Modification of polarization through de-Gaussification

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We analyze the polarization of a quantum radiation field under a de-Gaussification process. Specifically, we consider the addition of photons to a two-mode thermal state to get mixed non-Gaussian and nonclassical states which are still diagonal in the Fock basis. Stokes-operator-based degrees of polarization and two distance-type measures defined with Hilbert-Schmidt and Bures metrics are investigated. For a better insight we here introduce a polarization degree based on the relative entropy. Polarization of the thermal states is fully investigated and simple closed expressions are found for all the defined degrees. The evaluated degrees for photon-added states are then compared to the corresponding ones for the two-mode thermal states they originate from. We present interesting findings which tell us that some popular degrees of polarization are not fully consistent. However, the most solidly defined degrees, which are based on the Bures metric and relative entropy, clearly indicate an enhancement of polarization through de-Gaussification. This conclusion is supported by the behavior of the degrees of polarization of the Fock states, which are finally discussed as a limit case.

I. INTRODUCTION

An important concept for classical optics, polarization of light has quite recently become of interest in quantum information processing as well. Its usefulness in this area arises especially from the robustness of light as an information carrier. This allows easy manipulation and transmission of polarization-encoded information with negligible losses, thus providing an appreciated experimental convenience. Indeed, polarization encoding was considered to be the optimal choice in many recent experiments: the quantum key distribution required in cryptography [1,2], polarization entanglement [3], superdense coding [4], quantum teleportation of polarization [5], entanglement swapping [6], quantum tomography [7], and quantum computation [8]. Due to the relevance of polarization in such a large number of quantum processes, one needs to find proper measures for description of polarization in the quantum realm.

Classically, the definition of the degree of polarization is obtained by using the Stokes parameters [9,10]. In the quantum domain, the standard degree of polarization was defined by replacing the Stokes variables by the expectation values of the Stokes operators [11–14]. This definition based on first-order moments cannot give a complete description for all quantum fields, since it assigns the zero value in some cases of pure polarized states. Therefore, the idea of construction of the polarization measures by using second-order moments of the Stokes operators occurred [15,16]. Moreover, a provisionally improved characterization of polarization has recently been obtained with the help of higher-order moments [17–22]. Collaterally, it was recently proved that the

Stokes-operator measurements have great importance for the estimation of the covariance matrix of macroscopic quantum states [23].

Taking inspiration from the quantum information toolbox, the degree of polarization was quantified as the distance between the given quantum field state and the set of all unpolarized states. These definitions have considered several metrics: Hilbert-Schmidt, Bures [24,25], and Chernoff [26–30].

Two recent reviews by Chirkin and Luis present possible definitions of the degree of polarization of a quantum field and applications of the polarized states [31,32]. An overview of the difficulties encountered in using various types of polarization degrees was recently given in Ref. [33].

A lot of attention has been devoted in recent years to the study of polarization of Gaussian states of light fields [34–37]. On the other hand, one would find little investigation of polarization for non-Gaussian field states. See, however, the interesting findings on the polarization of pure Schrödinger cat or catlike states and entangled bimodal coherent states in Ref. [38]. Since the non-Gaussian states were proven to be more efficient resources in some quantum information processes [39–43], we felt the need to analyze their polarization more deeply using some degrees originating from both the classical and the quantum perspectives. Our aim here is thus twofold. First, we are interested in observing the behavior of polarization under de-Gaussification as an interesting process in its own right. Second, we want to figure out the consistency of the results given by differently defined degrees of polarization and eventually draw a conclusion on their usefulness. Specifically, we use a description of the polarization for a product of mixed states which are Fock diagonal. With the unique exception of the thermal states, these are definitely non-Gaussian.

The paper is organized as follows. In Sec. II we review the traditional Stokes definitions based on the first- and

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second-order moments. We further consider two distance-type measures based on the Hilbert-Schmidt and Bures metrics. In Sec. III, we introduce a new definition of the quantum degree of polarization which is based on the relative entropy. The above-mentioned quantum degrees of polarization are applied to a tensor product of Fock-diagonal states in Sec. IV. The obtained closed expressions are state-dependent expansions on N-photon manifolds. In Sec. V we give our exact findings on the degrees of polarization of two-mode thermal states. These are important results because we then compare them to the degrees of polarization of some non-Gaussian states resulting from the addition of photons to thermal states in Sec. VI. Our conclusions are drawn in Sec. VII.

II. QUANTUM DEGREES OF POLARIZATION

In any discussion of the polarization of the quantum radiation field, a quasimonochromatic light beam is decomposed into two orthogonal transverse oscillating modes which are described by a definite two-mode state $\hat{\rho}$. The quantum treatment of polarization starts from the Stokes operators built with the amplitude operators of the conventional horizontally (H) and vertically (V) oscillating modes:

$$\hat{S}_1 := \hat{a}_H^\dagger \hat{a}_V + \hat{a}_H \hat{a}_V^\dagger, \quad \hat{S}_2 := \frac{1}{i}(\hat{a}_H^\dagger \hat{a}_V - \hat{a}_H \hat{a}_V^\dagger),$$

$$\hat{S}_3 := \hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V, \quad \hat{S}_0 := \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V. \quad (2.1)$$

Their expectation values correspond to the classical Stokes parameters:

$$\langle \hat{S}_0 \rangle = \mathrm{Tr}(\hat{\rho}\hat{S}_0), \quad \langle \hat{S}_j \rangle = \mathrm{Tr}(\hat{\rho}\hat{S}_j) \quad (j = 1, 2, 3). \quad (2.2)$$

A. Previously defined measures

The first proposal for defining a quantum degree of polarization based on Stokes variables was a direct generalization of the classical measure [11–14]:

$$\mathbb{P}_1(\hat{\rho}) := \frac{\sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}}{\langle \hat{S}_0 \rangle}. \quad (2.3)$$

The index 1 emphasizes that this definition considers only first-order moments of the Stokes operators. Accordingly, all the product-states $|\psi\rangle_H |0\rangle_V$ have a degree of polarization equal to unity. For the two-mode state close to the vacuum, i.e., $|\psi\rangle_H \to |0\rangle_H$, one obtains also $\mathbb{P}_1 = 1$, which is an unphysical result [16].

