Understanding the electromagnetic 4-potential in the tetrad bundle

Yufang Hao,* Jiayin Shen, and Jianhong Ruan†
Department of Physics, East China Normal University, Shanghai 200241, China

(Received 13 June 2018; published 10 September 2018)

Separation of the spin and orbital angular momenta of the electromagnetic field has been discussed frequently in recent years. The spin and orbital angular momenta cannot be made simultaneously gauge invariant and Lorentz covariant and are not conserved separately. After analyzing the source of the problem, we find that the electromagnetic 4-potential depends on the local reference frame instead of the global reference frame. The transformation of the local reference frame is the intrinsic degree of freedom of the electromagnetic field. Therefore, considering only the Lorentz transformation of the global reference frame and neglecting the Lorentz transformation of the local reference frame may lead to the noncovariance of the electromagnetic 4-potential. Accordingly, we redescribe these difficulties of the electromagnetic field from the perspective of quantum field theory. By using the behavior of the electromagnetic 4-potential that satisfies the Coulomb gauge in Lorentz coordinate transformation, we can construct the electromagnetic vector in the tetrad bundle. The various physical quantities that are induced by this electromagnetic vector satisfy Lorentz covariance in the tetrad bundle. This electromagnetic vector, which is projected onto space-time, is an electromagnetic 4-potential that satisfies the Coulomb gauge; thus, the electromagnetic vector is gauge invariant.

DOI: 10.1103/PhysRevA.98.033809

I. INTRODUCTION

There has been substantial discussion about the separation of the angular momentum (AM) of the electromagnetic field into its spin and orbital parts [1,2]. The orbital AM density of the electromagnetic field is defined as $\mathbf{L} = E_j(\mathbf{r}\times\nabla)\mathscr{A}_j$ and the spin AM density as $\mathbf{S} = \mathbf{E}\times\mathbf{A}$, where $\mathscr{A}$ is the electromagnetic 4-potential and $\mathbf{A}$ is the spatial part of $\mathscr{A}$ [3]. According to this definition, neither the spin nor the orbital AM satisfies [U(1)-group] gauge invariance [4,5]. However, gauge invariance is an inevitable requirement of an observable physical quantity and the spin and orbital AMs of the electromagnetic field can be observed in various optical experiments [6,7].

This problem can be solved by replacing $\mathscr{A}_\mu$ with only its transverse part, which is denoted $A_\mu = (0, \mathbf{A}^\perp)$ [8–10], where $\mathbf{A}^\perp$ satisfies the Coulomb gauge: $\nabla\cdot\mathbf{A}^\perp = 0$. Namely, the definitions of the spin and orbital AM densities are modified to $\mathbf{L} = E_j(\mathbf{r}\times\nabla)A_j$, $\mathbf{S} = \mathbf{E}\times\mathbf{A}^\perp$. However, this definition violates the Lorentz covariance: the Coulomb gauge is not Lorentz covariant. Lorentz covariance is a requirement of the principle of relativity: physical laws should not depend on the reference frame.

The spin and orbital AMs cannot simultaneously be Lorentz covariant and gauge invariant and they are not conserved separately [11,12]. Bliokh *et al.* constructed a set of conserved spin and orbital AM densities [12]. However, this structure also depends on the Coulomb gauge. Therefore, it is not Lorentz covariant. Bliokh *et al.* remark that this phenomenon is consistent with the experimental operation because a local probe particle will always identify a special laboratory reference frame in which it is at rest. This explanation is not convincing. Any observable quantity must be observed and measured in a special laboratory reference frame; however, most of them do not have a Lorentz-violating mathematical form because the Lorentz violation of observation methods would not cause a Lorentz violation of physical laws. In other words, the mathematical form of the measurement result cannot depend on the reference frame, which is known as observer Lorentz covariation [13]. Furthermore, according to the gauge theory of gravitation [14], similarly to U(1)-group gauge invariance, Lorentz covariance is the gauge invariance of the SO(1,3) group; hence, Lorentz covariance is also an inevitable requirement of observable quantities.

However, Bliokh's point of view provides two main inspirations: One is the specificity of the Coulomb gauge. Not only can the Coulomb gauge be used to construct conserved spin and orbital AMs, but also the canonical quantization procedure performs well in this gauge [15]. The other is that the optical phenomenon is closely related to the reference frame. Physical laws are local [15,16]. What if an observable quantity of the electromagnetic field depends not on the global reference frame (coordinate system) but on the local one (tetrad field)? We suspect that the origin of the noncovariance of the electromagnetic 4-potential, which satisfies the Coulomb gauge, is that we have not taken the transformation of the tetrad field into account. In detail, when the coordinate system is transformed, the choice of tetrad changes, and although the Coulomb gauge is broken, the transformation of the tetrad will produce a phase that corrects the deviation [see Eq. (40)].

