Testing atomic wave functions in the nuclear vicinity: The hyperfine structure with empirically deduced nuclear and quantum electrodynamic effects

J. S. M. Ginges[1,2,*] and A. V. Volotka[3]

[1]*School of Mathematics and Physics, The University of Queensland, Brisbane QLD 4072, Australia*

[2]*Centre for Engineered Quantum Systems, School of Physics, The University of Sydney, Sydney NSW 2006, Australia*

[3]*Helmholtz-Institut Jena, Fröbelstieg 3, D-07743 Jena, Germany*

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Calculations of the magnetic hyperfine structure rely on the input of nuclear properties—nuclear magnetic moments and nuclear magnetization distributions—as well as quantum electrodynamic radiative corrections for high-accuracy evaluation in heavy atoms. The uncertainties associated with assumed values of these properties limit the accuracy of hyperfine calculations. For example, for the heavy alkali-metal atoms Cs and Fr, these uncertainties may amount collectively to almost 1% or 2%, respectively. In this paper, we propose a method for removing the dependence of hyperfine structure calculations on assumed values of nuclear magnetic moments and nuclear magnetization distributions by determining these effects empirically from measurements of the hyperfine structure for high states. The method is valid for s, $p_{1/2}$, and $p_{3/2}$ states of alkali-metal atoms and alkali-metal-like ions. We have shown that for s states, the dependence on QED effects may also be removed to high accuracy. The ability to probe the electronic wave functions, through hyperfine comparisons, with significantly increased accuracy is important for the analysis of atomic parity violation measurements, and it may enable the accuracy of atomic parity violation calculations to be improved. More broadly, it paves the way for further development of high-precision atomic many-body methods.

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I. INTRODUCTION

Studies of atomic parity violation provide important low-energy tests of the standard model of particle physics. The largest effect in atoms arises from the nuclear weak charge, which depends on a unique combination of fundamental coupling constants. This makes atomic parity violation measurements uniquely sensitive to certain types of new physics, complementing searches for new physics performed at high energies [1–3].

Extraction of the nuclear weak charge from atomic measurements requires high-precision atomic calculations (see, e.g., reviews [4,5]). To gauge the accuracy of these calculations, a comparison of theoretical and experimental determinations of electric dipole transition amplitudes, energy intervals, and hyperfine structure constants is made [6,7]. The hyperfine structure is sensitive to the atomic wave functions in the nuclear region, and it is the hyperfine structure for which the largest deviations are seen.

The atoms and ions of interest for atomic parity violation measurements include Cs [8,9], Fr [10–12], Ba^+ [13,14], and Ra^+ [15]. The highest precision in atomic parity violation studies has been reached for ^{133}Cs, the measurement accurate to 0.35% [16], and calculations accurate to within 0.5% [6,7,17,18]. The future of nuclear spin-independent atomic parity violation studies on a single isotope depends on the ability to control the error of atomic calculations to ≈0.1%.

In the current work, we pave the way for significantly improved understanding of the electronic wave functions in the nuclear region. We propose a method for empirically correcting the unknown or neglected nuclear properties and quantum electrodynamic (QED) radiative corrections in hyperfine structure calculations by exploiting the scaling of different effects for higher states.

Beyond the uncertainties associated with many-electron correlations in the atomic theory evaluation of the hyperfine structure, there are several other sources of uncertainty related to assumed values of (i) the nuclear magnetic moment, (ii) the finite-magnetization distribution of the nucleus, and (iii) QED radiative corrections (or their neglect). The size of the error associated with each of these may be several 0.1%, or even ≈1%, for the systems of interest for parity violation studies. Controlling these errors is crucial if 0.1% tests of the atomic theory are to be performed.