Since $\mathbb{P}_1$ cannot be regarded as a proper definition of the quantum degree of polarization, proposals based on higher-order moments have to be considered. A second-order quantum degree of polarization was introduced in Ref. [15]:

$$\mathbb{P}_2(\hat{\rho}) := \sqrt{1 - \frac{(\Delta\hat{\mathbf{S}})^2}{\langle \hat{\mathbf{S}}^2 \rangle}}. \quad (2.4)$$

Here $\hat{\mathbf{S}}$ is the Stokes vector whose components are written in Eqs. (2.1) and

$$(\Delta\hat{\mathbf{S}})^2 := (\Delta\hat{S}_1)^2 + (\Delta\hat{S}_2)^2 + (\Delta\hat{S}_3)^2$$

is the total variance of the Stokes operators. We get, further,

$$\mathbb{P}_2(\hat{\rho}) = \sqrt{\frac{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}{\langle \hat{\mathbf{S}}^2 \rangle}}. \quad (2.5)$$

This definition gives the correct answer for the state close to the vacuum: $\mathbb{P}_2(|\psi\rangle_H |0\rangle_V) \to 0$ when $|\psi\rangle_H \to |0\rangle_H$.

More recently, various distance-type degrees of polarization have also been investigated [24,27]. Recall that, in general, the distance of a given state having a specific property to a reference set of states not having it has been recognized as a measure of that property. The essence of defining a reliable distance-type measure consists in choosing a convenient metric and identifying an appropriate reference set of states. Application of this recipe to the polarization issues is greatly facilitated by precise knowledge of the set $\mathcal{U}$ of unpolarized states. Indeed, any unpolarized two-mode state $\hat{\sigma}$ has a block-diagonal sector $\hat{\sigma}_b$ [26] which is SU(2) invariant [44,45]:

$$\hat{\sigma}_b = \sum_{N=0}^{\infty} \pi_N \frac{1}{N+1} \hat{P}_N. \quad (2.6)$$

In Eq. (2.6),

$$\hat{P}_N := \sum_{n=0}^{N} |n, N-n\rangle\langle n, N-n| \quad (2.7)$$

is the projection operator onto the vector subspace of the N-photon states, called the Nth excitation manifold. We have denoted $|n, N-n\rangle := |n\rangle_H \otimes |N-n\rangle_V$. Further, π_N are the photon-number probabilities in the SU(2)-invariant state $\hat{\sigma}_b$ and they satisfy the normalization condition

$$\sum_{N=0}^{\infty} \pi_N = 1. \quad (2.8)$$

Note that any SU(2)-invariant state (2.6) is Fock diagonal and, except for the vacuum, is mixed. We recall two distance-type measures for the quantum degree of polarization based on Hilbert-Schmidt and Bures metrics, which were defined in Ref. [24]:

$$\mathbb{P}_{\mathrm{HS}}(\hat{\rho}) = \inf_{\hat{\sigma} \in \mathcal{U}} \mathrm{Tr}[(\hat{\rho} - \hat{\sigma})^2], \quad (2.9)$$

$$\mathbb{P}_{\mathrm{B}}(\hat{\rho}) = 1 - \sup_{\hat{\sigma} \in \mathcal{U}} \sqrt{\mathcal{F}(\hat{\rho}, \hat{\sigma})}, \quad (2.10)$$

where $\mathcal{U}$ represents the set of all unpolarized two-mode states, while $\mathcal{F}$ stands for the fidelity between two states [46],

$$\mathcal{F}(\hat{\rho}_1, \hat{\rho}_2) := \left[\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2} \hat{\rho}_2 \hat{\rho}_1^{1/2}} \right]^2. \quad (2.11)$$

In Refs. [26] and [27] a quite different approach to defining quantum degrees of polarization was proposed. The polarization properties of the given state $\hat{\rho}$ were delegated to its block-diagonal sector,

$$\hat{\rho}_b := \sum_{N=0}^{\infty} \hat{P}_N \hat{\rho} \hat{P}_N. \quad (2.12)$$

The polarization-relevant state (2.12) is the result of an ideal nonselective measurement of the observable $\hat{N} := \hat{N}_H + \hat{N}_V$

which preserves the photon-number distribution of the given two-mode state $\hat{\rho}$ [26]. In particular, the quantum degree of polarization is defined as

$$\mathbb{P}(\hat{\rho}) := \mathbb{P}(\hat{\rho}_b), \qquad (2.13)$$

so that in Eqs. (2.9) and (2.10) one should replace $\hat{\rho}$ with $\hat{\rho}_b$ and $\hat{\sigma}$ with $\hat{\sigma}_b$.

Let us denote $\mu_{N,n}$, $(n = 0, 1, \cdots, N)$, the eigenvalues of the positive operator $\hat{P}_N \hat{\rho} \hat{P}_N$. Their sum is precisely the probability of the Nth excitation manifold:

$$p_N = \text{Tr}(\hat{\rho} \hat{P}_N) = \sum_{n=0}^{N} \mu_{N,n}. \qquad (2.14)$$

The N-photon state $\hat{\rho}_N := \frac{1}{p_N} \hat{P}_N \hat{\rho} \hat{P}_N$, $(p_N > 0)$, commutes with any SU(2)-invariant state $\hat{\sigma}_b$, Eq. (2.6), and so does the polarization state $\hat{\rho}_b$:

$$[\hat{\rho}_b, \hat{\sigma}_b] = \hat{0}. \qquad (2.15)$$

Extremization of expressions (2.9) and (2.10) of the Hilbert-Schmidt and Bures measures was previously carried out [24,26,27] by applying the method of Lagrange multipliers. An important and helpful property for the ongoing evaluations is the commutation relation (2.15). We write here the following general expressions in terms of the photon-number probabilities p_N and the eigenvalues $\mu_{N,n}$:

$$\mathbb{P}_{\text{HS}}(\hat{\rho}) = \sum_{N=0}^{\infty} \sum_{n=0}^{N} \mu_{N,n}^2 - \sum_{N=0}^{\infty} \frac{p_N^2}{N+1}, \qquad (2.16)$$

$$\mathbb{P}_{\text{B}}(\hat{\rho}) = 1 - \sqrt{\sum_{N=0}^{\infty} \frac{1}{N+1} \left(\sum_{n=0}^{N} \sqrt{\mu_{N,n}} \right)^2}. \qquad (2.17)$$