According to this view, we must shift the perspective from space-time to the tetrad bundle (see [17], [18]). The set of all local reference frames of a space-time point q constitutes q's fiber. As a result, each transformation Λ of a local

*haoyufang712@foxmail.com
†Corresponding author: jhruan@phy.ecnu.edu.cn

reference frame becomes an intrinsic degree of freedom of the electromagnetic field. The electromagnetic 4-potential that we observed is the projection to space-time of a high-dimensional electromagnetic vectors in the tetrad bundle. This electromagnetic vector has Lorentz covariance in the high-dimensional tetrad bundle. The main essence of Lorentz covariance is that the mathematical form of the physical quantity cannot depend on a reference frame; hence, the physical quantity should be a "geometric invariant." Not all geometric invariants must be vectors (or tensors) in space-time; however, we used to replace "Lorentz covariance" with "Lorentz covariance of space-time vectors (tensors)" narrowly. The electromagnetic vector in the tetrad bundle that we construct is such an example; it is a geometric invariant but does not have Lorentz covariance when it is projected to space-time, where we cannot obtain all its information.

The remainder of the paper is organized as follows: In Secs. II and III, we restate the issue about Lorentz covariance and gauge invariance that relates to the AM of the electromagnetic field from the perspective of quantum field theory, which lays the groundwork for Sec. IV. In Sec. IV, we present the revised definitions of various physical quantities of the electromagnetic field in the tetrad bundle and the relationship with the corresponding classical definition in space-time. Section V provides a summary. Throughout the text, we use Einstein's sum rule; natural electrodynamical units, namely, $\mu_0 = \varepsilon_0 = c = 1$; the Minkowski metric, which is expressed as $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$; Greek indices $\rho, \mu, \nu, \ldots = 0, 1, 2, 3$; and Latin indices $i, j, k, \ldots = 1, 2, 3$. We do not distinguish between the notion of a "vector" in tangent space and that of a "1-form" in the cotangent space; both are called vectors.

II. ENERGY-MOMENTUM TENSOR

We assume that ϕ^ρ is a spin-1 vector field with a mass under the Lorentz transformation of the reference frame $x \to \Lambda x$, which is transformed as a vector representation of the Lorentz group [19]:

$$U(\Lambda)\phi^\rho(x)U^{-1}(\Lambda) = \Lambda_\sigma{}^\rho \phi^\sigma(\Lambda x). \tag{1}$$

Hence, ϕ^ρ is a 4-vector. The canonical energy-momentum tensor of ϕ^ρ, which is denoted T_N, is usually defined as

$$T_N^{\mu\nu} = \eta^{\mu\nu}\mathscr{L} - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^\rho)}\partial^\nu \phi^\rho, \tag{2}$$

where $\mathscr{L}$ is the Lagrangian density.

Using Eq. (1), we can prove that under the coordinate transformation $x \to \Lambda x$,

$$U(\Lambda)T_N^{\mu\nu}(x)U^{-1}(\Lambda) = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu T_N^{\rho\sigma}(\Lambda x). \tag{3}$$

Therefore, T_N is a Lorentz tensor with Lorentz covariance. However, T_N does not have local U(1) gauge invariance because under the gauge transformation $\phi^\rho(x) \to e^{i\epsilon(x)}\phi^\rho(x)$, $\partial^\nu\phi^\rho$ will produce an additional factor, namely, $\partial^\nu\epsilon(x)$, which cannot be canceled unless ∂_μ is replaced with covariant derivative D_μ in the u(1) algebra [20].

Using the canonical energy-momentum tensor, we can construct a Noether flow, namely, the Lorentz generator density:

$$M_N^{\rho\mu\nu} = x^\mu T_N^{\rho\nu} - x^\nu T_N^{\rho\mu}. \tag{4}$$

By integrating this Noether flow (volume integrals for sufficiently localized fields are assumed), we can obtain the generator of the Lorentz group,

$$L^{\mu\nu} = \int M_N^{0\mu\nu} d^3x, \tag{5}$$

where the spatial part, namely, L^{ij} of $L^{\mu\nu}$, is the AM generator. However, the canonical energy-momentum tensor does not satisfy index symmetry generally, i.e., $T_N^{\mu\nu} \neq T_N^{\nu\mu}$. Index symmetry is a necessary and sufficient condition for the conservation of $M^{\rho\mu\nu}$ because

$$\partial_\rho M_N^{\rho\mu\nu} = 2T_N^{[\mu\nu]}, \tag{6}$$

where $[\cdot]$ represents the tensor's anticommutator. Therefore, the Belinfante energy-momentum tensor was introduced by adding an intrinsic spin term to the canonical energy-momentum tensor [21],

$$T_B^{\mu\nu} = T_N^{\mu\nu} + \tfrac{1}{2}\partial_\rho(S^{\rho\mu\nu} - S^{\mu\rho\nu} - S^{\nu\rho\mu}), \tag{7}$$

where the spin term, which is denoted $\partial_\rho S^{\rho\mu\nu}$, cancels out the antisymmetric part of T_N, thereby leaving only the symmetric part:

