 0.2770|66\rangle + 0.3288|77\rangle + 0.4690|88\rangle, \tag{13}$$

$$\left|\psi_{\rm MVS}^{\rm rank=10}\right\rangle = 0.4587|00\rangle + 0.3212|11\rangle + 0.2690|22\rangle + 0.2440|33\rangle + 0.2334|44\rangle + 0.2334|55\rangle + 0.2440|66\rangle + 0.2690|77\rangle + 0.3212|88\rangle + 0.4587|99\rangle. \tag{14}$$

The first three states above have been calculated by Zohren and Gill in [26]. The MES and MVS coincide for $d = 2$, and $p_v(2) \approx 0.32$, which shows that the restriction to MBSPS measurements increases the probability of violation. For general measurements, the probability of violation is around 0.28 for maximally entangled states since the CGLMP and the CHSH inequalities are equivalent for $d = 2$. A similar result appears when, in the CHSH scenario, the parties previously agree on one of the measurement directions. With this, the inequality becomes the first Bell inequality, for which $p_v = 1/3 \approx 0.33$ [41].

Regarding two qudits, MES are also maximally symmetric. However, one can consider maximally symmetric states (MSS) with Schmidt ranks such that $r < d$, which are not maximally entangled. In this case, the inequivalence between MSS's and states that maximize p_v reappears for the CGLMP inequality. In spite of the balancedness of states like $(|00\rangle + |11\rangle + \cdots + |(r-1)(r-1)\rangle)/\sqrt{r}$, due to the fact that the basis kets $|rr\rangle, \ldots, |(d-1)(d-1)\rangle$ are missing, they are not maximally nonlocal, in the CGLMP scenario. However, this does not constitute a true anomaly since the symmetric low-rank states cannot be considered maximally entangled. The investigation of states with lower ranks will provide a clear illustration of how different the results can be when a single Bell inequality is considered instead of the full space of behaviors.

As an example, let us consider the family of states (with zero as the coefficient of $|33\rangle$):

$$\cos\theta_0|00\rangle + \sin\theta_0\cos\theta_1|11\rangle + \sin\theta_0\sin\theta_1|22\rangle. \tag{15}$$

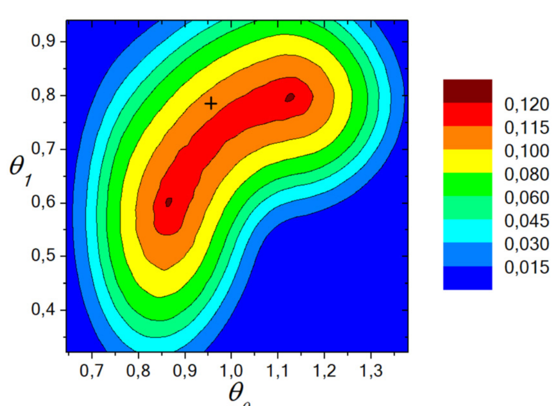

FIG. 3. Probability of violation (%) for rank-3 states with $d = 4$ in the context of the CGLMP inequality. The cross corresponds to the state $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$, and the lower-left darker spot corresponds to state (16).

In Fig. 3 we plot p_v for the above rank-3 states with $d = 4$, as a function of θ_0 and θ_1. The balanced state is identified by the cross, while the two states that maximize the probability of the violation are given by $(\theta_0, \theta_1) \approx (0.864, 0.604)$,

$$0.647|00\rangle + 0.628|11\rangle + 0.431|22\rangle; \tag{16}$$

and $(\theta_0, \theta_1) \approx (1.126, 0.798)$ (equivalent to the above state with $|00\rangle \leftrightarrow |22\rangle$), with $p_v \approx 0.224 \times p_v(\text{MES})$, where $p_v(\text{MES})$, refers to the full-rank maximally entangled state. Similar results are obtained for $r = 3$ and $d = 5$, in which case the state with larger probability of violation corresponds to $(\theta_0, \theta_1) \approx (0.840, 0.585)$. For $r = 3$ and $d \geqslant 6$ we did not find any violation.