An interesting case to examine is the polarization of an arbitrary pure state conveniently written as an expansion in pure N-photon states

$$|\Psi\rangle = \sum_{N=0}^{\infty} c_N |\Psi_N\rangle. \qquad (2.18)$$

We have $p_N = |c_N|^2$ and $\sum_{N=0}^{\infty} p_N = 1$. Accordingly, the associated state $\hat{\rho}_b$ is the mixture

$$\hat{\rho}_b = \sum_{N=0}^{\infty} |c_N|^2 |\Psi_N\rangle\langle\Psi_N|. \qquad (2.19)$$

Equations (2.16) and (2.17) greatly simplify to

$$\mathbb{P}_{\text{HS}}(|\Psi\rangle\langle\Psi|) = \sum_{N=0}^{\infty} |c_N|^4 \frac{N}{N+1}, \qquad (2.20)$$

$$\mathbb{P}_{\text{B}}(|\Psi\rangle\langle\Psi|) = 1 - \sqrt{\sum_{N=0}^{\infty} \frac{|c_N|^2}{N+1}}. \qquad (2.21)$$

III. RELATIVE ENTROPY AS A MEASURE OF QUANTUM POLARIZATION

Despite not being a true metric, the relative entropy is acceptable as a measure of polarization due to its outstanding

distinguishability properties as discussed in Ref. [47]. Recall that the relative entropy between state $\hat{\sigma}'$ and state $\hat{\sigma}''$ is defined as the difference [48]

$$\mathcal{S}(\hat{\sigma}'|\hat{\sigma}'') := \text{Tr}[\hat{\sigma}' \ln(\hat{\sigma}')] - \text{Tr}[\hat{\sigma}' \ln(\hat{\sigma}'')]. \qquad (3.1)$$

The relative entropy was successfully used as a measure of entanglement for pure bipartite states providing one of the few exact and general evaluations [49]. A more recent general result [50] finds the relative entropy to be an exact measure of non-Gaussianity.

In view of the preceding discussion on the appropriateness of the associate block-diagonal density operator $\hat{\rho}_b$ in describing polarization, we define a degree of polarization based on the relative entropy as

$$\mathbb{P}_{\text{RE}}(\hat{\rho}) := \inf_{\hat{\sigma} \in \mathcal{U}} \frac{S(\hat{\rho}_b|\hat{\sigma}_b)}{1 + S(\hat{\rho}_b|\hat{\sigma}_b)}. \qquad (3.2)$$

The relative entropy between the commuting states $\hat{\rho}_b$ and $\hat{\sigma}_b$, Eq. (2.6), is

$$S(\hat{\rho}_b|\hat{\sigma}_b) = -S(\hat{\rho}_b) - \sum_{N=0}^{\infty} p_N \ln \left(\frac{\pi_N}{N+1} \right). \qquad (3.3)$$

In Eq. (3.3), $S(\hat{\rho}) = -\text{Tr}[\hat{\rho} \ln(\hat{\rho})]$ is the von Neumann entropy of state $\hat{\rho}$ and p_N is given by Eq. (2.14). Our task is to evaluate the parameters $\tilde{\pi}_N$ of the unpolarized state $\hat{\tilde{\sigma}}_b$ for which the infimum in Eq. (3.2) is realized. As far as we know at this moment, the present work is the first to look for the closest unpolarized two-mode state through relative entropy. We have to minimize function (3.3) with respect to the probabilities π_N under constraint (2.8). Similarly to what was previously discussed regarding other distance-type measures [24,26,27], the extremization is easily performed by applying the method of Lagrange multipliers. We easily get the conditions of minimum

$$\tilde{\pi}_N = p_N, \quad (N = 0, 1, 2, 3, \dots). \qquad (3.4)$$

Interestingly, the closest unpolarized state is the same as for the Hilbert-Schmidt polarization measure first written in Ref. [24]. It has the same photon-number distribution as the given state,

$$\hat{\tilde{\sigma}}_b = \sum_{N=0}^{\infty} p_N \frac{1}{N+1} \hat{P}_N, \qquad (3.5)$$

and leads to the final formula

$$S(\hat{\rho}_b|\hat{\tilde{\sigma}}_b) = -S(\hat{\rho}_b) - \sum_{N=0}^{\infty} p_N \ln \left(\frac{p_N}{N+1} \right). \qquad (3.6)$$

For the pure state (2.18), the minimal relative entropy (3.6) simplifies to

$$S(\hat{\rho}_b|\hat{\tilde{\sigma}}_b) = \sum_{N=0}^{\infty} |c_N|^2 \ln(N+1). \qquad (3.7)$$

IV. POLARIZATION OF FOCK-DIAGONAL STATES

Obviously, any Fock-diagonal state $\hat{\rho}$ coincides with its polarization sector, namely, $\hat{\rho}_b = \hat{\rho}$. For the sake of simplicity, in this paper we deal with a mixed Fock-diagonal product

state,

$$\hat{\rho} = \hat{\rho}_H \otimes \hat{\rho}_V, \tag{4.1}$$

where

$$\hat{\rho}_H = \sum_{m=0}^{\infty} \xi_m |m\rangle\langle m|, \quad \hat{\rho}_V = \sum_{n=0}^{\infty} \eta_n |n\rangle\langle n|. \tag{4.2}$$