$$\tfrac{1}{2}\partial_\rho S^{\rho\mu\nu} = -T_N^{[\mu\nu]}. \tag{8}$$

Thus, the Belinfante energy-momentum tensor is a symmetric tensor and the Lorentz-group generator density, denoted M_B, which is induced by the Belinfante energy-momentum tensor, satisfies

$$\begin{aligned} M_B^{\rho\mu\nu} &= x^\mu T_B^{\rho\nu} - x^\nu T_B^{\rho\mu} \\ &= M_N^{\rho\mu\nu} + S^{\rho\mu\nu} + \partial_\kappa \Psi^{\kappa\rho\mu\nu}, \end{aligned} \tag{9}$$

where $\Psi^{\kappa\rho\mu\nu}$ is a surface term that consists of coordinates and spins. We interpret M_N^{0ij} as the orbital AM density and S^{0ij} as the spin AM density. The total Noether flow is conserved:

$$\partial_\rho M_B^{\rho\mu\nu} = 0. \tag{10}$$

The famous physicist S. Weinberg presented the expression of spin $S^{\rho\mu\nu}$ in Ref. [15]:

$$S^{\rho\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\rho\phi_\nu)}\phi^\mu - \frac{\partial\mathscr{L}}{\partial(\partial_\rho\phi_\mu)}\phi^\nu. \tag{11}$$

The Lorentz covariance condition of the spin is

$$U(\Lambda)S^{\rho\mu\nu}U^{-1}(\Lambda) = \Lambda_\gamma{}^\rho \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu S^{\gamma\alpha\beta}(\Lambda x). \tag{12}$$

III. RESTATEMENT OF THE ANGULAR MOMENTUM PROBLEMS

Problems arise in the construction of a massless vector field $A_\mu(x)$ that is modeled on a mass vector field ϕ^ρ [15]. Simply using the creation and annihilation operators of the photon, which are denoted $a^\dagger$ and a, we cannot construct a 4-vector that satisfies (1); $A_\mu(x)$ can only have the form

$$A_\mu(x) = (2\pi)^{-\frac{3}{2}} \sum_{h=\pm 1} \int \frac{d^3p}{\sqrt{2p^0}} (e_\mu(\mathbf{p}, h) \times e^{ip_\mu x^\mu} a(\mathbf{p}, h) + e^*_\mu(\mathbf{p}, h) e^{-ip_\mu x^\mu} a^\dagger(\mathbf{p}, h)), \quad (13)$$

where $\mathbf{p}$ is the 3-momentum, $p = (p^0, \mathbf{p})$ is a lightlike 4-momentum, and h is the helicity of the photon. The coefficient $e_\mu(\mathbf{p}, h) = R(\hat{\mathbf{p}})_\mu{}^\nu e_\nu(\mathbf{k}, h)$, where $R(\hat{\mathbf{p}})$ is a rotational transformation that rotates the spatial part $\mathbf{k} = (0, 0, 1)$ of the standard momentum $k = (1, 0, 0, 1)$ to the direction of $\mathbf{p}$, which is written as $\hat{\mathbf{p}}$, and $e_\nu(\mathbf{k}, h)$ can be expressed as

$$e_\nu(\mathbf{k}, h) = \frac{1}{\sqrt{2}}(0, 1, ih, 0). \quad (14)$$

In this configuration, A_μ satisfies the Coulomb gauge (in vacuum):

$$A_0 = 0, \ \partial^j A_j = 0. \quad (15)$$

Under a reference-frame transformation $x \to \Lambda x$, A_μ behaves as follows:

$$U(\Lambda)A_\rho(x)U^{-1}(\Lambda) = \Lambda^\sigma{}_\rho A_\sigma(\Lambda x) + \Lambda^\sigma{}_\rho(\partial_\sigma \Omega)(\Lambda x, \Lambda). \quad (16)$$

The transformation of the reference frame leads A_μ to produce a gauge $\partial_\rho \Omega_\Lambda$ that depends on the reference frame, where $\Omega(x, \Lambda)$ can be expressed as a linear combination of creation and annihilation operators. Reference [15] did not present an explicit expression for Ω. According to our calculations,

$$\Omega(x, \Lambda) = (2\pi)^{-\frac{3}{2}} \sum_{h=\pm 1} \int \frac{d^3p}{2\sqrt{p^0}} ([\alpha(p, \Lambda) + ih\beta(p, \Lambda)] e^{ip_\mu x^\mu} a(\mathbf{p}, h) + [\alpha(p, \Lambda) - ih\beta(p, \Lambda)] e^{-ip_\mu x^\mu} a^\dagger(\mathbf{p}, h)), \quad (17)$$

where α and β are parameters that depend on the Lorentz transformation Λ and the 4-momentum p. The little-group representation of Λ is expressed as $W = L^{-1}(\Lambda p)\Lambda L(p)$, where $L(p)$ is the standard Lorentz transformation, which is applied to boost the standard momentum k^μ to the 4-momentum p^μ, that is, $L(p)k = p$. The definitions of α and β are $\alpha = W^0{}_1$ and $\beta = W^0{}_2$.