Indeed, (i) in the comparison between theory and experiment for the hyperfine structure, a value for the nuclear magnetic moment is assumed. However, for the Fr isotopes, these are not known to better than 1–2% [19,20]. (ii) An assumed magnetization distribution is used in atomic calculations, the most routinely used being the uniformly magnetized sphere. There are data, however, for the hyperfine structure for the neutron-deficient isotopes of Fr [21,22] that support the validity of the single-particle model for that system. In our recent work on the ground-state hyperfine structure [23], we demonstrated that using the single-particle model gives a result for ^{211}Fr that differs by 1.3% from that found using the sphere; for ^{133}Cs this difference is 0.5%. (iii) Rigorous calculations

*Corresponding author: j.ginges@uq.edu.au

of QED radiative corrections to the ground-state hyperfine structure have been performed at the one-loop level for the alkali-metal atoms [23,24] and only recently for alkali-metal-like ions Ba^+ and Ra^+ [23], with contributions entering at around 0.5% for Cs, Ba^+, Fr, and Ra^+. Overwhelmingly, these effects have been neglected or crudely estimated in theoretical determinations of the hyperfine structure.

The uncertainty associated with the nuclear magnetization distribution also poses a problem in the area of QED tests in few-electron highly charged ions. That problem is addressed by constructing a difference between hyperfine intervals of the ion in question and the hydrogen-like ion that cancels the effect [25]. This difference, however, relies on precise knowledge of the nuclear magnetic moment. This proved problematic in studies of Li-like ^{208}Bi, where the accepted value of this moment turned out to be inaccurate [26].

II. THE HYPERFINE STRUCTURE ACROSS PRINCIPAL QUANTUM NUMBER

A. Factorizing the hyperfine structure

The magnetic hyperfine interaction is given by

$$h_{\rm hfs} = c\boldsymbol{\alpha}\cdot\mathbf{A} = \frac{1}{c}\frac{\boldsymbol{\mu}\cdot(\boldsymbol{r}\times\boldsymbol{\alpha})}{r^3}F(r), \tag{1}$$

where $\boldsymbol{\alpha}$ is a Dirac matrix, $\mathbf{A}$ is the nuclear vector potential, $\boldsymbol{\mu} = \mu\mathbf{I}/I$ is the nuclear magnetic moment (and $\mathbf{I}$ is the nuclear spin), and $F(r)$ describes the magnetization distribution. We use atomic units ($|e| = m = \hbar = 4\pi\epsilon_0 = 1$, $c = 1/\alpha$). $F(r) = 1$ corresponds to the case of point-nucleus magnetization. For a finite-nucleus magnetization distribution, the value $F(r) - 1$ differs from zero within the nucleus, $r \leqslant r_n$. Account of finite-nucleus magnetization gives a correction to the (point-magnetization) hyperfine structure—the Bohr-Weisskopf (BW) effect [27].

The splitting due to the magnetic hyperfine interaction (1) may be quantified in terms of the magnetic constant A. In the zeroth-order approximation (lowest-order in the atomic potential and for point-nucleus magnetization), the hyperfine A constant for the state with principal quantum number n and angular momentum quantum number κ ($\kappa = -1, 1, -2, \ldots$ for s, $p_{1/2}$, $p_{3/2}$, $\ldots$) is given by

$$A_{n\kappa} = -\frac{\alpha^2}{m_p}\frac{g_I\kappa}{J(J+1)}\int_0^\infty dr\, f(r)g(r)/r^2. \tag{2}$$

Here J is the electronic angular momentum, m_p is the proton mass, $g_I = \mu/(\mu_N I)$ is the nuclear g-factor, and $f(r)$ and $\alpha g(r)$ are the upper and lower radial components of the relativistic wave functions φ that satisfy the Dirac equation,

$$[c\boldsymbol{\alpha}\cdot\mathbf{p} + (\beta - 1)c^2 + V_{\rm nuc}(r) + V_{\rm el}]\varphi = \epsilon\varphi, \tag{3}$$

where β is a Dirac matrix. In our many-body calculations for the hyperfine structure, we use the Hartree-Fock potential as our starting potential, $V_{\rm el} = V_{\rm HF}$. Finite nuclear charge distribution is included in the determination of the wave functions, with $V_{\rm nuc}$ corresponding to a two-parameter Fermi distribution in our calculations.