III. NONLOCALITY OF TWO ENTANGLED QUDITS IN THE SPACE OF BEHAVIORS

In this section we will consider the nonlocality of two entangled qudits, with $2 \leqslant d \leqslant 10$, in a more general way, by calculating the probability of violation without referring to a particular Bell inequality. This can be done by means of linear programming [42]. In this scenario the observers can choose between two arbitrary observables defined by orthogonal projections (specified by general unitary transformations). The integration in (1) is now defined in the space of behaviors, characterized by the joint conditional probabilities $\{p(ab|xy)\}$. Each $p(ab|xy)$ defines an axis in this space, whose dimension is given by $4d^2$, e.g., for two inputs and d possible outputs for each of the two parties. This dimension can be lowered if we take into account the normalization of probabilities and the no-signaling condition. With these physical constraints the effective dimension becomes $4d(d-1)$ [16]. Practically, we check how many (in percents) projective measurement operators sampled according to Haar measure lead to violation of local realism. The numerical procedures are described in detail in [23]. The results of this section are summarized in Tables I and II.

In accordance with the results of the previous section, the probability of violation decreases as d grows, for two observables per party for the investigated values of d. However, the fact that there is no restriction to a particular Bell inequality (all relevant scenarios with a fixed number of observables per party are simultaneously considered), makes the decrease in p_v qualitatively different. Instead of an exponential drop we find an initially linear decay for $2 \leqslant d \leqslant 5$. In Fig. 4 we display the probability of violation (this time in the space of behaviors) for the MES's, red triangles, and, for the sake of comparison, for the MVS considered in the previous section, black squares. Here, no anomaly shows up.

Differently from what we observed in the CGLMP-MBSPS scenario, we found that balanced states with any rank larger

TABLE I. Probability of violation with two measurement settings per party for two qudits MSS and MVS states of different rank. For $r = d$ the MSS's are also MES's (see the bold entries).

		sample	$p_v(\%)$	
d	r	size	$\lvert\psi_{\mathrm{MSS}}^{\mathrm{rank=r}}\rangle$	$\lvert\psi_{\mathrm{MVS}}^{\mathrm{rank=r}}\rangle$
2	2	10^{10}	**28.318**	
3	2	10^9	10.757	
3	3	10^9	**24.011**	22.317
4	2	5×10^8	3.548	
4	3	5×10^8	11.206	9.749
4	4	5×10^8	**18.667**	16.252
5	2	10^8	0.734	
5	3	10^8	2.858	2.423
5	4	10^8	5.228	3.713
5	5	10^8	**12.709**	10.863
6	2	10^7	0.173	
6	3	10^7	0.397	0.322
6	4	10^7	1.390	0.930
6	5	10^7	1.748	1.139
6	6	10^7	**9.300**	7.738
7	2	10^6	0.034	
7	3	10^6	0.044	0.029
7	4	10^6	0.215	0.134
7	5	10^6	0.435	0.258
7	6	10^6	0.679	0.400
7	7	10^6	**8.132**	6.537
8	2	10^5	0.003	
8	3	10^5	0.008	0.004
8	4	10^5	0.020	0.013
8	5	10^5	0.107	0.040
8	6	10^5	0.128	0.061
8	7	10^5	0.422	0.246
8	8	10^6	**7.380**	5.734
9	2	10^5	0.003	
9	3	10^5	0.000	0.000
9	4	10^5	0.001	0.002
9	5	10^5	0.015	0.007
9	6	10^5	0.033	0.017
9	7	10^5	0.067	0.032
9	8	10^5	0.358	0.206
9	9	5×10^5	**7.047**	5.198
10	2	10^4	0.000	
10	3	10^4	0.000	0.000
10	4	10^4	0.000	0.000
10	5	10^4	0.000	0.000
10	6	10^4	0.000	0.000
10	7	10^4	0.000	0.010
10	8	10^4	0.050	0.000
10	9	10^4	0.420	0.220
10	10	10^5	**6.671**	4.773

than 1, present a nonvanishing probability of violation. For instance, with $r = 2$ and $d = 6$ we found that 0.173% of the possible behaviors are outside the local polytope, while for $r = d = 6$ this percentage is about 9.3%. In Fig. 5 we plot p_v against the dimension d for MSS with ranks ranging from $r = 2$ to $r = 10$.