From the above discussion, although both the electromagnetic 4-potential A_μ and the Lorentz vector ϕ^ρ have the same index, their transformation properties are different. According to gauge field theory [20], A_μ is a gauge potential in the principal bundle whose structure group is U(1), whereas ϕ^ρ is a vector (component) in the representation space. Therefore, if we directly apply the definitions of energy-momentum tensor, spin, Lorentz generator density, etc., of ϕ^ρ to the electromagnetic 4-potential A_μ, problems will inevitably be encountered. However, researchers typically apply these definitions directly to electromagnetic fields. We believe that this is why the orbital and spin AMs of the electromagnetic field do not have Lorentz covariance or gauge invariance.

The Lagrangian and action of the electromagnetic field in a vacuum (with no matter) can be expressed as

$$\mathscr{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x),$$
$$S_\gamma = -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu}, \quad (18)$$

where $F_{\mu\nu}(x) = \partial_\mu \mathscr{A}_\nu(x) - \partial_\nu \mathscr{A}_\mu(x)$ is the electromagnetic field tensor and $\mathscr{A}_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)$ is an electromagnetic 4-potential under any gauge. We temporarily imitate Eq. (2) to calculate the canonical energy-momentum tensor of the electromagnetic field,

$$T_N^{\mu\nu} = \eta^{\mu\nu} \mathscr{L} - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \mathscr{A}_\rho)} \partial^\nu \mathscr{A}_\rho = -\frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F^{\mu\rho} \partial^\nu \mathscr{A}_\rho, \quad (19)$$

and imitate Eq. (11) to calculate the spin term $S^{\rho\mu\nu}$ of the electromagnetic field:

$$S^{\rho\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\rho \mathscr{A}_\nu)} \mathscr{A}^\mu - \frac{\partial \mathscr{L}}{\partial(\partial_\rho \mathscr{A}_\mu)} \mathscr{A}^\nu = F^{\rho\mu} \mathscr{A}^\nu - F^{\rho\nu} \mathscr{A}^\mu. \quad (20)$$

According to Eqs. (19) and (20), it is easy to calculate the three-dimensional form of the AM density; the calculation is typically presented in textbooks [22]

$$\mathbf{L} = E_j(\mathbf{r} \times \nabla)\mathscr{A}_j, \quad \mathbf{S} = \mathbf{E} \times \mathbf{A}, \quad (21)$$

where $\mathscr{A}_\mu = (\varphi, \mathbf{A})$. There is no local gauge invariance in Eqs. (19) and (20). Unlike the gauge transformation of a vector field ϕ^ρ, the gauge transformation of $\mathscr{A}_\mu(x)$ is not performed by multiplying a local phase factor $e^{i\epsilon(x)}$. $\mathscr{A}_\mu(x)$ is the gauge potential in the $\mathfrak{u}(1)$ algebra, i.e. [20],

$$\mathscr{A}'_\rho(x) = \mathscr{A}_\rho(x) + \partial_\rho \epsilon(x). \quad (22)$$

Substituting (22) into (19) and (20) will yield two additional terms: $F^{\mu\rho} \partial^\nu \partial_\rho \epsilon$ and $2F^{\rho[\mu} \partial^{\nu]} \epsilon$. The canonical energy-momentum tensor and spin do not satisfy U(1) gauge invariance. We cannot change this phenomenon even if we replace the derivative operator ∂_μ in Eq. (19) with the covariant derivative D_μ.

The typical solution is to replace $\mathscr{A}_\rho$ by its transverse part, which is denoted A_ρ [8–10] and always satisfies the Coulomb gauge; hence, it is invariant under the gauge transformation in Eq. (22):

$$T_N^{\mu\nu} = -\tfrac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F^{\mu\rho} \partial^\nu A_\rho, \quad (23)$$

$$S^{\rho\mu\nu} = F^{\rho\mu} A^\nu - F^{\rho\nu} A^\mu. \quad (24)$$

The three-dimensional form of the orbital and spin AMs is corrected by

$$\mathbf{L} = E_j(\mathbf{r} \times \nabla) A_j, \quad \mathbf{S} = \mathbf{E} \times \mathbf{A}^\perp, \quad (25)$$

where $A_\mu = (0, \mathbf{A}^\perp)$ and $\nabla \cdot \mathbf{A}^\perp = 0$.

However, the frame transformation, (16), of A_ρ causes a new problem immediately: $T_N^{\mu\nu}$ and $S^{\rho\mu\nu}$ no longer satisfy the Lorentz covariance conditions in Eqs. (3) and (12) and neither do the orbital and spin AM densities that are induced by them. The source of the difficulty is the dependency of the Coulomb gauge on the reference frame: if we perform a boost transformation to the reference frame $\bar{x} = \Lambda x$ while transforming $A_\mu(x)$ via the classic approach, namely, $\bar{A}_\mu(\bar{x}) = \Lambda_\mu{}^\rho A_\rho(\Lambda^{-1}\bar{x})$, then $\bar{A}_\mu(\bar{x})$ no longer satisfies this gauge. However, the Belinfante energy-momentum tensor that sums the contributions of the orbital and spin parts (ignoring the surface terms), which is expressed as

$$T_B^{\mu\nu} = -\tfrac{1}{4}\eta^{\mu\nu} F_{\rho\sigma}F^{\rho\sigma} + F^{\mu\rho}F^\nu{}_\rho, \tag{26}$$

is index symmetric, gauge invariant, and Lorentz covariant. Hence, it is a well-defined observable quantity.

