# Bending of light in a Coulomb gas

Taekoon Lee*

*Department of Physics, Kunsan National University, Kunsan 54150, Korea*

(Received 12 August 2018; published 10 September 2018)

Photons traveling in a background electromagnetic field may bend via the vacuum polarization effect with the background field. The bending in a Coulomb field by a heavy nucleus is small even at a large atomic number, rendering it difficult to detect experimentally. As an amplifying mechanism of the effect we consider the bending of light traveling in a chamber of Coulomb gas. The Gaussian nature of the bending in the gas increases the total bending angle in proportion to the square root of the photon travel distance. The enhancement can be orders of magnitude over the bending by a single nucleus at a small impact parameter, which may help experimental observation of the Coulombic bending.

DOI: 10.1103/PhysRevA.98.033811

The vacuum polarization effect of the quantum electrodynamics renders a photon traveling in a background electromagnetic field bend. For a photon moving in a slowly varying background field, with energy smaller than the electron rest mass, the bending may be described by a low-energy effective Lagrangian that encodes the vacuum polarization effect. At the leading order of the fine-structure constant the polarization effect is given by a box diagram with four external photon lines, which gives rise to the nonlinear interaction of Euler-Heisenberg [1,2]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha^2\hbar^3}{90m_e^4c^5}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right], \tag{1}$$

where $\hbar$, $c$, $\alpha$, and $m_e$ are the Planck constant, the speed of light, the fine-structure constant, and the electron mass, respectively, and $\tilde{F}_{\mu\nu}$ denotes the dual of the field strength tensor $F_{\mu\nu}$.

A linearization of the Euler-Heisenberg interaction in a slowly varying background field yields a photon dispersion relation in which the background field is encapsulated in an index of refraction $n$ [3–7]:

$$n = 1 + \frac{a\alpha^2\hbar^3}{45m_e^4c^5}[\boldsymbol{E}^2 + \boldsymbol{B}^2 - (\hat{\boldsymbol{k}}\cdot\boldsymbol{E})^2 - (\hat{\boldsymbol{k}}\cdot\boldsymbol{B})^2 - 2\hat{\boldsymbol{k}}\cdot(\boldsymbol{E}\times\boldsymbol{B})], \tag{2}$$

where $\hat{\boldsymbol{k}}$ is the unit vector in the direction of photon propagation, and $a$ is the birefringence constant that is either 8 or 14, depending on the photon polarization.

Thus a photon moving in a background field behaves as if it is traveling in a dielectric medium with a refractive index that depends on the background field strength, and consequently the photon bends when the field strength is nonuniform.

This light bending has been studied in relation to astronomical objects with strong electromagnet fields, such as magnetars or black holes [8–10]. On the opposite scale, at a microscopic level a particularly interesting problem is the bending in a Coulomb field. Because the field-dependent index of refraction becomes larger at a stronger field, the incoming photon bends toward the charge, in a fashion reminiscent of the gravitational bending in general relativity.

The bending angle can be easily calculated in geometrical optics. For a photon with the impact parameter $b$ in the Coulomb field by a nucleus of charge $Ze$, it is given by [11]

$$\theta(b) = \frac{aZ^2\alpha^3}{160}\left(\frac{\lambda\!\!\!^{-}_e}{b}\right)^4, \tag{3}$$

where $\alpha$ is the fine-structure constant, and $\lambda\!\!\!^{-}_e = \hbar/m_e c$ is the reduced electron Compton length.

The impact parameter in Eq. (3) cannot be arbitrarily small, putting a limit on the size of the bending angle. Requiring that the Euler-Heisenberg interaction be a small perturbation to the Maxwell theory places a constraint on the field strength [3]:

$$\frac{2a\alpha^2\hbar^3}{45m_e^4c^5}|F_{\mu\nu}|^2 \ll 1, \tag{4}$$

where $F_{\mu\nu}$ denotes the background field strength. For the Coulomb field

$$E(r) = \frac{Ze}{4\pi r^2}, \tag{5}$$

the constraint requires the radius to satisfy

$$r \gg \lambda\!\!\!^{-}_e\left(\frac{aZ^2\alpha^3}{90}\right)^{\frac{1}{4}}, \tag{6}$$

where the fine-structure constant is given by $\alpha = e^2/4\pi\hbar c$. Even for a large $Z$, the radius satisfying the constraint can be fairly small. For instance, at $Z = 100$,

$$r \gg 0.14\lambda\!\!\!^{-}_e. \tag{7}$$

Also the requirement that the background field be slowly varying demands [3] the following:

$$|\partial_\lambda F_{\mu\nu}| \ll \frac{m_e c}{\hbar}|F_{\mu\nu}|,$$

*tlee@kunsan.ac.kr

 

