Family of coherence measures and duality between quantum coherence and path distinguishability

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Coherence measures and their operational interpretations lay the cornerstone of coherence theory. In this paper, we introduce a class of coherence measures with α affinity, say α affinity of coherence for $\alpha \in (0, 1)$. Furthermore, we obtain the analytic formulas for these coherence measures and study their corresponding convex roof extension. We provide an operational interpretation for 1/2 affinity of coherence by showing that it is equal to the error probability to discrimination a set of pure states with the least-square measurement. By employing this relationship we regain the optimal measurement for equiprobable quantum state discrimination. Moreover, we compare these coherence quantifiers and establish a complementarity relation between the 1/2 affinity of coherence and path distinguishability for some special cases.

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I. INTRODUCTION

Quantum coherence [1] is one of the fundamental features in quantum mechanics and characterizes the wave-like property for all objects. It is also a necessary condition for entanglement and other quantum correlations which manifests its core position in quantum information theory [2,3]. As a key quantum resource, coherence may lead to an operational advantage over classical physics, and its important role in quantum algorithms has been investigated [4–7]. Hence, for a given quantum state, it is important to ask the amount of coherence it has and if the quantifier of coherence has any operational meaning. In Ref. [1], the authors have established a resource theory of coherence which is a rigorous framework to quantify coherence. In this theory, coherence characterizes the superposition of a quantum state relative to a fixed orthogonal basis and thereafter a lot of work has been done to enrich this theory [8–12]. This framework places certain important constraints on the measures of coherence, and different coherence measures may reflect different physical aspects of the quantum system [13–16]. Like other resource theories, the resource theory of coherence is composed of "free states" and "free operations."

Let $\mathcal{H}$ be a finite-dimensional Hilbert space with an orthogonal basis $\{|i\rangle\}_{i=1}^{d}$. Density matrices that are diagonal in this basis are free states and we call them incoherent states because they do not possess any coherence. We label this set of incoherent quantum states by $\mathcal{I}$. That is,

$$\mathcal{I} = \left\{\sigma|\sigma = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|\right\}. \tag{1}$$

Free operations in coherence theory are the completely positive and trace-preserving (CPTP) maps which admit an incoherent Kraus representation. That is, there always exists a set of Kraus operators $\{K_i\}$ such that

$$\frac{K_i \sigma K_i^\dagger}{\mathrm{Tr} K_i \sigma K_i^\dagger} \in \mathcal{I}, \tag{2}$$

for each i and any incoherent state σ. These operations are also called incoherent operations and we label these operations by Φ.

Analogous to the quantification of entanglement [17–20], any measure of coherence C should satisfy the following axioms [1]:

(C1) Faithfulness. $C(\rho) \geqslant 0$ with equality if and only if ρ is incoherent.

(C2) Monotonicity. C does not increase under the action of an incoherent operation, i.e., $C(\Phi(\rho)) \leqslant C(\rho)$ for any incoherent operation Φ.

(C3) Strong monotonicity. C does not increase on average under selective incoherent operations, i.e., $\sum_i p_i C(\sigma_i) \leqslant C(\rho)$ with probabilities $p_i = \mathrm{Tr} K_i \rho K_i^\dagger$, postmeasurement states $\sigma_i = p_i^{-1} K_i \rho K_i^\dagger$, and incoherent operators K_i.

(C4) Convexity. Nonincreasing under mixing of quantum states, i.e.,$\sum_i p_i C(\rho_i) \geqslant C(\sum_i p_i \rho_i)$ for any set of states $\{\rho_i\}$ and $p_i \geqslant 0$ with $\sum_i p_i = 1$.

Conditions (C1) and (C2) highlight the role of free states and free operations in the coherence theory, i.e., the free states have zero coherence and the free operations cannot increase the coherence of any state. (C3) and (C4) are two constraints imposed on coherence measures. Like in entanglement theory, a coherence quantifier which satisfies non-negativity and (strong) monotonicity is called a (strong) coherence monotone. Furthermore, if it also satisfies convexity, we call it a convex (strong) coherence monotone.

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The authors of Ref. [21] provided a simple and interesting condition to replace (C3) and (C4) with the additivity of coherence for block-diagonal states,

$$C(p\rho \oplus (1-p)\sigma) = pC(\rho) + (1-p)C(\sigma), \tag{3}$$

for any $p \in [0,1]$, $\rho \in \mathcal{E}(\mathcal{H}_1)$, $\sigma \in \mathcal{E}(\mathcal{H}_2)$, and $p\rho \oplus (1-p)\sigma \in \mathcal{E}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where $\mathcal{E}(\mathcal{H})$ denotes the set of density matrices on $\mathcal{H}$.