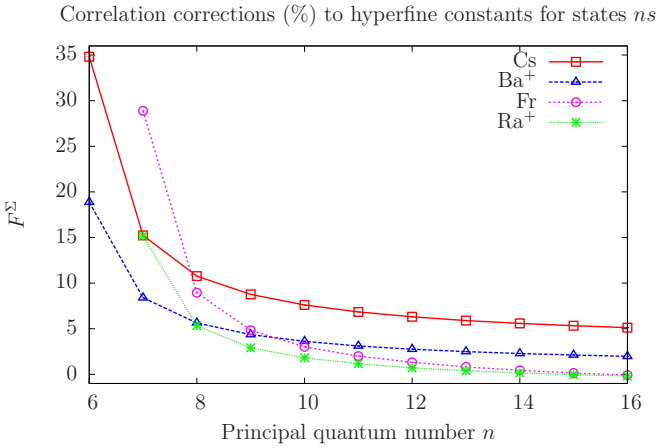

FIG. 1. Relative correlation corrections F^Σ to the hyperfine structure (in %) for Cs, Ba^+, Fr, and Ra^+. This corresponds to evaluation of $\langle\varphi_{\rm Br}|h_{\rm hfs} + \delta V_{\rm hfs}|\varphi_{\rm Br}\rangle - \langle\varphi|h_{\rm hfs} + \delta V_{\rm hfs}|\varphi\rangle$ relative to $\langle\varphi|h_{\rm hfs} + \delta V_{\rm hfs}|\varphi\rangle$.

We introduce the following parametrization for the hyperfine A constant:

$$A_{n\kappa} = A^{\rm MB}_{n\kappa}\frac{\mu}{\mu_N}\left(1 + \frac{\alpha}{\pi}F^{\rm BW}_{n\kappa} + \frac{\alpha}{\pi}F^{\rm QED}_{n\kappa}\right). \tag{4}$$

We explicitly separate different aspects of the hyperfine problem. The nuclear magnetic moment μ is factored out, and the value for $A^{\rm MB}_{n\kappa}$ comes only from electronic many-body calculations with the nuclear magnetic moment set to μ_N and for point-nucleus magnetization. The parameters $F^{\rm BW}_{n\kappa}$ and $F^{\rm QED}_{n\kappa}$ give the relative Bohr-Weisskopf correction and the relative QED correction, respectively. In the following, we will consider how the relative correlation, BW, and QED corrections scale with principal quantum number.

B. Correlation corrections

Our many-body calculations are carried out using the correlation potential approach [28]. A nonlocal, energy-dependent correlation potential $\Sigma(\mathbf{r}, \mathbf{r}', \epsilon)$ is constructed such that, in lowest order, the average value of this potential coincides with the second-order correlation correction to the energy. We use the Feynman diagram technique to include in Σ the electron-electron screening and the hole-particle interaction to all orders in the Coulomb interaction [29]. This potential is added to the relativistic Hartree-Fock equation (3), with $V_{\rm HF} \to V_{\rm HF} + \Sigma^{(\infty)}$, and correlation-corrected (Brueckner) energies $\epsilon_{\rm Br}$ and orbitals $\varphi_{\rm Br}$ are obtained.

The dominant part of the external-field correlation corrections—the core polarization—is included using the random-phase approximation with exchange (RPA). From this we get a correction to the hyperfine operator that corresponds to a hyperfine-modified Hartree-Fock potential, $h_{\rm hfs} + \delta V_{\rm hfs}$ [28]. Inclusion of the correlation potential and RPA corrections corresponds to evaluation of the matrix element $\langle\varphi_{\rm Br}|h_{\rm hfs} + \delta V_{\rm hfs}|\varphi_{\rm Br}\rangle$.

In Fig. 1 we plot the relative correlation corrections for ns states from the ground state to principal quantum number $n = 16$ for alkali-metal atoms and alkali-metal-like