which, for the Coulomb field (5), is satisfied when

$$r \gg \lambda\!\!\!^{-}_e\,. \tag{8}$$

We also note that at a very small impact parameter where the electric field becomes strong the corrections to the Euler-Heisenberg interaction can be significant. The corrections arise from the box diagrams with more than four external photon lines. A simple dimensional analysis shows that these give rise to effective interactions in powers of

$$\frac{\alpha\hbar^3}{m_e^4 c^5}|F_{\mu\nu}|^2, \tag{9}$$

relative to the Euler-Heisenberg term. For these corrections to be small on the Coulomb field (5) the radius must satisfy

$$r \gg \sqrt{Z\alpha}\lambda\!\!\!^{-}_e. \tag{10}$$

The combined constraints of Eq. (6), (8), and (10) on the radius put a limit on the impact parameter in the bending angle (3). For heavy nuclei with $Z\alpha \sim 1$, it requires

$$b \gg \lambda\!\!\!^{-}_e. \tag{11}$$

The bending angle under this constraint is quite small, even at a large $Z$ and a small impact parameter. For instance, for $Z = 100$ and $b = 10\lambda\!\!\!^{-}_e$, the bending is 34 nano rad.

Because of the smallness of the effect detecting the bending experimentally may be challenging. It may thus be interesting to study an amplifying mechanism for the effect. As such a mechanism we consider in this paper a photon (in a collimated beam) traveling in a chamber of Coulomb gas that comprises heavy nuclei.

As the photon travels in the chamber it will bend off each nuclei in the gas, and because the bendings are random in the impact parameter as well as in the azimuthal angle to the beam axis, the distribution of the total bending angles will be Gaussian and the root-mean-square (rms) angle be proportional to the square root of the number of nuclei in the gas. An experimental consequence of this Gaussian bending will be a broadening of the photon beam as it travels through the gas.

Though securing the Coulomb gas is beyond the scope of this paper, we could imagine obtaining it by blowing off the valence electrons of the heavy atoms through an illumination of x rays or $\gamma$ rays, perhaps as well with the help of an electric field applied to the chamber to separate the electrons from the nuclei. Further, it may not be necessary to separate the electrons from the ionized nuclei, because the bendings off the electrons would be ignorable compared to those off the heavy nuclei. In this case ionization of the heavy atoms alone would suffice for the purpose. Furthermore, even ionization may not be necessary with high-energy photons (x ray or $\gamma$ ray), as in experiments for the Delbrück scattering [12], because the Coulombic interactions of the photons with the nuclei would occur deep inside the atoms near the nuclei. In this case the Rayleigh scattering off the electrons must be accurately subtracted.

Now to compute the rms angle we consider a cylindrical chamber of length $L$ and radius $R$ filled with a Coulomb gas and assume the photon travels along the axis of the chamber. Although we start with this particular form of chamber the final result will be independent of the chamber geometry.

To be specific, for a photon with incoming velocity $\vec{c} = c\hat{n}$, where $\hat{n}$ denotes the unit vector in the beam direction, the exit velocity $\vec{v}_e$ off the chamber can be written as

$$\vec{v}_e \approx \vec{c} + \sum_{i=1}^{N} \vec{v}_\perp^{\,i}, \tag{12}$$

where $N$ is the number of nuclei in the Coulomb gas, and $\vec{v}_\perp^{\,i}$ denotes the perpendicular component of the beam axis of the deflected velocity vector off the $i$th nucleus. The total rms angle $\bar{\Theta}$ is then given by

$$\bar{\Theta} \equiv \sqrt{\left\langle \frac{(\vec{v}_e - \vec{c})^2}{c^2} \right\rangle} = \sqrt{\left\langle \left( \sum_{i=1}^{N} \frac{\vec{v}_\perp^{\,i}}{c} \right)^2 \right\rangle} = \sqrt{\sum_{i=1}^{N} \left\langle \left( \frac{\vec{v}_\perp^{\,i}}{c} \right)^2 \right\rangle} = \sqrt{N}\bar{\theta}, \tag{13}$$

where

$$\bar{\theta} \equiv \sqrt{\left\langle \left( \frac{\vec{v}_\perp^{\,i}}{c} \right)^2 \right\rangle} \tag{14}$$

is the rms angle of the bending off a single nucleus. Noticing that

$$\left| \frac{\vec{v}_\perp^{\,i}}{c} \right|$$

is nothing but the bending angle $\theta(b)$ in Eq. (3), the $\bar{\theta}(b)$ can be computed by

$$\bar{\theta} = \sqrt{\langle \theta^2 \rangle}, \tag{15}$$

where

$$\langle \theta^2 \rangle = \int_{\Lambda}^{R} \langle \theta(b)^2 \rangle_{\text{spin}} P(b) db. \tag{16}$$