The photon-number distributions ξ_m and η_m satisfy the normalization conditions

$$\sum_{m=0}^{\infty} \xi_m = 1, \quad \sum_{m=0}^{\infty} \eta_m = 1.$$

Our first aim here is to write the quantum degrees of polarization reviewed in the previous section for this type of Fock-diagonal state. With the notable exception of the thermal states, we thus deal with non-Gaussian density operators which are known to be important in several protocols of quantum information. To begin, let us recall the photon-number operators in the two modes, i.e., $\hat{N}_H = \hat{a}_H^\dagger \hat{a}_H$ and $\hat{N}_V = \hat{a}_V^\dagger \hat{a}_V$. The expectation values of the Stokes operators, Eq. (2.1), are found to be

$$\langle \hat{S}_1 \rangle = \langle \hat{S}_2 \rangle = 0, \quad \langle \hat{S}_3 \rangle = \langle \hat{N}_H \rangle - \langle \hat{N}_V \rangle,$$

and further,

$$\langle \hat{\mathbf{S}}^2 \rangle = 2(\langle \hat{N}_H \rangle \langle \hat{N}_V \rangle + \langle \hat{N}_H \rangle + \langle \hat{N}_V \rangle) + \langle \hat{N}_H^2 \rangle + \langle \hat{N}_V^2 \rangle.$$

In general we simply find

$$\langle \hat{N}_H^j \rangle = \sum_{m=0}^{\infty} \xi_m m^j, \quad \langle \hat{N}_V^j \rangle = \sum_{m=0}^{\infty} \eta_m m^j \quad (j = 1, 2, \dots).$$

By using Eq. (2.3) we get the first-order Stokes degree of polarization

$$\mathbb{P}_1(\hat{\rho}) = \frac{|\langle \hat{N}_H \rangle - \langle \hat{N}_V \rangle|}{\langle \hat{N}_H \rangle + \langle \hat{N}_V \rangle}. \tag{4.3}$$

Further, Eq. (2.4) becomes in this case

$$\mathbb{P}_2(\hat{\rho}) = \frac{|\langle \hat{N}_H \rangle - \langle \hat{N}_V \rangle|}{\sqrt{\langle \hat{\mathbf{S}}^2 \rangle}}. \tag{4.4}$$

As regards the distance-type measures, they are simply written by setting $\mu_{N,n} = \xi_n \eta_{N-n}$ into Eqs. (2.16), (2.17), and (3.6).

As an example, the entropic degree of polarization is given by Eq. (3.2) after inserting the explicit relative entropy of polarization for state (4.1):

$$S(\hat{\rho}|\hat{\bar{\sigma}}_b) = \sum_{m=0}^{\infty} [\xi_m \ln(\xi_m) + \eta_m \ln(\eta_m)] - \sum_{N=0}^{\infty} p_N \ln\left(\frac{p_N}{N+1}\right). \tag{4.5}$$

In the following we specialize the above final expressions of the quantum degrees of polarization to two interesting Fock-diagonal states: the unique Gaussian case, which is a two-mode thermal state, and a non-Gaussian example prepared by adding photons to a thermal state.

V. POLARIZATION OF TWO-MODE THERMAL STATES

We here compute the quantum degrees of polarization for the relevant class of two-mode thermal states, whose density operators depend only on the mean occupancies $\bar{n}_1 := \langle \hat{N}_H \rangle$, $\bar{n}_2 := \langle \hat{N}_V \rangle$ of the modes,

$$\hat{\rho}_{\text{th}}(\overline{n}_1, \overline{n}_2) := \hat{\rho}_{\text{th}}(\overline{n}_1) \otimes \hat{\rho}_{\text{th}}(\overline{n}_2), \tag{5.1}$$

with

$$\hat{\rho}_{\text{th}}(\overline{n}_j) = \frac{1}{\overline{n}_j + 1} \sum_{n=0}^{\infty} \left(\frac{\overline{n}_j}{\overline{n}_j + 1}\right)^n |n\rangle\langle n| \quad (j = 1, 2).$$

It is convenient to rewrite the thermal state (5.1) as follows:

$$\hat{\rho}_{\text{th}}(\overline{n}_1, \overline{n}_2) = (1 - q_1)(1 - q_2)$$
$$\times \sum_{N=0}^{\infty} \sum_{n=0}^{N} q_1^n q_2^{N-n} |n, N-n\rangle\langle n, N-n|, \tag{5.2}$$

where the notation $q_j := \overline{n}_j/(\overline{n}_j + 1)$ has been introduced. Before proceeding with the evaluation of various polarization degrees we have to note an important property. At thermal equilibrium $(q_1 = q_2 := q)$, the density operator (5.2) simplifies to

$$\hat{\rho}_{\text{th}}(\bar{n}, \bar{n}) = (1 - q)^2 \sum_{N=0}^{\infty} q^N \hat{P}_N. \tag{5.3}$$

According to Eq. (2.6), the thermal state at equilibrium, Eq. (5.3), is unpolarized and therefore its degree of polarization should be equal to 0, regardless of the type of measure we use. In the following we take the thermal equilibrium limit as a useful test of our results.

A. Degrees of polarization based on the Stokes operators

We insert the thermal mean occupancies and the expectation values

$$\langle \hat{N}_H^2 \rangle = \bar{n}_1(2\bar{n}_1 + 1), \quad \langle \hat{N}_V^2 \rangle = \bar{n}_2(2\bar{n}_2 + 1)$$

in Eqs. (4.3) and (4.4) to easily get

$$\mathbb{P}_1(\hat{\rho}_{\text{th}}(\overline{n}_1, \overline{n}_2)) = \frac{|\overline{n}_1 - \overline{n}_2|}{\overline{n}_1 + \overline{n}_2}, \tag{5.4}$$

$$\mathbb{P}_2(\hat{\rho}_{\text{th}}(\overline{n}_1, \overline{n}_2)) = \frac{|\overline{n}_1 - \overline{n}_2|}{\sqrt{2\,\overline{n}_1^2 + 2\,\overline{n}_2^2 + 2\,\overline{n}_1\,\overline{n}_2 + 3\,\overline{n}_1 + 3\,\overline{n}_2}}. \tag{5.5}$$

Both expressions were first written in Ref. [36]. They obviously meet the requirement of being equal to 0 at thermal equilibrium.