ions of interest for parity violation studies, Cs, Fr, Ba^+, and Ra^+. These corrections correspond to the difference $\langle\varphi_{\rm Br}|h_{\rm hfs}+\delta V_{\rm hfs}|\varphi_{\rm Br}\rangle - \langle\varphi|h_{\rm hfs}+\delta V_{\rm hfs}|\varphi\rangle$ relative to $\langle\varphi|h_{\rm hfs}+\delta V_{\rm hfs}|\varphi\rangle$, which we denote by F^Σ. Since most of the uncertainty in many-body calculations is associated with evaluation of correlations, the smaller the relative size of the correlations F^Σ, the smaller the anticipated error in the many-body calculations. Our calculations for smaller correlation corrections (structural radiation and normalization of states that enter at around 1% or less and are not included in $\langle\varphi_{\rm Br}|h_{\rm hfs}+\delta V_{\rm hfs}|\varphi_{\rm Br}\rangle$) also demonstrate a reduction in relative size with an increase in n. We may therefore expect that significantly higher accuracy may be achieved for calculations of $A^{\rm MB}_{n'\kappa}$ for the higher states compared to calculations of $A^{\rm MB}_{n\kappa}$ for the ground or lower level, where $n' > n$ and particularly for $n' \gg n$.

C. Bohr-Weisskopf corrections

We have studied the Bohr-Weisskopf effect in two very different models: (i) the uniform spherical distribution, where $F(r) = (r/r_n)^3$, and (ii) the nuclear single-particle model, with spin nucleon g-factors found from measured nuclear magnetic moments. We refer the reader to, e.g., Ref. [30] for the single-particle model expressions for $F(r)$, derived in Refs. [27,31–33]. We performed calculations for ^{133}Cs, ^{211}Fr, $^{135}Ba^+$, and $^{225}Ra^+$ with 2×10^4 grid points and a grid radius of $r = 230$ a.u. for the atoms and $r = 120$ a.u. for the ions. Calculations were performed for states ns, $np_{1/2}$, and $np_{3/2}$, with $n = 6$–10 for Cs and Ba^+ and $n = 7$–11 for Fr and Ra^+. Account of core polarization is important for all non-s states, and for states with $j > 1/2$ it is responsible for the effect entirely. We observe that for the s, $p_{1/2}$, and $p_{3/2}$ states, the Bohr-Weisskopf effect $(\alpha/\pi)F^{\rm BW}_{n\kappa}$ stays the same across n to within 0.05% for all considered isotopes and magnetization models (the variability tends to be significantly smaller for the s and $p_{1/2}$ states). For example, in the nuclear single-particle model for ^{133}Cs, the values for $F^{\rm BW}$ at the RPA level are -0.898, -0.056, and -0.247 for $6s$, $6p_{1/2}$, and $6p_{3/2}$ states, and the variation across $n = 6$–10 for $\alpha/\pi\ F^{\rm BW}$ is $\ll 0.1\%$; in the spherical model, the values are about 3.4 times larger. Account of correlations hardly influences the $F^{\rm BW}$ values for s states and changes the values for p states only on the level of 1%, while the variation across n stays about the same. Note that in Rb the n-independence of the Bohr-Weisskopf effect for s-states has been observed experimentally across $n = 5$–7 through isotopic ratios of the hyperfine constants [34].

Therefore, for the excited states $n's_{1/2}$, $n'p_{1/2}$, and $n'p_{3/2}$ we may write the hyperfine constant as

$$A_{n'\kappa} = A^{\rm MB}_{n'\kappa}\frac{\mu}{\mu_N}\left(1+\frac{\alpha}{\pi}F^{\rm BW}_{n\kappa}+\frac{\alpha}{\pi}F^{\rm QED}_{n'\kappa}\right), \tag{5}$$

where the relative BW effect coincides with that for the ground state. Combining Eqs. (4) and (5), it is seen that *the dependence on nuclear properties may be completely eliminated*,

$$A_{n\kappa} = A^{\rm MB}_{n\kappa}\left(A^{\rm exp}_{n'\kappa}/A^{\rm MB}_{n'\kappa}\right)\left(1-\frac{\alpha}{\pi}\delta F^{\rm QED}_{\kappa}\right). \tag{6}$$

We have replaced the total hyperfine constant $A_{n'\kappa}$ with its experimentally determined value $A^{\rm exp}_{n'\kappa}$ and $\delta F^{\rm QED}_\kappa = F^{\rm QED}_{n'\kappa} - F^{\rm QED}_{n\kappa}$. The nuclear physics input, which has until now been required in hyperfine structure calculations, may instead be determined implicitly through hyperfine structure measurements for high states, electronic many-body calculations for high states, and the difference in QED contributions for the states being compared. We are confident that these may all be determined with sub-0.1% uncertainty.