They proved that conditions (C1) and (C2) and Eq. (3) are equivalent to conditions (C1) through (C4). This is surprising because Eq. (3) is an operation-independent equality, whereas strong monotonicity and convexity are operation-dependent inequalities. In general, it is relatively easy to check whether a coherence quantifier satisfies Eq. (3) more than (C3) and (C4).

In this paper we introduce a class of coherence measures and attempt to answer the question posed in the beginning by linking this coherence measure to ambiguous quantum state discrimination (QSD). QSD, as a fundamental problem in quantum mechanics, has been studied extensively [22–29]. It is not only an important problem of theoretical research, but also plays a key role in quantum communication and quantum cryptography [30–34].

We briefly review the ambiguous QSD. Suppose there are two persons, Alice and Bob. Alice chooses a state ρ_i from a set of states $\{\rho_i\}_{i=1}^N$ with probability η_i and sends it to Bob. Now Bob's job is to determine which state he has received, as accurately as possible. To do this, Bob performs a positive-operator valued measure (POVM) on each ρ_i and declares that the state is ρ_j when the measurement outcome reads j. The POVM is a set of positive operators $\{M_i\}$ satisfying $\sum_i M_i = I$. As the probability to get the result j with state ρ_i is $p_{j|i} = \mathrm{Tr}(M_j\rho_i)$, the corresponding maximal success probability is

$$P_S^{\mathrm{opt}}(\{\rho_i,\eta_i\}) = \max_{\{M_i\}} \sum_i \eta_i \mathrm{Tr}(M_i\rho_i), \tag{4}$$

where the maximization is done over all POVMs. For the $N=2$ case, the analytic formula of P_S^{opt} and the optimal measurement are known. However, no solution about optimal probability and measurement is known for the general $N > 2$ case.

As a suboptimal choice, the least square measurement (LSM) is an alternative to discriminate quantum states [35–41]. In comparison with the optimal measurement, the LSM has several nice properties. First, its construction is relatively simple because it can be determined directly from the given ensemble. Second, it is very close to the optimal measurement when the states to be distinguished are *almost orthogonal* [37,42]. The construction of the LSM is as follows:

Given an ensemble $\{\rho_i,\eta_i\}_{i=1}^N$ and denoting $\rho_{\mathrm{out}} = \sum_i \eta_i\rho_i$, the least square measurements are [43]

$$M_i^{lsm} = \eta_i \rho_{\mathrm{out}}^{-1/2}\rho_i\rho_{\mathrm{out}}^{-1/2}, \quad i = 1,2,\ldots,N. \tag{5}$$

As a result, the minimal error probability of this measurement is

$$P_E^{lsm}(\{\rho_i,\eta_i\}) = 1 - \sum_i \eta_i \mathrm{Tr}\big(M_i^{lsm}\rho_i\big). \tag{6}$$

The paper is structured as follows: In Sec. II we introduce the α affinity of coherence. We reveal the connection between the $1/2$ affinity of coherence and QSD with least square measurements in Sec. III. Furthermore, we deal with quantum state discrimination with coherence theory in Secs. IV and V. Besides, we establish a duality between the $1/2$ affinity of coherence and the path distinguishability in Sec. VI, and finally conclude in Sec. VII with a summary and outlook.

II. QUANTIFYING COHERENCE WITH AFFINITY

A. α affinity and α affinity of distance

Distances in state space are good candidates for quantifying quantum correlations. In this section, we introduce a distance that we can use to establish a bona fide measure of quantify coherence. In classical statistical theory [44], affinity is defined as

$$A(f,g) = \sum_x \sqrt{f(x)g(x)},$$

where f and g are discrete probability distributions. This definition is alike the Bhattacharyya coefficient [45] between two probability distributions (discrete or continuous) in classical probability theory. Classical affinity quantifies the closeness of two probability distributions. Borrowing the notion from classical statistical theory, Luo and Zhang [46] have introduced quantum affinity as follows: Let $\mathcal{H}$ be a d-dimensional Hilbert space and $\mathcal{E}(\mathcal{H})$ be the set of density matrices on $\mathcal{H}$. For any $\rho,\sigma \in \mathcal{E}(\mathcal{H})$, quantum affinity is defined as

$$A(\rho,\sigma) := \mathrm{Tr}(\sqrt{\rho}\sqrt{\sigma}). \tag{7}$$

Quantum affinity, similar to fidelity [47], describes how close two quantum states are. We drop the adjective "quantum" in the rest of this paper unless there is any ambiguity.