The origin of these difficulties in the final analysis is that $A_\mu(x)$'s transformation, (16), differs substantially from the 4-vector ϕ^ρ. We cannot simply assert that $A_\mu(x)$ does not have Lorentz covariance. After careful consideration, we find that the essential requirement of Lorentz covariance is that the definition of a physical quantity be independent of the reference frames, that is, the physical quantity is a geometric invariant. The geometric invariants of different structures have different Lorentz transformation forms. The most familiar example, namely, the Christoffel symbol, which is denoted $\Gamma^\rho{}_{\nu\mu}$ is a projection to space-time of the torsion-vanishing metric connection ω in the tetrad bundle. This projection mapping depends on the reference frames, whereas the mathematical definition of ω is independent of the reference frames [17]. As a projection of a geometric invariant, the Christoffel symbol's transformation differs substantially from that of a tensor. We cannot simply treat the "Lorentz transformation" as the "Lorentz transformation of tensors."

If we can identify a geometric invariant whose projection to space-time is the electromagnetic 4-potential, then we will prove that the electromagnetic 4-potential remains Lorentz covariant and the problem will be solved. Starting from the Lorentz transformation in Eq. (16) of $A_\mu(x)$, we modify the definitions of various physical quantities of the electromagnetic field.

IV. PROMOTION OF THE ELECTROMAGNETIC 4-POTENTIAL TO A BUNDLE VECTOR

According to Eq. (16) and the little-group representation of the unit Lorentz transformation $\mathbb{1}$, which is expressed as $W(p,\mathbb{1}) = L^{-1}(\mathbb{1}p)\mathbb{1}L(p) = \mathbb{1}, \alpha(p,\mathbb{1}) = 0, \beta(p,\mathbb{1}) = 0$, which leads to $\Omega(x,\mathbb{1}) = 0$. We rewrite Eq. (16) as

$$\begin{aligned} U(\Lambda)[A_\rho(x) + \partial_\rho\Omega(x,\mathbb{1})]U^{-1}(\Lambda) \\ = \Lambda^\sigma{}_\rho[A_\sigma(\Lambda x) + (\partial_\sigma\Omega)(\Lambda x,\Lambda)] \end{aligned} \tag{27}$$

and observe that $A_\rho + \partial_\rho\Omega$ behaves similarly to a 4-vector under a reference-frame transformation. We define

$$\mathcal{A}_\rho(x,\Lambda) = A_\rho(x) + \partial_\rho\Omega(x,\Lambda). \tag{28}$$

Then, according to Eq. (27), the Lorentz transformation of $\mathcal{A}_\rho$ and its derivative can be obtained directly:

$$\begin{aligned} U(\Lambda)\mathcal{A}_\rho(x,\mathbb{1})U^{-1}(\Lambda) &= \Lambda^\sigma{}_\rho\mathcal{A}_\sigma(\Lambda x,\Lambda), \\ U(\Lambda)\partial_\sigma\mathcal{A}_\rho(x,\mathbb{1})U^{-1}(\Lambda) &= \Lambda^\nu{}_\sigma\Lambda^\mu{}_\rho(\partial_\nu\mathcal{A}_\mu)(\Lambda x,\Lambda). \end{aligned} \tag{29}$$

Although $\mathcal{A}_\rho$ is similar to a 4-vector, $\mathcal{A}_\rho$ depends on both x and Λ. Hence, $\mathcal{A}_\rho$ is not a vector field in space-time. If we consider $(x^\rho, \Lambda^\mu{}_\nu)$ as a coordinate, x and Λ will constitute a coordinate domain of the tetrad bundle. For a point (q, e_0, e_1, e_2, e_3) in the tetrad bundle, where q is a point in space-time whose coordinate is x^ρ, any orthonormal basis (e_0, e_1, e_2, e_3) of the tangent space of q can be expressed as $e_\nu = \Lambda^\mu{}_\nu\partial_\mu$. Hence, $\Lambda^\mu{}_\nu$ is selected as the coordinate of this basis which is called a tetrad of q. Therefore, $\mathcal{A}_\rho$ is a vector field in the tetrad bundle. Strictly, $\mathcal{A}_\rho$ is a component of the vector field and $\mathcal{A} = \mathcal{A}_\rho\, dx^\rho$ is a vector field where dx^ρ is a vector not in space-time but in the tetrad bundle. Both dx^μ and $d\Lambda^\rho{}_\sigma$ constitute the vector basis of the cotangent space of (q, e_0, e_1, e_2, e_3). We rewrite Eq. (28) in a coordinate-independent form and (q, e_0, e_1, e_2, e_3) can be simplified as (q, e)