B. Distance-type degrees of polarization

Two distance-type measures of polarization can be evaluated analytically for a thermal state, Eq. (5.2), due to the privilege of getting a closed form for the probability of an N-photon state:

$$p_N = (1 - q_1)(1 - q_2) \sum_{n=0}^{N} q_1^n q_2^{N-n}$$
$$= (1 - q_1)(1 - q_2) \frac{q_1^{N+1} - q_2^{N+1}}{q_1 - q_2}. \tag{5.6}$$

Above we have simply used the geometric sequence

$$\sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1 - q}.$$

For $0 \leqq q < 1$ and $N \to \infty$, we get the well-known geometric series

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1 - q}, \quad (0 \leqq q < 1), \tag{5.7}$$

whose term-by-term integration yields another useful power series:

$$\sum_{n=0}^{\infty} \frac{1}{n+1} q^{n+1} = -\ln(1 - q), \quad (0 \leqq q < 1). \tag{5.8}$$

Evaluation of the Hilbert-Schmidt polarization degree, Eq. (2.16), is routinely performed via Eqs. (5.6) and (5.8). In terms of thermal mean occupancies we nicely get

$$\mathbb{P}_{\text{HS}}(\hat{\rho}_{\text{th}}(\overline{n}_1, \overline{n}_2)) = \frac{1}{(2\overline{n}_1 + 1)(2\overline{n}_2 + 1)}$$
$$- \frac{1}{(\overline{n}_1 - \overline{n}_2)^2} \ln\left[\frac{(\overline{n}_1 + \overline{n}_2 + 1)^2}{(2\overline{n}_1 + 1)(2\overline{n}_2 + 1)}\right]. \tag{5.9}$$

In the symmetric case $\bar{n}_1 = \bar{n}_2$ the obtained degree goes to 0 as it should. However, Eq. (5.9) reveals a nonmonotonic behavior of the Hilbert-Schmidt degree which has the limit 0 when the difference between the two thermal mean occupancies is very large. This suggests that the Hilbert-Schmidt metric is not a reliable measure of polarization.

According to Eq. (2.17), in order to obtain the Bures degree of polarization for a two-mode thermal state we need to use once more the geometric sequence to write

$$\sum_{n=0}^{N} \sqrt{\mu_{N,n}} = [(1 - q_1)(1 - q_2)]^{1/2} \frac{q_2^{\frac{N+1}{2}} - q_1^{\frac{N+1}{2}}}{q_2^{1/2} - q_1^{1/2}}.$$

By replacing the above result in Eq. (2.17) and again taking advantage of the series (5.8), we write the Bures degree of polarization:

$$\mathbb{P}_{\text{B}}(\hat{\rho}_{\text{th}}(\overline{n}_1, \overline{n}_2)) = 1 - \frac{\sqrt{2 \ln\left[\sqrt{(\overline{n}_1 + 1)(\overline{n}_2 + 1)} - \sqrt{\overline{n}_1 \overline{n}_2}\right]}}{|\sqrt{\overline{n}_1(\overline{n}_2 + 1)} - \sqrt{\overline{n}_2(\overline{n}_1 + 1)}|}. \tag{5.10}$$

The Bures degree (5.10) appears to be monotonic and is 0 in the symmetric case.

The last degree of polarization we have to examine is the entropic one, Eq. (4.5). With the von Neumann entropy of a one-mode thermal state,

$$S(\hat{\rho}_{\text{th}}(\overline{n})) = (\overline{n} + 1) \ln(\overline{n} + 1) - \overline{n} \ln(\overline{n}), \tag{5.11}$$

and the explicit expression of the probability of an N-photon state, Eq. (5.6), we are left with the numerical evaluation of the sum appearing in the expression of the minimal relative
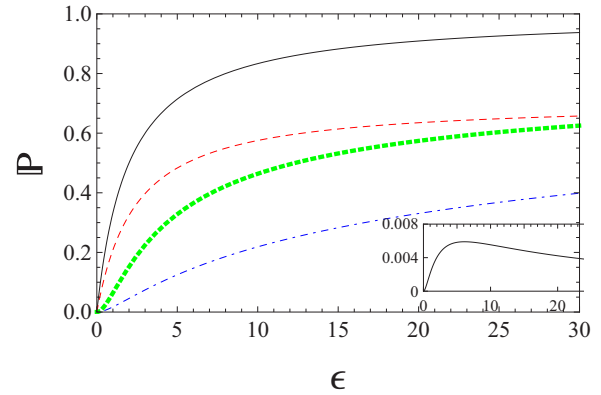

FIG. 1. Different degrees of polarization of the two-mode thermal state in terms of $\epsilon = |\overline{n}_1 - \overline{n}_2|$: Stokes degrees $\mathbb{P}_1$ (solid black curve) and $\mathbb{P}_2$ (dashed red curve), relative entropy (dotted green line), Bures (dot-dashed blue line). Inset: Nonmonotonic behavior of the Hilbert-Schmidt measure. We have considered $\overline{n}_2 = 1$.

entropy:

$$S(\hat{\rho}|\hat{\bar{\sigma}}_b) = -S(\hat{\rho}_{\text{th}}(\overline{n}_1)) - S(\hat{\rho}_{\text{th}}(\overline{n}_2)) - \sum_{N=0}^{\infty} p_N \ln\left(\frac{p_N}{N+1}\right). \tag{5.12}$$

Figure 1 shows a monotonic aspect of either Stokes-operator-based degrees or Bures and entropic ones versus the relative mean occupancy $\epsilon := |\overline{n}_1 - \overline{n}_2|$ of the two modes. These degrees are consistent, having a similar behavior with respect to the same parameter. Quite the reverse, in the inset we see the Hilbert-Schmidt degree of polarization, Eq. (5.9), displaying a flat maximum and then slowly decreasing to 0. The results obtained in this section on the polarization of the thermal states are compared in the following to the similar degrees for a class of non-Gaussian ones prepared by adding photons to thermal states.