TABLE II. Probability of violation for MES with 3×3 measurement settings per party. Compare to the bold entries of Table I.

d	2	3	4	5
$p_v(\lvert\psi_{\mathrm{mes}}^{\mathrm{rank}} = d\rangle)$	78.219	78.675	71.478	56.681
sample size	10^9	10^8	10^7	2.25×10^5

Another interesting feature is the strong enhancement in our ability to detect nonlocality by increasing the number of observables per party from 2 to 3 (see Table II). In the simplest case of two entangled qubits, this amounts to a change from $p_v \approx 28.3\%$ to $p_v \approx 78.2\%$ for MES. For $d = r = 5$, the probabilities of violation for 2 and 3 observables per party are 12.7% and 56.5%, respectively. In fact, very recently, this tendency towards large probabilities of violation for an increasing number of observables has been expressed rigorously in [43]. The property demonstrated in this reference is that, for any pure bipartite entangled state, p_v tends to unity whenever the number of measurement choices (of the two parties) tends to infinity [43].

Finally, we address the family of states in (15), this time considering all possible behaviors. The results for the probability of violation are given in the contour plot in Fig. 6. It is much more symmetric than the corresponding plot, restricted to the CGLMP-MBSPS scenario, Fig. 3. Due to statistical fluctuations, we were not able to determine the exact location of the state that maximizes the probability, rather we determined a region in the θ_0-θ_1 plane which contains such a state. The boundary of this region is the innermost contour in Fig. 6, and the MSS with $r = 3$ ($d = 4$) is identified by the cross. We thus conclude that the apparent asymmetry revealed in Fig. 3 is mainly due to the bias introduced by the choice of a particular facet of the local polytope. Since the number of relevant Bell inequalities grows with the dimension, the effect of this bias tends to increase with d. We believe that this is the cause for the reappearance of the anomaly in the nonlocality of two qudits for $d > 7$, in the context of a single Bell scenario, as reported in [40]. We, therefore, conclude that the most consistent definition for the probability of violation is that concerning integrations in the space of behaviors.

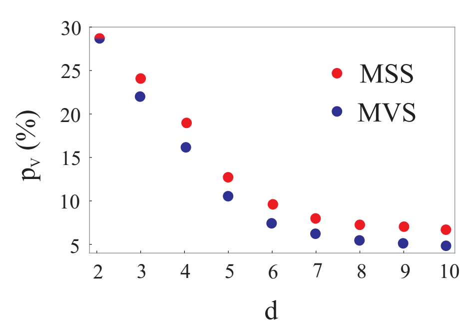

FIG. 4. Probability of violation of MES's and MVS's as functions of the dimension d, in the space of behaviors. The nonlocality decreases slowly with the dimension. Note that apart from the qubit case ($d = 2$), the MES presents more nonlocality than the MVS. Compare to the monolog plot of Fig. 2.

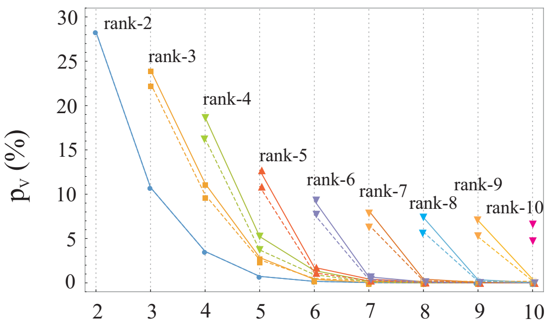

FIG. 5. Probability of violation of maximally symmetric states of several ranks r as a function of d, in the space of behaviors. Despite the strong decrease in p_v as d grows, all states with $r \geqslant 2$ present nonvanishing nonlocality.