$$\mathcal{A}(q,e) = \mathcal{A}_\rho(q,e)dx^\rho + \mathcal{A}^\nu{}_\mu(q,e)d\Lambda^\mu{}_\nu, \tag{30}$$

where

$$\mathcal{A}_\rho(q,e) = A_\rho(x) + \partial_\rho\Omega(x,\Lambda), \tag{31}$$

$$\mathcal{A}^\nu{}_\mu(q,e) = 0. \tag{32}$$

Although it vanishes, $\mathcal{A}^\nu{}_\mu$ is written in Eq. (30) to emphasize the difference between the electromagnetic vector $\mathcal{A}(q,e) = \mathcal{A}_\rho(q,e)dx^\rho$ and the electromagnetic 4-potential $A(q) = A_\rho(x)dx^\rho$, which have different dimensions. To observe the difference more clearly, we consider their relationships with the electromagnetic field tensor $F_{\mu\nu}(q)$. The electromagnetic field tensor $F(q)$ in space-time is defined as

$$\begin{aligned} F(q) &= \tfrac{1}{2}F_{\rho\sigma}(q)dx^\rho\wedge dx^\sigma \\ &= \tfrac{1}{2}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)dx^\rho\wedge dx^\sigma \\ &= dA. \end{aligned} \tag{33}$$

Since $F_{\rho\sigma} = 2\partial_{[\rho}A_{\sigma]} = 2\partial_{[\rho}\mathcal{A}_{\sigma]}$, $F(q)$ can be promoted to the bundle tensor $\mathcal{F}(q,e)$:

$$\begin{aligned} \mathcal{F}(q,e) &= \tfrac{1}{2}\mathcal{F}_{\rho\sigma}(q,e)dx^\rho\wedge dx^\sigma \\ &= \tfrac{1}{2}(\partial_\rho\mathcal{A}_\sigma - \partial_\sigma\mathcal{A}_\rho)dx^\rho\wedge dx^\sigma. \end{aligned} \tag{34}$$

Then $\mathcal{F}_{\rho\sigma}(q,e) = F_{\rho\sigma}(q)$, that is, $\mathcal{F}$ is independent of the selection of the tetrad. However,

$$\begin{aligned} d\mathcal{A} &= d\big(\mathcal{A}_\rho(q,e)dx^\rho\big) = d\mathcal{A}_\rho\wedge dx^\rho \\ &= \frac{1}{2}(\partial_\rho\mathcal{A}_\sigma - \partial_\sigma\mathcal{A}_\rho)dx^\rho\wedge dx^\sigma + \frac{\partial\mathcal{A}_\rho}{\partial\Lambda^\mu{}_\nu}d\Lambda^\mu{}_\nu\wedge dx^\rho \\ &= \mathcal{F}(q,e) + \left(\frac{\partial}{\partial\Lambda^\mu{}_\nu}\frac{\partial}{\partial x^\rho}\Omega\right)d\Lambda^\mu{}_\nu\wedge dx^\rho. \end{aligned} \tag{35}$$

We observe that $\mathcal{F} \neq d\mathcal{A}$ and $\mathcal{F} = \mathrm{D}\mathcal{A}$, where D is the exterior covariant derivative in the tetrad bundle, whose definition can be found in the literature [17].

It is necessary to discuss the performance of the projection of $\mathcal{A}$ to space-time. Only this projection can be directly observed. If we choose a fixed tetrad $e(q)$ at each space-time point q, mapping $e(q)$ is called a tetrad field.

Applying the pullback mapping e^*, which is induced by the tetrad $e(q)$ to $\mathcal{A}$, we obtain a vector field $e^*\mathcal{A}$ in space-time. Via Eq. (30), we obtain the component form of $e^*\mathcal{A}$ by setting

$e_\nu(q) = \Lambda^\mu{}_\nu(q)\partial_\mu$, where $\Lambda^\mu{}_\nu(q)$ is related to q,

$$(e^*\mathcal{A})_\rho(q) = \mathcal{A}_\rho(q, e(q)) + \mathcal{A}^\nu{}_\mu(q, e(q))\frac{\partial \Lambda^\mu{}_\nu(q)}{\partial x^\rho} = \mathcal{A}_\rho(q, e(q)); \quad (36)$$

that is,

$$(e^*\mathcal{A})(q) = \mathcal{A}_\rho(q, e(q))dx^\rho = (A_\rho(x) + \partial_\rho\Omega(x, \Lambda))dx^\rho, \quad (37)$$

where dx^ρ becomes a vector in space-time again.

Pullback e^* is a projection. The tetrad $e(q)$ represents the (local) laboratory reference frame that we are observing. We regard $e(q)$ as a Lorentz gauge.