The notion of affinity has been extended to α affinity ($0 < \alpha < 1$) and is defined as

$$A^{(\alpha)}(\rho,\sigma) := \mathrm{Tr}\rho^\alpha\sigma^{1-\alpha}.$$

For each $\alpha \in (0,1)$, $A^{(\alpha)}(\rho,\sigma)$ satisfies the following properties: (1) Boundedness. $A^{(\alpha)}(\rho,\sigma) \in [0,1]$ with $A^{(\alpha)}(\rho,\sigma) = 1$ if and only if $\rho = \sigma$. (2) Monotonicity. $A^{(\alpha)}(\rho,\sigma) \leqslant A^{(\alpha)}(\Phi(\rho),\Phi(\sigma))$ for any CPTP map Φ. (3) Joint concavity. If $\rho_i,\sigma_i \in \mathcal{E}(\mathcal{H})$ and $p_i \geqslant 0$, $\sum_i p_i = 1$, then $A^{(\alpha)}(\sum_i p_i\rho_i, \sum_i p_i\sigma_i) \geqslant \sum_i p_i A^{(\alpha)}(\rho_i,\sigma_i)$. The proof of property (1) is given in Appendix A. See Ref. [48] for property (2), and property (3) is the result of Lieb's concavity theorem [49].

It is well known that α affinity plays an important role in quantum hypothesis testing. For two-state discrimination with many identical copies, one has [50,51]

$$-\lim_{N\to\infty}\frac{1}{N}P_{E,N}^{\mathrm{opt}}\big(\{\rho_i^{\otimes N},\eta_i\}_{i=1}^2\big) = -\inf_{\alpha\in(0,1)}\big\{\ln\big(\mathrm{Tr}\rho_1^\alpha\rho_2^{1-\alpha}\big)\big\}.$$

This limit defines a function of α affinity, and

$$Q(\rho,\sigma) := \min_{\alpha\in(0,1)} A^{(\alpha)}(\rho,\sigma), \tag{8}$$

is the nonlogarithmic version of the quantum Chernoff bound (QCB) [51].

Moreover, we can see that α affinity is related to α-z-relative Rényi entropy [48],

$$S_{\alpha,z}(\rho \parallel \sigma) = \frac{1}{\alpha - 1} \ln F_{\alpha,z}(\rho \parallel \sigma),$$

where

$$F_{\alpha,z}(\rho \parallel \sigma) := \mathrm{Tr}\big(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}}\big)^z, \tag{9}$$

and

$$A^{(\alpha)}(\rho, \sigma) = F_{\alpha,1}(\rho, \sigma). \tag{10}$$

Note that the family of α-z-relative Rényi entropies includes relative entropy S and max-relative entropy $S_{\max}$ [48]:

$$S = \lim_{\alpha\to 1} S_{\alpha,\alpha}, \quad S_{\max} = \lim_{\alpha\to\infty} S_{\alpha,\alpha}.$$

It is worth noting that several coherence measures like relative entropy [1], geometric coherence [9], and max-relative entropy [16] are related to α-z-relative Rényi entropy. In the next section, we introduce yet another measure of coherence; namely, the α affinity of coherence which is related to α-z-relative Rényi entropy.

Based on the α affinity, we introduce the α affinity of distance as

$$d_a^{(\alpha)}(\rho, \sigma) := 1 - [A^{(\alpha)}(\rho, \sigma)]^{1/\alpha}, \tag{11}$$

where $\rho, \sigma \in \mathcal{E}(\mathcal{H})$. Obviously, the α affinity of distance satisfies the following properties:

(P1) $d_a^{(\alpha)}(\rho, \sigma) \geqslant 0$ with equality if and only if $\rho = \sigma$.

(P2) $d_a^{(\alpha)}$ is contractive under CPTP maps.

B. Quantifying coherence

Quantification of entanglement from the geometric point of view began in Refs. [18,19]. The authors of these two papers put forward the scheme to quantify entanglement with the minimal distance between a given quantum state and all separable states with relative entropy and Bures distance. Later, Luo and Zhang [46] studied the quantification of entanglement by using the Hellinger distance. The Bures distance and the Hellinger distance have been proven to be good choices to quantify quantum discord [52–54].