IV. CLOSING REMARKS

The goal of the present paper was to study quantum nonlocality in bipartite systems of high dimensionality. The results showed that the extent of nonlocality decreases with the dimension of the qudits in both, the CGLMP scenario ($d \leqslant 7$) and in the space of behaviors ($d \leqslant 10$). The decay being exponential for the particular Bell inequality we addressed and much slower, at most linear, when all possible behaviors are considered. It was additionally shown that no anomaly of nonlocality showed up in the space of behaviors, with p_v as the figure of merit.

The qualitative agreement between the two approaches ceases to hold when maximally symmetric states of lower rank ($r < d$) are considered and for MES with $d > 7$ [40]. While in the fixed Bell scenario we observed that the MSS are not maximally nonlocal, we found numerical evidence that, whenever the entire local polytope is considered this is no longer true. This may be understood as an effect of the increasing (as d grows) bias introduced by the choice of a particular facet. This is an indication that the probability of violation defined in the space of behaviors is a more fundamental quantity as compared to the volume of violation of a particular Bell scenario.

The regime of large d may be, at least in some sense, considered as a classical limit, and then, we should observe local behaviors as the dominant ones. However, we may as well conceive the classical limit as a large gathering of two-level systems, which leads to an apparent contradiction. It has been shown that the probability of violation strongly increases with the number N of qubits, and two observables per party [23,44]. In fact, random states of 5 qubits typically present $p_v > 0.99$ [23] and nonlocality becomes completely dominant for large N. We remark that this is not a loose comparison because there is an isomorphism between the Hilbert space of a system with N qubits (for simplicity we assume N to be even) and the Hilbert space of two qudits with $d = 2^{N/2}$ levels, each. How do we get opposite trends in the limit $N \to \infty$, and consequently in the limit $d \to \infty$?

The point is that, in both cases, we have two observables per party, but this amounts to quite different physical situations. In the N-qubit case we have two observables per qubit, say $A_1, A_2; B_1, B_2; C_1, C_2$; and so on. Since each observable is dichotomous, we have 4 possibilities involving the choice of observables and potential outcomes for every qubit. This leads to a total of $4^N = 2^{2N}$ independent possibilities. In the case of 2 qudits with dimension $d = 2^{N/2}$ we only have four observables: $\mathcal{A}_1, \mathcal{A}_2; \mathcal{B}_1, \mathcal{B}_2$, each with $2^{N/2}$ outputs, leading to a total of $4 \times 2^{N/2} \times 2^{N/2} = 2^{N+2}$ possibilities. So, the four many-output observables in the latter case are not sufficient to compensate for the $2N$ dichotomic observables in the former situation. Of course, in practice, it may become increasingly hard to address individual qubits in the large-N regime.

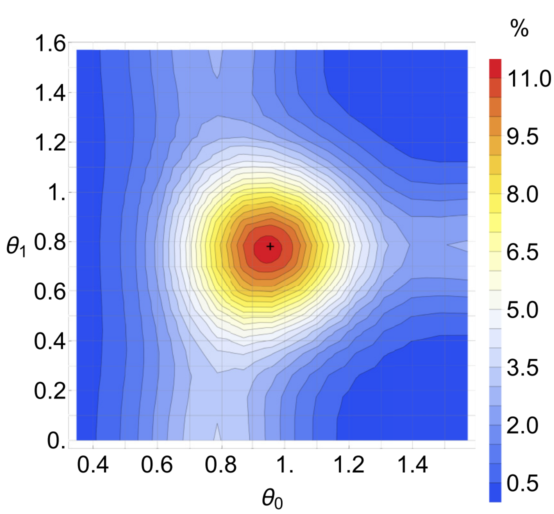

FIG. 6. Probability of violation for rank-3 states with $d = 4$ in the space of behavoirs. The cross corresponds to the state $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$. Note how symmetric this plot is in comparison to that of Fig. 3.