Here the averaging over spins is over the photon polarizations, which applies to the birefringence constant, $\Lambda$ is the lower cutoff in the impact parameter, and $P(b)$ denotes the probability density for a particular nucleus to fall at the impact parameter $b$ with the photon. The cutoff $\Lambda$ should be subject to the bound on the impact parameter (11). Because in a uniformly distributed gas $P(b)$ should be proportional to $b$, the normalized density is given by

$$P(b) = \frac{2}{R^2} b, \tag{17}$$

which satisfies

$$\int_0^R P(b) db = 1. \tag{18}$$

Then we get

$$\bar{\theta} = \frac{\bar{a} Z^2 \alpha^3}{160\sqrt{3}} \frac{\lambda\!\!\!^{-}_e{}^4}{R\Lambda^3}, \tag{19}$$

where

$$\bar{a} = \sqrt{(8^2 + 14^2)/2} = \sqrt{130} \tag{20}$$

is the rms of the birefringence constant. The total rms angle is then given by

$$\bar{\Theta} = \frac{Z^2\alpha^3}{160}\sqrt{\frac{130\pi}{3}}\sqrt{\frac{L}{L_0}}\left(\frac{\lambdabar_e}{\Lambda}\right)^4, \tag{21}$$

where $L_0 = 1/\rho_{\mathrm{N}}\Lambda^2$, with $\rho_{\mathrm{N}} = N/V$ denoting the number density of the gas in volume $V = \pi R^2 L$.

Now to estimate the amplification effect of this result we need to express the number density in terms of temperature and pressure using the equation of state of the Coulomb gas, which at high density is not known. However, this problem can be avoided if we assume that the Coulomb gas was obtained in the manner described before, by stripping the electrons off neutral heavy atoms. Then the number density of the Coulomb gas is identical to that of the atomic gas, for which we may assume the ideal gas law $\rho_{\mathrm{N}} = P/k_{\mathrm{B}}T$, with $T$ and $P$ denoting the temperature and the pressure of the atomic gas and $k_{\mathrm{B}}$ being the Boltzmann constant. We then have

$$\bar{\Theta} = \frac{Z^2\alpha^3}{160}\sqrt{\frac{130\pi}{3}}\sqrt{\frac{L}{L_0(P,T,\Lambda)}}\left(\frac{\lambdabar_e}{\Lambda}\right)^4, \tag{22}$$

where

$$L_0(P,T,\Lambda) = \frac{k_{\mathrm{B}}T}{P\Lambda^2}.$$

The result shows that at a given temperature and pressure the bending angle increases in proportion to the square root of the photon travel distance. As asserted, the result is independent of the geometry of the chamber, as there is no geometry-dependent parameter except for $L$, which, however, being the distance of the photon traveled, is not particular to the geometry.

To see the amplifying effect at some readily available parameter values we write Eq. (22) as

$$\bar{\Theta} = 2.945\times10^{-6}\left(\frac{Z}{100}\right)^2\left(\frac{10\lambdabar_e}{\Lambda}\right)^3 \times\sqrt{\left(\frac{300\,\mathrm{K}}{T}\right)\left(\frac{P}{1\,\mathrm{bar}}\right)\left(\frac{L}{30\,\mathrm{m}}\right)}\ (\mathrm{rad}). \tag{23}$$

It shows the rms angle is about 3 $\mu$rad at $Z = 100$, $T = 300$ K, $P = 1$ bar, and $L = 30$ m, with the cutoff at $\Lambda = 10\lambdabar_e$. Note that this value is 2 orders of magnitude larger than the bending angle by a single nucleus of the same $Z$ value and at the impact parameter $b = \Lambda$. Obviously, a greater amplification can be obtained with a colder, higher pressure gas in a longer chamber. This demonstrates that a Coulomb gas can be an amplifier for the light bending in a Coulomb background.

Let us now focus on the cutoff $\Lambda$. For a photon at the impact parameter $b$ with a nucleus there are two constraints on the photon wavelength $\lambda$, arising from the Euler-Heisenberg Lagrangian (1) and the bending angle (3). For the local Euler-Heisenberg interaction to be valid the wavelength should be larger than the electron Compton length $\lambdabar_e$, and for the bending angle (3) be valid the wavelength should be smaller than the impact parameter $b$, so that the geometrical optics will be applicable. Thus

$$\lambdabar_e \ll \lambda \ll b, \tag{24}$$

which indicates the cutoff $\Lambda$, the lower bound of $b$, should be a multiple of the photon wavelength. Putting in Eq. (22)

$$\Lambda = \mu\lambda, \tag{25}$$

where $\mu$ is a constant larger than unity, we get the rms angle for a photon beam of wavelength $\lambda$:

$$\bar{\Theta} = \frac{Z^2\alpha^3}{160\mu^3}\sqrt{\frac{130\pi}{3}}\sqrt{\frac{L}{L_0(P,T,\lambda)}}\left(\frac{\lambdabar_e}{\lambda}\right)^4, \tag{26}$$

where $L_0(P,T,\lambda) = k_BT/P\lambda^2$, and

$$\bar{\Theta} = 2.945\times10^{-6}\left(\frac{Z}{100}\right)^2\left(\frac{10\lambdabar_e}{\mu\lambda}\right)^3 \times\sqrt{\left(\frac{300\,\mathrm{K}}{T}\right)\left(\frac{P}{1\,\mathrm{bar}}\right)\left(\frac{L}{30\,\mathrm{m}}\right)}\ (\mathrm{rad}). \tag{27}$$

The magnitude of the rms angle has a sharp dependence on the cutoff parameter $\mu$. This clearly results from the limitation of the bending angle (3) which is obtained in the geometrical optics and is valid only for $b \gg \lambda$. At $b$ not larger than the wavelength, a more complete formula for the bending angle may be obtained using the wave optics, which is beyond the scope of this paper. Physically, however, it is clear that the quartic divergence at small $b$, which gives the strong $\mu$ dependence, should disappear in a complete formula, because at $b = 0$ the bending angle must vanish for symmetry reasons. We thus expect the bending angle will have a maximum at the impact parameter $b = \mu_0\lambda$, where $\mu_0$ is a constant. We may then identify $\mu_0$ with the cutoff parameter $\mu$. It may not be unreasonable to assume $\mu$ to be of the order of unity; so if we put $\mu = 5$ and $\lambda = 5\lambdabar_e$, as an example, the rms angle is a few tenths of $\mu$rad at the reference values of $Z$, $P$, $T$ and $L$ in Eq. (27).

An experimental consequence of the bending will be a broadening of the beam as it travels through the gas. This beam broadening may be exploited to detect the light bending, by studying its dependence on the temperature, the pressure, the beam's travel distance, and the wavelength of the photon in the beam, and see if it follows Eq. (26). Another interesting signal would be the intensity profile of the beam cross section. Because of the random nature of the bendings off the nuclei, the intensity profile should be Gaussian.

To conclude, we have shown that the Coulombic light bending can be amplified by orders of magnitude with a photon beam traveling in a Coulomb gas. The rms bending angle at a given temperature and pressure is proportional to the square root of the distance the beam traveled. Although we considered a Coulomb gas as the amplifying medium for the light bending, it is conceivable that a similar amplification would occur as well in other mediums, like solid metals of large atomic number such as gold. Such an amplification might be investigated in experiments for precision measurement of refractive index with $\gamma$ rays [13–15], as the Coulombic bending can have a dominating effect on the Delbrück scattering [16].

I am thankful to S. Han for encouragement. The early stages of this work were done in collaboration with N. Ahmadiniaz. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), funded by the Ministry of Education, Science, and Technology (Grant No. 2012R1A1A2044543).

[1] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
[2] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).
[3] Z. Bialynicka-Birula and I. Bialynicki-Birula, Phys. Rev. D **2**, 2341 (1970).
[4] S. L. Adler, Ann. Phys. **67**, 599 (1971).
[5] J. S. Heyl and L. Hernquist, J. Phys. A **30**, 6485 (1997).
[6] D. Boer and J.-W. van Holten, arXiv:hep-ph/0204207.
[7] V. A. De Lorenci, R. Klippert, M. Novello, and J. M. Salim, Phys. Lett. B **482**, 134 (2000).
[8] V. I. Denisov, I. P. Denisova, and S. I. Svertilov, Dokl. Akad. Nauk. Ser. Fiz. **380**, 435 (2001).
[9] V. A. De Lorenci, N. Figueiredo, H. H. Fliche, and M. Novello, Astron. Astrophys. **369**, 690 (2001).
[10] J. Y. Kim and T. Lee, J. Cosmol. Astropart. Phys. **2011**, 017 (2011).
[11] J. Y. Kim and T. Lee, Mod. Phys. Lett. A **26**, 1481 (2011).
[12] A. I. Milstein and M. Schumacher, Phys. Rep. **243**, 183 (1994).
[13] J. K. Koga and T. Hayakawa, Phys. Rev. Lett. **118**, 204801 (2017).
[14] T. Kawasaki, S. Naito, Y. Sano, T. Hayakawa, T. Shizuma, R. Hajima, and S. Miyamoto, Phys. Lett. A **381**, 3129 (2017).
[15] M. M. Günther, M. Jentschel, A. J. Pollitt, P. G. Thirolf, and M. Zepf, Phys. Rev. A **95**, 053864 (2017).
[16] T. Lee, arXiv:1711.00160.