TABLE I. Relative QED radiative corrections $F^{\rm QED}$ to hyperfine constants for ns states of ^{133}Cs found in core-Hartree and Kohn-Sham potentials, $V_{\rm CH}$ and $V_{\rm KS}$.

	$F^{\rm QED}$				
	$6s$	$7s$	$8s$	$9s$	$10s$
$V_{\rm CH}$	−1.64	−1.49	−1.44	−1.39	−1.31
$V_{\rm KS}$	−1.91	−1.77	−1.71	−1.65	−1.59

D. QED radiative corrections

We have gone further in this work and explicitly calculated the QED radiative corrections for ns states for ^{133}Cs over $n = 6$–10. Rigorous calculations of the one-loop self-energy and vacuum polarization corrections were performed using the extended Furry picture with the core-Hartree (CH) potential and the Kohn-Sham (KS) potential generated for the ground state, as in Ref. [23]. In Table I we present our results. The correction changes relatively significantly from $n = 6$ to 10 (20% for CH and 17% for KS), however a difference of 20% in $F^{\rm QED}$ corresponds to $|(\alpha/\pi)\delta F^{\rm QED}_{-1}| \approx 0.05\%$, and we expect similar QED differences across n for s states of other heavy alkali-metal atoms and alkali-metal-like ions.

E. The hyperfine ratio method

Therefore, we have shown—accurate to within 0.1%—that we may simplify Eq. (6) further for the s states,

$$A_{ns} = A^{\rm MB}_{ns}\left(A^{\rm exp}_{n's}/A^{\rm MB}_{n's}\right). \tag{7}$$

(We replace the notation A_{n-1} with A_{ns}.) The ratio in parentheses in Eq. (7) corresponds to the nuclear and QED aspects of the problem, $\mu/\mu_N(1+\alpha/\pi\ F^{\rm BW}+\alpha/\pi\ F^{\rm QED})$.

The right-hand side of Eq. (7) contains the ratio of the results of two many-body calculations, $A^{\rm MB}_{ns}/A^{\rm MB}_{n's}$. We may write

$$A^{\rm MB}_{ns} \approx A^{\rm HF}_{ns}\left(1+F^{\delta V}_{ns}\right)\left(1+F^{\Sigma}_{ns}\right), \tag{8}$$

where $A^{\rm HF}_{ns}$ is the hyperfine constant found in the relativistic Hartree-Fock approximation, and $F^{\delta V}_{ns}$, F^{Σ}_{ns} are the relative RPA and correlation corrections. The RPA correction is essentially the same for all principal quantum numbers—for Cs, $F^{\delta V}_{ns} \approx 0.2$—and it cancels in the ratio Eq. (7). It means that the ratio method is largely insensitive to account of core polarization.

The atomic theory error, however, is mainly associated with the evaluation of the electron-electron correlations, most of which may be represented by a correlation potential Σ. For the correlation potential, and smaller correlation corrections such as the structural radiation and normalization of states, the relative correction is not the same for different n—as we saw

TABLE II. Magnetic hyperfine A constants for $6s$ and $7s$ states of ^{133}Cs from Refs. [6,7]. Raw values and percentage deviations (% dev.) from experiment are shown in the first rows of data, values with QED and Bohr-Weisskopf (single-particle Woods-Saxon) corrections from Ref. [23] are shown in the following rows, and values for $6s$ using the ratio involving $7s$, $(A_{7s}^{\rm exp}/A_{7s}^{\rm th})\,A_{6s}^{\rm th}$, and vice versa for $7s$ are shown in the final rows. Units: MHz.