For any $\alpha \in (0, 1)$, we define the α affinity of coherence as the minimal α affinity of distance over all incoherent states,

$$\begin{aligned} C_a^{(\alpha)}(\rho) &:= \min_{\sigma\in\mathcal{I}} d_a^{(\alpha)}(\rho, \sigma) \\ &= 1 - \max_{\sigma\in\mathcal{I}} (\mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha}))^{1/\alpha}. \end{aligned} \tag{12}$$

An advantage of $C_a^{(\alpha)}$ over geometric coherence, $C_g(\rho) := 1 - \max_{\sigma\in\mathcal{I}}(\mathrm{Tr}(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}))^2$ [9], is that it is relatively easy to compute. Let $\sigma = \sum_i \mu_i |i\rangle\langle i|$ be an incoherent state. Then

$$\begin{aligned} A^{(\alpha)}(\rho) &\equiv \max_{\sigma\in\mathcal{I}} \mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha}) = \max_{\mu_i} \left(\sum_i \mu_i^{1-\alpha} \langle i|\rho^\alpha|i\rangle \right) \\ &\leqslant \max_{\mu_i} \left(\sum_i \mu_i \right)^{1-\alpha} \left(\sum_i \langle i|\rho^\alpha|i\rangle^{1/\alpha} \right)^\alpha \\ &= \left(\sum_i \langle i|\rho^\alpha|i\rangle^{1/\alpha} \right)^\alpha, \end{aligned} \tag{13}$$

where the inequality follows from Hölder's inequality: $\sum_{i=1}^n |x_i y_i| \leqslant (\sum_{i=1}^n |x_i|^p)^{1/p} (\sum_{i=1}^n |y_i|^q)^{1/q}$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Here, $p = \frac{1}{1-\alpha} > 1$ and $q = \frac{1}{\alpha} > 1$. Inequality (13) gives an upper bound on $A^{(\alpha)}(\rho)$. This suggests that we can choose suitable $\{\mu_i\}$ such that the above inequality becomes an equality. As a result, we obtain the analytic expression for $C_a^{(\alpha)}$ as

$$C_a^{(\alpha)}(\rho) = 1 - \sum_i \langle i|\rho^\alpha|i\rangle^{1/\alpha}, \tag{14}$$

and the closest incoherent state which minimizes $C_a^{(\alpha)}(\rho)$ is

$$\sigma_\rho = \sum_i \frac{\langle i|\rho^\alpha|i\rangle^{1/\alpha}}{\sum_j \langle j|\rho^\alpha|j\rangle^{1/\alpha}} |i\rangle\langle i|. \tag{15}$$

With (P1), (P2), and Eq. (3), we have the following theorem:

Theorem 1. The α affinity of coherence is a coherence measure.

Proof. First, it is obvious that $C_a^{(\alpha)}(\rho) \geqslant 0$. Since $d_a^{(\alpha)}(\rho, \sigma) = 0$ if and only if $\rho = \sigma$, one has $C_a^{(\alpha)}(\rho) = 0$ if and only if $\rho \in \mathcal{I}$. In addition, since $d_a^{(\alpha)}(\rho, \sigma)$ obeys monotonicity under CPTP maps, we have $C_a^{(\alpha)}(\rho) \geqslant C_a^{(\alpha)}(\Phi(\rho))$ for any incoherent operation Φ. Now, instead of (C3) and (C4), we prove that $C_a^{(\alpha)}$ satisfies additivity of coherence for block-diagonal states. We have

$$\begin{aligned} &C_a^{(\alpha)}(p\rho \oplus (1-p)\sigma) \\ &= 1 - \sum_i \langle i|[p\rho \oplus (1-p)\sigma]^\alpha|i\rangle^{1/\alpha} \\ &= 1 - \sum_i \langle i|(p\rho)^\alpha \oplus [(1-p)\sigma]^\alpha|i\rangle^{1/\alpha} \\ &= p\left(1 - \sum_i \langle i|\rho^\alpha|i\rangle^{1/\alpha}\right) + (1-p)\left(1 - \sum_i \langle i|\sigma^\alpha|i\rangle^{1/\alpha}\right) \\ &= pC_a^{(\alpha)}(\rho) + (1-p)C_a^{(\alpha)}(\sigma). \end{aligned}$$