VI. POLARIZATION OF NON-GAUSSIAN STATES

It is well known that various excitations on a single-mode thermal state of the type $\hat{\rho} \sim (\hat{a}^\dagger)^k \hat{a}^l \hat{\rho}_{\text{th}} (\hat{a}^\dagger)^l \hat{a}^k$ are diagonal in the Fock basis. Here $\hat{a}$ and $\hat{a}^\dagger$ are the amplitude operators of the field mode. In general, states with added photons are nonclassical and non-Gaussian. We choose to apply now both Stokes-operator-based and distance-type degrees of polarization to a class of states of this type, that is, their density operator is the tensor product shown in Eqs. (4.1) and (4.2). Specifically, we work here with a tensor product of multi-photon-added thermal states. Single-mode photon-added thermal states (PATSs) were introduced in Ref. [51], where their nonclassicality expressed by the negativity of the P function was studied. Quite recently, thermal states with one added photon were experimentally prepared and their nonclassical properties were investigated [52–54]. The present authors were interested in the nonclassicality of PATSs and looked at their evolution during the interaction of the field mode with a thermal reservoir. Basically we have investigated two processes: loss of nonclassicality indicated by the time development of the Wigner and P functions and loss of

non-Gaussianity shown by some recently introduced distance-type measures [55,56].

Adding M photons to a thermal state $\hat{\rho}_{\mathrm{th}}(\overline{n})$ results in the density operator

$$\hat{\rho}^{(M)}(\overline{n}) = \frac{1}{M!\,(\overline{n}+1)^M}\,(\hat{a}^\dagger)^M\,\hat{\rho}_{\mathrm{th}}(\overline{n})\,\hat{a}^M, \qquad (6.1)$$

which in the Fock basis is easily written as the mixture [55]

$$\hat{\rho}^{(M)}(\overline{n}) = \sum_{l=M}^{\infty} \binom{l}{M} \frac{\overline{n}^{l-M}}{(\overline{n}+1)^{l+1}} \,|\,l\,\rangle\langle\,l\,|. \qquad (6.2)$$

The purity of a PATS was found in Ref. [56] as a function of the thermal ratio $q := \overline{n}/(\overline{n}+1)$, which involves a Legendre polynomial

$$\mathrm{Tr}\big\{[\hat{\rho}^{(M)}(\overline{n})]^2\big\} = \left(\frac{1-q}{1+q}\right)^{M+1}\mathcal{P}_M\left(\frac{1+q^2}{1-q^2}\right). \quad (6.3)$$

Note that the above Legendre polynomial $\mathcal{P}_M$ is strictly positive because its argument is at least equal to 1.

For polarization issues we consider the tensor product of two PATSs:

$$\hat{\rho} := \hat{\rho}^{(M)}(\overline{n}_1) \otimes \hat{\rho}^{(S)}(\overline{n}_2). \qquad (6.4)$$

A. Degrees of polarization based on the Stokes operators

In order to evaluate the quantum degrees of polarization based on the Stokes variables we only need the expectation values of the operators $\hat{N}$ and $\hat{N}^2$ of the PATS $\hat{\rho}^{(M)}(\overline{n})$. These can be obtained with the photon-number distribution of PATSs arising from Eq. (6.2). More elegantly, we can use the generating function of PATSs written in Ref. [56] to get

$$\langle\,\hat{N}\,\rangle = M(\overline{n}+1) + \overline{n},$$
$$\langle\,\hat{N}^2\,\rangle = \overline{n}(M+1)[(M+2)\overline{n} + 2M+1] + M^2. \quad (6.5)$$

In the following, all figures representing various polarization degrees are plotted versus the relative thermal mean occupancy $\epsilon = |\overline{n}_1 - \overline{n}_2|$ of the two modes. We used the same values of the parameters of PATSs in order to facilitate the comparison of their behaviors. Specifically, the black plots are characterized by $\overline{n}_2 = 0$. That is, we deal with the particular state

$$\hat{\rho} := \hat{\rho}^{(M)}(\overline{n}_1) \otimes |S\rangle\langle S|, \qquad (6.6)$$

which is a tensor product of a PATS and a Fock state. The red plots describe the polarization of a proper two-mode PATS, Eq. (6.4), for a fixed value $\overline{n}_2 = 1$. For both sets of plots (black and red), we examine both symmetric PATSs ($M = S = 2$) and nonsymmetric ones ($M = 1$, $S = 2$). Also plotted (solid lines) in all subsequent figures are the degrees of polarization for the thermal states from which the corresponding PATSs are prepared.

Inserting Eqs. (6.5) written for both the H and the V modes into formulas (4.3) and (4.4), one obtains the expression of the two Stokes degrees of polarization described in Sec. II. In Fig. 2 we plot the degrees $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively, under the conditions and parameters described above. What can we see in these plots? The degree $\mathbb{P}_1$ based on only first-order moments of the Stokes operators indicates that by adding
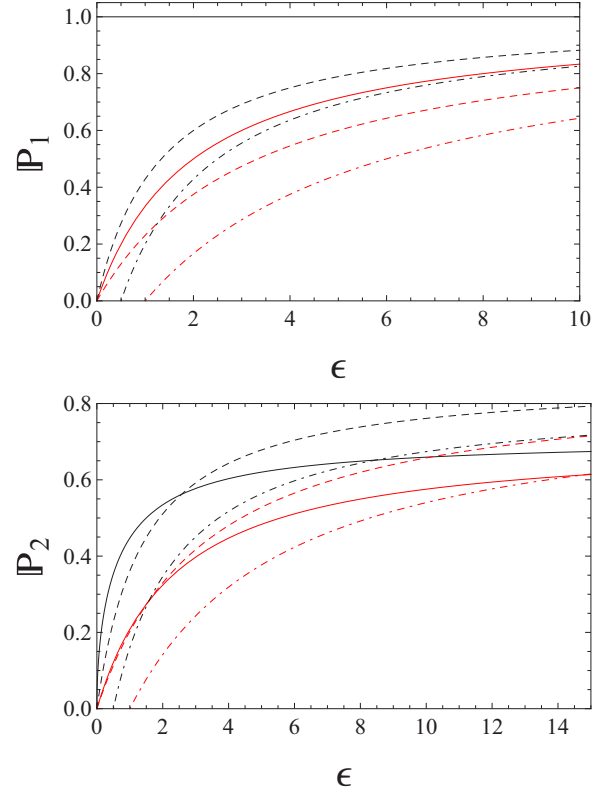