Reference	A_{6s}	% dev.	A_{7s}	% dev.
Raw				
Theory [6], unfitted	2315.0	0.73	545.3	−0.09
Theory [6], fitted	2300.3	0.09	543.8	−0.37
Theory [7]	2306.6	0.37	544.59	−0.23
With QED, BW(SP)				
Theory [6], unfitted	2318.4	0.88	546.1	0.06
Theory [6], fitted	2303.8	0.24	544.6	−0.22
Theory [7]	2319.7	0.94	547.69	0.34
Ratio				
Theory [6], unfitted	2317.1	0.82	541.3	−0.82
Theory [6], fitted	2308.8	0.46	543.3	−0.46
Theory [7]	2311.8	0.59	542.61	−0.59
Experiment	2298.157[a]		545.818(16)[b]	

[a]Reference [35].
[b]Reference [36].

earlier—and we have

$$A_{ns}^{\rm MB}\big/A_{n's}^{\rm MB} \approx A_{ns}^{\rm HF}\big/A_{n's}^{\rm HF}\left(1+F_{ns}^{\Sigma}-F_{n's}^{\Sigma}\right). \tag{9}$$

The ratio, therefore, depends on the difference in the relative correlation corrections between states ns and $n's$, $F_{ns}^{\Sigma}-F_{n's}^{\Sigma}$. If we are interested in preserving the dependence on the theoretical account of the electron correlations—*i.e., we want to test the accuracy of atomic calculations*—then this difference should be as large as possible, $F_{ns}^{\Sigma}\gg F_{n's}^{\Sigma}$. The most suitable state to test is therefore the ground state, with the other state chosen to be as high as possible (see Fig. 1). On the other hand, for the case in which $F_{ns}^{\Sigma}-F_{n's}^{\Sigma}\approx 0$ (e.g., both states with high principal quantum number), the dependence on the correlations is largely removed in the ratio, and we may perform high-precision predictions of the hyperfine structure. Note that it may happen that sub-1% many-body contributions do not appear with the same trend across n—including, e.g., when fitting procedures are used and states are treated differently—and the hyperfine structure for the lower states may be evaluated more accurately than for higher states.

III. APPLICATION OF THE RATIO METHOD

We apply the ratio method to the hyperfine structure results for states $6s$ and $7s$ of ^{133}Cs used to test the accuracy of the most precise calculations for atomic parity violation performed in Refs. [6,7]. We are limited to these lowest ns states, as only data for the states involved in the parity violation transition were presented in these works. The data are shown in Table II. We include two sets of results from Ref. [6] corresponding to *ab initio* values and values obtained with modified wave functions found by fitting the calculated binding energies to measured data. The original theory results are presented in the first rows. The results of both works [6,7] were obtained by modeling the nuclear magnetization distribution as a uniform sphere. The QED radiative corrections were not included in Ref. [6], while subsequently determined rigorous calculations for $6s$ [24] were included in Ref. [7] (with relative QED corrections for $7s$ assumed to be equal to those for $6s$). The percentage deviations of these values from experiment are shown in the following columns. For the hyperfine structure for $6s$, the deviations of the unfitted and fitted results of Ref. [6] are 0.73% and 0.09%, while the deviation of the result of Ref. [7] is 0.37%; for $7s$, the deviations are −0.09%, −0.37%, and −0.23%, respectively. We expect that a better indication of the quality of the wave functions may be obtained by employing the more well-motivated single-particle model for the magnetization distribution, as well as accounting for the QED corrections where these have been omitted. For the s-states of ^{133}Cs, adjusting the magnetization model accordingly amounts to a correction of 0.53%, and inclusion of QED radiative effects amounts to a correction of −0.38% [23]. With these corrections, we obtain values for the magnetic hyperfine constants—shown in Table II—with the following deviations from experiment for the unfitted and fitted results of Ref. [6] and the results of Ref. [7], respectively: 0.88%, 0.24%, and 0.94% for the $6s$ state; 0.06%, −0.22%, and 0.34% for the $7s$ state. The corrections to the hyperfine results of Ref. [7] are sizeable enough to affect the error assignment for the parity violating amplitude computed in that work.