When a set of coordinates $x(q)$ is selected, a natural selection of a tetrad is $e(q) = (\partial_0, \partial_1, \partial_2, \partial_3)$. This is the default choice of optical experiments so that the influence of the local reference frame (tetrad) can be ignored. The coordinate of $(q, e(q))$ is $(x^\sigma, \delta^\mu{}_\nu)$. Since $\partial_\rho\Omega(x^\sigma, \delta^\mu{}_\nu) = 0$, $(e^*\mathcal{A})_\rho(q) = A_\rho(x)$. If we choose another coordinate $\bar{x} = \Lambda x$, the tetrad field will be reselected as $\bar{e}(q) = (\bar{\partial}_0, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3)$. Naturally, $\mathcal{A}$'s projection under $\bar{e}(q)$ is

$$(\bar{e}^*\mathcal{A})(q) = \bar{A}_\rho(q, \bar{e}(q))d\bar{x}^\rho = \bar{A}_\rho(\bar{x})d\bar{x}^\rho. \quad (38)$$

From another perspective, to calculate $\bar{e}^*\mathcal{A}$, we observe that $\bar{e}(q) = e(q)\Lambda^{-1}$. Substituting $x = \Lambda^{-1}\bar{x}$ and $\bar{e}(q)$ into Eq. (37), we obtain

$$\begin{aligned}(\bar{e}^*\mathcal{A})(q) &= \mathcal{A}_\rho(q, \bar{e}(q))dx^\rho \\ &= (A_\rho(x) + \partial_\rho\Omega(x, \Lambda^{-1}))dx^\rho \\ &= (A_\sigma(\Lambda^{-1}\bar{x}) + (\partial_\sigma\Omega)(\Lambda^{-1}\bar{x}, \Lambda^{-1}))\Lambda_\rho{}^\sigma d\bar{x}^\rho. \end{aligned} \quad (39)$$

Contrasting Eqs. (38) and (39), there must be

$$\bar{A}_\rho(\bar{x}) = \Lambda_\rho{}^\sigma A_\sigma(\Lambda^{-1}\bar{x}) + \Lambda_\rho{}^\sigma(\partial_\sigma\Omega)(\Lambda^{-1}\bar{x}, \Lambda^{-1}), \quad (40)$$

where both $A_\rho(x)$ and $\bar{A}_\rho(\bar{x})$ satisfy the Coulomb gauge, (15); hence, they are not Lorentz vectors in space-time. This is expected because the electromagnetic 4-potential that we have observed is a projection of the vector in the tetrad bundle; thus, it is one-sided and, naturally, cannot satisfy Lorentz covariance. Vector $\mathcal{A}(q, e)$ provides a complete description of the electromagnetic field. Equation (40) proves that the definition of $\mathcal{A}$ is independent of the reference frames. For any reference frame x, $A_\rho(x)$ in the frame that satisfies the Coulomb gauge is promoted to the vector $\mathcal{A}$. Hence, $\mathcal{A}$ is the geometric invariant for which we are looking.

Since $A_\rho(x)$ in Eq. (31) must satisfy the Coulomb gauge, (15), $\mathcal{A}$ does not change when $\mathscr{A}_\rho(x)$ is subjected to gauge transformation $\mathscr{A}_\rho(x) + \partial_\rho\epsilon(x)$. Thus, $\mathcal{A}$ has U(1) gauge invariance.

It is natural to replace $\mathscr{A}_\rho$ and A_ρ with $\mathcal{A}_\rho$ in the definition of each physical quantity. The form of the electromagnetic field tensor will be unchanged under the replacement since $\mathcal{F}_{\rho\sigma} = F_{\rho\sigma}$ and $e^*\mathcal{F} = F$. In addition, the form of the Lagrangian will be unchanged, in particular, for Maxwell's equation. However, this substitution results in the promotion of scalars, vectors, and tensors from space-time to the tetrad bundle.

The definition of the canonical energy-momentum tensor T_N is modified to

$$\mathcal{T}_N^{\mu\nu}(q, e) = -\tfrac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}(q)F^{\rho\sigma}(q) + F^{\mu\rho}(q)\partial^\nu\mathcal{A}_\rho(q, e). \quad (41)$$

Then we can prove that $\mathcal{T}_N$ is Lorentz covariant:

$$U(\Lambda)\mathcal{T}_N^{\mu\nu}(x, \mathbb{1})U^{-1}(\Lambda) = \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu\mathcal{T}_N^{\rho\sigma}(\Lambda x, \Lambda). \quad (42)$$

From the U(1) gauge invariance of $\mathcal{A}$, the gauge invariance of $\mathcal{T}_N$ follows.

Similarly, the spin $S^{\rho\mu\nu}$ is modified to

$$\mathcal{S}^{\rho\mu\nu} = F^{\rho\mu}\mathcal{A}^\nu - F^{\rho\nu}\mathcal{A}^\mu. \quad (43)$$

The Belinfante energy-momentum tensor T_B remains unchanged under $\mathscr{A} \to \mathcal{A}$.