FIG. 2. Stokes degrees of polarization, (2.3) (top) and (2.5) (bottom), in terms of $\epsilon = |\overline{n}_1 - \overline{n}_2|$ for two classes of states: the two-mode thermal state (solid lines) and two-mode photon-added thermal state (dashed and dot-dashed lines). We have considered $\overline{n}_2 = 0$ (black curves) and $\overline{n}_2 = 1$ (red curves) and, in addition, two cases of the two-mode photon-added thermal state: the symmetric one $M = S = 2$ (dashed curves) and the nonsymmetric one $M = 1$, $S = 2$ (dot-dashed curves).

photons to a thermal state we are decreasing its polarization. This effect is stronger for nonsymmetric addition. However, $\mathbb{P}_1$ appears to be monotonic and consistent for both fixed values of $\overline{n}_2$. Unlike the aspect of $\mathbb{P}_1$, the degree $\mathbb{P}_2$ displays an acute lack of consistency. The difference in the polarization of thermal states vs PATSs described by $\mathbb{P}_2$ fluctuates with their relative thermal mean occupancy.

B. Distance-type degrees of polarization

In order to calculate the distance-type polarization degrees introduced in Sec. III for state (6.4) we first write the probability of its Nth excitation manifold, Eq. (2.14):

$$p_N = \sum_{l=0}^{N} \binom{l}{M}\binom{N-l}{S} \frac{\overline{n}_1^{l-M}}{(\overline{n}_1+1)^{l+1}} \frac{\overline{n}_2^{N-l-S}}{(\overline{n}_2+1)^{N-l+1}}. \quad (6.7)$$

This greatly simplifies for the particular state (6.6):

$$p_N = \binom{N-S}{M} \frac{\overline{n}_1^{N-S-M}}{(\overline{n}_1+1)^{N-S+1}}. \qquad (6.8)$$

For evaluating the Hilbert-Schmidt degree of polarization we use Eq. (2.16), as well as Eq. (6.3) for the degree of purity of a PATS. The Hilbert-Schmidt measure in terms of thermal
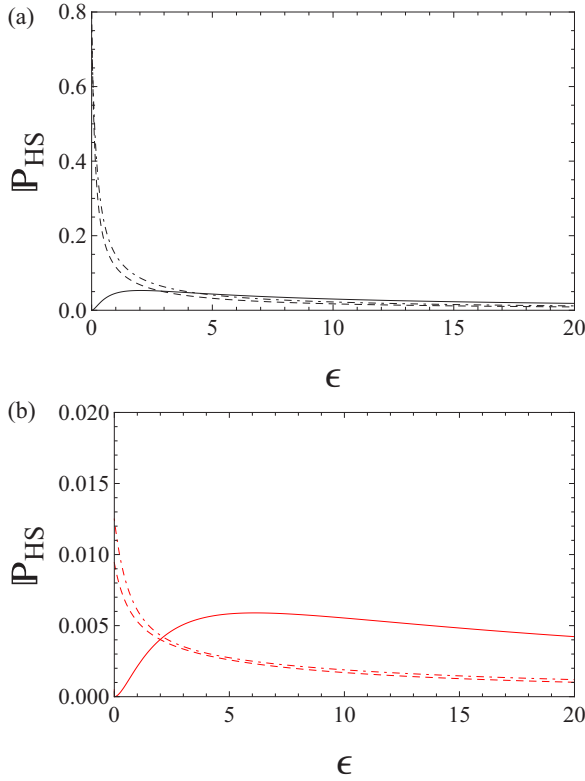
FIG. 3. Evaluation of the Hilbert-Schmidt degree of polarization in terms of $\epsilon = |\bar{n}_1 - \bar{n}_2|$ for two classes of states: the two-mode thermal state (solid lines) and two-mode photon-added thermal state (dashed and dot-dashed lines). The parameters of two-mode PATSs are the same as in Fig. 2.

mean occupancies $\bar{n}_1$ and $\bar{n}_2$ is then

$$\mathbb{P}_{HS}\big(\hat{\rho}^{(M)}(\bar{n}_1) \otimes \hat{\rho}^{(S)}(\bar{n}_2)\big) = \frac{1}{(2\bar{n}_1 + 1)^{M+1}} \frac{1}{(2\bar{n}_2 + 1)^{S+1}}$$
$$\times \mathcal{P}_M\left(1 + \frac{2\bar{n}_1^2}{2\bar{n}_1 + 1}\right) \mathcal{P}_S\left(1 + \frac{2\bar{n}_2^2}{2\bar{n}_2 + 1}\right) - \sum_{N=0}^{\infty} \frac{p_N^2}{N+1},$$

with $\mathcal{P}_L(x)$ being the Legendre polynomial of degree L.

Figure 3 presents plots of the Hilbert-Schmidt degree of polarization $\mathbb{P}_{HS}$ for two-mode PATSs compared to the corresponding one for thermal states. We can see that adding photons to a thermal state strongly modifies the aspect of this polarization degree, in contrast with the evolution shown in Fig. 2 for Stokes-operator-based degrees.