The nuclear and QED effects and their uncertainties are difficult to quantify, however, and application of the ratio method ensures that any deviations are associated with the uncertainties of the electronic calculations alone. The results for the magnetic hyperfine constants for the $6s$ state of ^{133}Cs, obtained from the ratio method with data from $7s$, are shown in the final rows of Table II; for illustration, we also present values for $7s$ found from the ratio involving $6s$. We obtain values for $6s$ with deviations 0.82%, 0.46%, and 0.59% for the unfitted and fitted data from Ref. [6] and the data from Ref. [7], respectively. In the general case, these deviations are attributable to the errors associated with *both* states in the ratio. Note that the procedure for testing the atomic theory is equivalent to comparing the theory ratio $A_{n\kappa}/A_{n'\kappa}$ or $A_{n\kappa}^{\rm MB}/A_{n'\kappa}^{\rm MB}$ with the empirical one, $A_{n\kappa}^{\rm exp}/A_{n'\kappa}^{\rm exp}$. Expressing the many-body value $A_{n\kappa}^{\rm MB}$ in terms of an "exact" value $A_{n\kappa}^{\rm MB,\,exact}$ and a deviation from this value, $\delta_{n\kappa}^{\rm MB}=A_{n\kappa}^{\rm MB}-A_{n\kappa}^{\rm MB,\,exact}$, the ratio $A_{n\kappa}^{\rm MB}/A_{n'\kappa}^{\rm MB}$ deviates from the exact value by $(\delta A_{n\kappa}^{\rm MB}/A_{n\kappa}^{\rm MB}-\delta A_{n'\kappa}^{\rm MB}/A_{n'\kappa}^{\rm MB})$, that is, by the *difference* in the relative deviations from the exact or experimental value. We emphasize that the dependence on QED corrections is removed in the ratio when both states are treated in the same way. Similarly, the dependence on the chosen magnetization distribution, and the nuclear magnetic moment, is also removed. The relatively large deviations seen above—*connected only with the uncertainties in the electronic wave functions of the two states*—indicate that there is room for improvement of the wave functions and calculations of atomic parity violation. It would be more useful, though, to isolate the uncertainty connected with *just one state*, that is, e.g., $(\delta A_{6s}^{\rm MB}/A_{6s}^{\rm MB})$ and $(\delta A_{7s}^{\rm MB}/A_{7s}^{\rm MB})$ and not their difference. To do this, high-precision measurements and calculations for high states are required, as discussed earlier in the paper. We propose such a prescription—a detailed study of the hyperfine structure for a number of states across principal quantum number—in future analyses of atomic parity violation calculations.

We would also like to demonstrate the predictive power of the ratio method. We may use the theory results of Ref. [19] for states $8s$ and $9s$ of ^{210}Fr where the deviations from experiment for both states are 0.4% and cancel in the ratio. These calculations were performed using a magnetic moment with 2% uncertainty, the nuclear magnetization taken to have a Fermi distribution, and QED corrections omitted, with raw values 1584 and 624.8 MHz. Using the data for $8s$ (with the measurement from Ref. [37]) to make a "prediction" for $9s$, we obtain a value for the hyperfine constant 622.4 MHz compared to the measured value 622.25(36) MHz [19].

IV. CONCLUSIONS

In summary, the ratio method advocated in the current work allows the electronic wave functions in the nuclear region to be probed, through hyperfine comparisons, free from uncertainties connected with the nuclear magnetic moment, nuclear magnetization distribution, and QED. We have demonstrated that by using this method, the remaining electronic uncertainties are associated with the difference in the relative errors associated with the two states. This may be reduced to the electronic uncertainty associated with one state by making that of the other very small, which we expect is possible for the highly excited states. This development is important for the interpretation of atomic parity violation measurements in several ways. Most directly, it will enable a more reliable error analysis of the hyperfine structure to be performed, leading to a more reliable error assignment for the theoretical parity violating amplitude and deduced nuclear weak charge. It may also enable improved control of the electronic wave functions in the nuclear region and higher accuracy in calculations of atomic parity violation and other short-distance effects. This may be achieved through empirical fitting of the hyperfine structure by introducing parameters to the correlation potential in a similar way to the method used currently for energies. Far more broadly, the proposed method opens up the possibility of using the hyperfine structure as a benchmark against which developments in precision atomic many-body theory may be tested. Indeed, the hyperfine structure has a different sensitivity to correlations compared to energies.

Therefore, high-precision measurements—and calculations—of hyperfine structure constants for high s, $p_{1/2}$, and $p_{3/2}$ states of alkali-metal atoms and alkali-metal-like ions would prove invaluable for replacing the unknown nuclear physics parameters and QED in atomic calculations of the hyperfine structure by means of the proposed ratio method.

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