In the laboratory reference frame, we select the tetrad $e(q) = (\partial_0, \partial_1, \partial_2, \partial_3)$ and the projection of the canonical energy-momentum tensor $\mathcal{T}_N$ is

$$(e^*\mathcal{T}_N)^{\mu\nu}(x) = T_N^{\mu\nu}(x) = -\tfrac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F^{\mu\rho}\partial^\nu A_\rho. \quad (44)$$

Naturally, the projection of spin $\mathcal{S}$ is

$$(e^*\mathcal{S})^{\rho\mu\nu} = F^{\rho\mu}A^\nu - F^{\rho\nu}A^\mu. \quad (45)$$

Similarly, we can formulate the projection of the AM density, which is the same as Eq. (25). These results are consistent with Eqs. (23) and (24). Equations (23) and (24) may have Lorentz covariance; however, T_N and S, which only reflect part of the information, do not provide a complete physical description, in contrast to $\mathcal{T}_N$ and $\mathcal{S}$.

Since the projection of $\mathcal{A}$ maintains the Coulomb gauge in any reference frame, after being promoted to bundle tensors, the conserved orbital and spin AMs that are constructed by Bliokh in [12] will be meaningful in any reference frame.

V. SUMMARY

The problem that the orbital and spin AMs of an electromagnetic field cannot be simultaneously gauge invariant and Lorentz covariant becomes clear in the framework of quantum field theory. With a Lorentz transformation, the electromagnetic 4-potential that satisfies the Coulomb gauge will produce a phase, denoted Ω, which depends on this transformation and breaks the Lorentz-covariant form of the electromagnetic 4-potential. After promotion as a vector in the tetrad bundle, the electromagnetic 4-potential can possess Lorentz covariance, which suggests that the noncovariance is due to our negligence of the transformation of the local reference frame. Equivalently, the electromagnetic object at (x^μ, ∂_ν) should be covariant with the object at $(\Lambda^\mu{}_\sigma x^\sigma, \Lambda_\nu{}^\sigma\partial_\sigma)$; however, we are accustomed to comparing the electromagnetic objects at (x^μ, ∂_ν) and $(\Lambda^\mu{}_\sigma x^\sigma, \partial_\nu)$. The local reference frame also plays a role as an internal degree of freedom.

The tetrad bundle is closely related to the gravitational effects [23]. Strictly, in curved space-time, the local frame $e(q) = (\partial_0, \partial_1, \partial_2, \partial_3)$ is not orthonormal; hence, it is not a tetrad so that the coordinate and tetrad diverge, which will generate observable differences between the electromagnetic 4-potential and the electromagnetic vector in a strong gravitational field. Therefore, the electromagnetic vector could play a role in the optical observation of astronomical objects.

In addition, the electromagnetic 4-potential is a gauge potential in the principal bundle whose structure group is U(1) and a projection of a vector in the tetrad bundle. We can view the electromagnetic 4-potential as a link that connects the two bundles. A natural question arises: What type of relationship has been established by the electromagnetic 4-potential between electromagnetic and gravitational interactions? We will explore this further.

[1] S. M. Barnett, J. Opt. B **4**, S7 (2001).

[2] K. Y. Bliokh, M. A. Alonso, E. A. Ostrovskaya, and A. Aiello, Phys. Rev. A **82**, 063825 (2010).

[3] C. Darwin, Proc. R. Soc. A **136**, 829 (1932).

[4] S. M. Barnett, L. Allen, R. P. Cameron, C. R. Gilson, M. J. Padgett, F. C. Speirits, and A. M. Yao, J. Opt. **18**, 064004 (2016).

[5] R. T. Hammond, arXiv:1706.07665 [physics.optics].

[6] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).

[7] D. L. P. Vitullo, C. C. Leary, P. Gregg, R. A. Smith, D. V. Reddy, S. Ramachandran, and M. G. Raymer, Phys. Rev. Lett. **118**, 083601 (2017).

[8] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atoms and Photons, Introduction to Quantum Electrodynamics* (Wiley-Interscience, New York, 1989).

[9] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, New J. Phys. **15**, 033026 (2013).

[10] S. Van Enk and G. Nienhuis, Europhys. Lett. **25**, 497 (1994).

[11] D. E. Soper, *Classical Field Theory* (Courier Dover, New York, 2008).

[12] K. Y. Bliokh, J. Dressel, and F. Nori, New J. Phys. **16**, 093037 (2014).

[13] R. Bluhm, Phys. World **17**, 41 (2004).

[14] R. Utiyama, Phys. Rev. **101**, 1597 (1956).

[15] S. Weinberg, *The Quantum Theory of Fields, Volume 1: Foundations* (Cambridge University Press, Cambridge, 1995).

[16] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras* (Springer Science & Business Media, Berlin, 2012).

[17] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1 (Interscience, New York, 1963).

[18] G. Rudolph and M. Schmidt, *Differential Geometry and Mathematical Physics* (Springer, Berlin, 2012).

[19] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, UK, 2007).

[20] E. Zeidler, *Quantum Field Theory III: Gauge Theory: A Bridge between Mathematicians and Physicists* (Springer Science & Business Media, Berlin, 2011).

[21] F. Belinfante, Physica **7**, 449 (1940).

[22] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999).

[23] Y. Tanii, *Introduction to Supergravity* (Springer, Berlin, 2014).