Using the probability of its Nth excitation manifold written in Eqs. (6.7) and (6.8), we have numerically calculated the Bures measure, Eq. (2.17), and the entropic one, Eq. (4.5), for the two-mode thermal state and two-mode PATSs. To accomplish this task we have used the von Neumann entropy of a PATS written in a simplified form,

$$S(\hat{\rho}^{(M)}(\bar{n})) = (M + 1)S(\hat{\rho}_{th}(\bar{n}))$$
$$- \sum_{l=M}^{\infty} \big(\hat{\rho}^{(M)}(\bar{n})\big)_{ll} \ln\left[\binom{l}{M}\right], \quad (6.9)$$

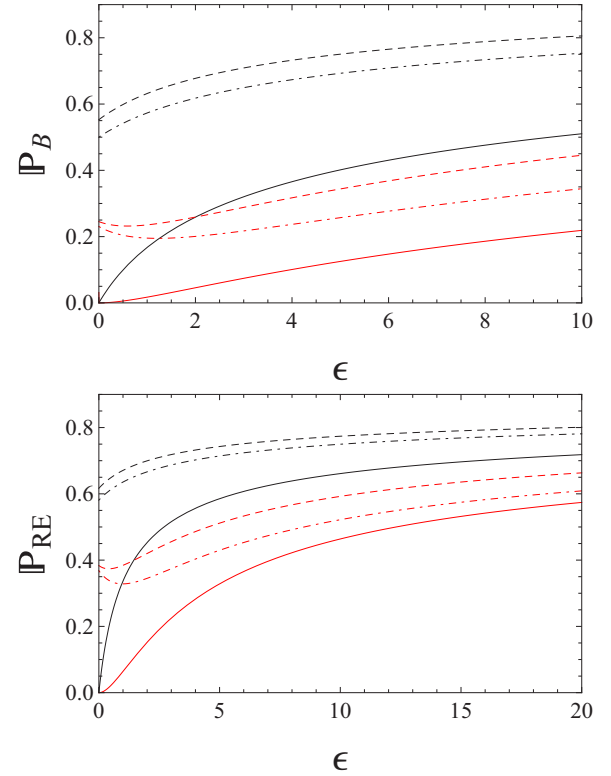with $S(\hat{\rho}_{th}(\bar{n}))$ being the von Neumann entropy of the thermal state, Eq. (5.11).

FIG. 4. Bures (top) and relative entropy (bottom) degrees of polarization as defined in the text vs the relative thermal mean occupancy $\epsilon = |\bar{n}_1 - \bar{n}_2|$ for the same states and parameters considered in Figs. 1–3.

For proper comparison of all these degrees, when plotting our final Fig. 4, we have considered the same states and parameters as in Figs. 2 and 3. What can we say about the plots in Fig. 4? They appear to be in agreement and manifest an overall consistency. Unlike the Hilbert-Schmidt degree, they show a monotonic behavior with respect to the relative thermal mean occupancy. Consistency means here the same ordering of the degrees for the same states in both cases.

C. A limit case: Polarization of a two-mode Fock state

When setting $\bar{n}_1 = 0$ in Eq. (6.6), we get the density operator of a two-mode Fock state $|M, S\rangle$. The expressions of the degrees of polarization for this pure state with $N = M + S$ photons easily emerge as follows. The series expansions on the right-hand side of Eqs. (2.20), (2.21), and (3.7) reduce to a single term:

$$\mathbb{P}_{HS} = \frac{N}{N+1}, \quad \mathbb{P}_B = 1 - \sqrt{\frac{1}{N+1}},$$
$$\mathbb{P}_{RE} = 1 - \frac{1}{1 + \ln(N+1)}. \quad (6.10)$$

Note the obvious inequalities

$$\mathbb{P}_{HS} \geqq \mathbb{P}_B, \quad \mathbb{P}_{HS} \geqq \mathbb{P}_{RE}. \quad (6.11)$$

While the only N dependence is common to the above three distance-type degrees, the Stokes-operator-based degrees depend on the difference between the occupancies of the two

modes, just as for classical light [10]. We get

$$\mathbb{P}_1 = \frac{|M - S|}{N}, \quad \mathbb{P}_2 = \frac{|M - S|}{\sqrt{N(N + 2)}},$$

$$\mathbb{P}_1 \geqq \mathbb{P}_2. \tag{6.12}$$

Comparing the above degrees to those for thermal light in Sec. V, it appears that the Stokes-operator-based ones are not quite sensitive to the statistical properties of the radiation. The distance-type degrees are all monotonic for the highly nonclassical Fock states, which is not the case for the thermal states, as shown in Fig. 1.

VII. CONCLUSIONS

We have examined the polarization of two-mode photon-added thermal states, which are known to be nonclassical, non-Gaussian, and diagonal in the Fock basis, in comparison to the thermal states from which they originate. The latter are classical and the only Fock-diagonal Gaussian states. Use was made of two types of degrees of polarization: one defined with Stokes operators by analogy with a classical description, the other being a bunch of distance-type measures based on Hilbert-Schmidt and Bures metrics as well as on the relative entropy. We have first given a more general treatment valid for any Fock-diagonal state. Specializing it to the case of thermal states, we have written for the first time closed and simple expressions for their Bures and Hilbert-Schmidt degrees of polarization.

Adding photons to thermal states is a de-Gaussification process which is currently being investigated in experiments. Modification of polarization due to this procedure is one of our interests in the present paper. We have found that, according to the Stokes-operator-based degrees, polarization decreases upon adding photons to thermal states, as shown in Fig. 2. On the contrary, when looking at the Bures and entropic degrees in Fig. 4, polarization is larger for photon-added states. The Hilbert-Schmidt degree is not reliable due to its lack of consistency. So we are now in a dilemma regarding the description of polarization with the two types of defined degrees. How can we solve this? The agreement between the Bures degree and the entropic one is quite remarkable. However, the only solid conclusion that one can come to is that a supplementary testing of polarization is, at this moment, highly desirable. Work along these lines is in progress.

ACKNOWLEDGMENTS

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