ACKNOWLEDGMENTS

A.F. and F.P. thank the financial support from Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Instituto Nacional de Ciência e Tecnologia-Informação Quântica (INCT-IQ). A.R and W.L. are supported by the National Science Center (NCN) Grant No. 2014/14/M/ST2/00818. T.V. is supported by the National Research, Development and Innovation Office NKFIH (Grants No. K111734 and No. KH125096).

APPENDIX: CGLMP INEQUALITY UNDER MULTIPORT BEAM-SPLITTERS–PHASE-SHIFTERS EXPERIMENTAL SETUP

Any probability term $P(A_x = B_y + k)$ in the CGLMP inequality may be written in function of joint probabilities as

$$P(A_x = B_y + k) = \sum_{j=0}^{d-1} P(A_x = j \oplus k, B_y = j)$$

$$= \sum_{j=0}^{d-1} P_{xy}(j \oplus k, j),$$

thus $\mathcal{B}_k$ may be written as

$$\mathcal{B}_k = \sum_{x,y=1}^{2} \sum_{j=0}^{d-1} P_{xy}(j \oplus \kappa_{xyk}, j \oplus \lambda_{xyk}),$$

with nonvanishing coefficients κ_{xyk} and λ_{xyk} given by $\kappa_{11k} = \kappa_{22k} = \lambda_{12k} = k$, and $\lambda_{21k} = k + 1$.

Joint probabilities for the experimental setup considered in this work [(3)] satisfy the following symmetry property:

$$\sum_{j=0}^{d-1} P_{xy}(j \oplus k, j \oplus l) = d P_{xy}(k, l),$$

taking this into account it is easy to see that $\mathcal{B}_k - \mathcal{B}_{-(k+1)}$ in the CGLMP inequality [(4)] reduces to

$$\frac{2}{d} \sum_{x,y=1}^{2} \sum_{m>n}^{d-1} \mathrm{Re}(\alpha_m \alpha_n^*) \{ \cos \Delta\beta_{xy}^{mn}(\kappa_{xyk}, \lambda_{xyk}) - \cos \Delta\beta_{xy}^{mn}[\kappa_{xy(-k-1)}, \lambda_{xy(-k-1)}] \}.$$

Using trigonometrical identities, the CGLMP function I_d takes the form

$$I_d = \sum_{x,y=1}^{2} \sum_{m>n=0}^{d-1} C_{xy}^{mn} \sin\left[\frac{\pi}{d}(m-n)\right] \times \{ \cos(\phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n) + A_{xy}^{mn} \sin(\phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n) \},$$

with

$$A_{xy}^{mn} = (-1)^{x(1+y)+1} \cot\left[\frac{\pi}{d}(m-n)\right]$$

and

$$C_{xy}^{mn} = \frac{4\mathrm{Re}(\alpha_m \alpha_n^*)}{d} (-1)^{y(1+x)} \mathcal{C}_{mn}, \tag{A1}$$

where

$$\mathcal{C}_{mn} = \sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) \sin\left[\frac{\pi}{d}(m-n)(2k+1)\right].$$

By using the harmonic addition theorem, the CGLMP function for quantum joint probabilities under a measurement scheme based on multiport beam splitters and phase shifters characterized by a set of angles (ϕ_x^n, φ_y^m) reduces to

$$I_d = \sum_{x,y=1}^{2} \sum_{m>n=0}^{d-1} C_{xy}^{mn} \cos(\phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n + \Psi_{xy}^{mn}),$$

with amplitude C_{xy}^{mn} given by (A1) and the phase coefficient

$$\Psi_{xy}^{mn} = (-1)^{x(1+y)} \left[\frac{\pi}{2} - \frac{\pi}{d}(m-n)\right]. \tag{A2}$$

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