Thus, $C_a^{(\alpha)}$ is a coherence measure for each $\alpha \in (0, 1)$. ■

Similarly, we define the quantum Chernoff bound of coherence, $C_{qcb}(\rho)$, and the affinity of coherence, $\widetilde{C}_a(\rho)$, respectively as

$$\begin{aligned} C_{qcb}(\rho) &:= \min_{\sigma\in\mathcal{I}} (1 - Q^{1/\alpha}(\rho, \sigma)) \\ &= 1 - \max_{\sigma\in\mathcal{I}} \min_{\alpha\in(0,1)} (\mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha}))^{1/\alpha} \\ &= \max_{\alpha\in(0,1)} C_a^{(\alpha)}(\rho), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \widetilde{C}_a(\rho) &:= \min_{\sigma\in\mathcal{I}} (1 - A(\rho, \sigma)) = 1 - \max_{\sigma\in\mathcal{I}} \mathrm{Tr}(\sqrt{\rho}\sqrt{\sigma}) \\ &= 1 - \sqrt{\sum_i \langle i|\sqrt{\rho}|i\rangle^2}, \end{aligned} \tag{17}$$

and the closest incoherent state is again σ_ρ in Eq. (15).

Note that Eq. (16) does not necessarily imply that C_{qcb} is a coherence measure for some $\alpha \in (0, 1)$ because C_a^α is a coherence measure. This can be argued as follows: for a given ρ, let α' be the value of α such that $C_a^{(\alpha')}(\rho) = \max_\alpha C_a^{(\alpha)}(\rho)$. Then, $C_{qcb}(\rho) = C_a^{(\alpha')}(\rho) \geqslant C_a^{(\alpha')}[\Phi(\rho)] \leqslant \max_\alpha C_a^{(\alpha)}[\Phi(\rho)]$, where Φ is an incoherent operation. Thus, it is not immediately clear that C_{qcb} is a coherence measure. Next, we can show that $\widetilde{C}_a$ is a convex weak coherence monotone. Following the same lines of the proof of Theorem 1, $\widetilde{C}_a$ satisfies (C1) and (C2). Moreover, the convexity of $\widetilde{C}_a$ can be derived from the joint concavity of $A(\rho,\sigma)$. However, $\widetilde{C}_a$ does not satisfy strong monotonicity.

$$\text{Let } \rho_1 = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \quad \text{and} \quad \rho_2 = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix},$$

then $\widetilde{C}_a(\rho_1) = 1 - \sqrt{\frac{1}{2}}$, $\widetilde{C}_a(\rho_2) = 1 - \sqrt{\frac{1}{3}}$, and

$$\widetilde{C}_a\left(\frac{1}{2}\rho_1 \oplus \frac{1}{2}\rho_2\right) = 1 - \sqrt{\frac{5}{12}} \neq \frac{1}{2}[\widetilde{C}_a(\rho_1) + \widetilde{C}_a(\rho_2)].$$

In conclusion, $\widetilde{C}_a$ is a convex weak coherence monotone.

C. Coherence for pure states and single-qubit states

In this section, we evaluate the α affinity of coherence for pure states and single-qubit states. For any pure state $|\psi\rangle$,

$$C_a^{(\alpha)}(|\psi\rangle) = 1 - \sum_i |\langle i|\psi\rangle|^{2/\alpha}, \tag{18}$$

is a nonincreasing function of α. We have $C_a^{\alpha}(|\psi\rangle) \to 1$ when $\alpha \to 0$. This is a very interesting observation that all coherent pure states are almost the maximally coherent states. If we consider the convex roof extension of the α affinity of coherence for a mixed state ρ as

$$C_a^{(\alpha)}(\rho) := \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C_a^{(\alpha)}(|\psi_i\rangle), \tag{19}$$

then $\lim_{\alpha\to 0} C_a^{(\alpha)}(\rho) = 1$. That is, $\lim_{\alpha\to 0} C_a^{(\alpha)}$ is a coherence measure which equals unity when the state is coherent and is zero otherwise. A similar measure; namely, the *trivial coherence measure*, was discussed in Ref. [55] for which similar consequences were observed.

For a single-qubit state $\rho = \frac{1}{2}(I + \sum_i c_i\sigma_i)$ with σ_i $(i = 1,2,3)$ being Pauli matrices, the eigenvalues are $\lambda_{1,2} = (1 \mp |\mathbf{c}|)/2$, and

$$\rho^\alpha = \begin{pmatrix} \frac{\lambda_1^\alpha+\lambda_2^\alpha}{2} + \frac{c_3(\lambda_2^\alpha-\lambda_1^\alpha)}{2|\mathbf{c}|} & \frac{(-c_1+ic_2)(\lambda_1^\alpha-\lambda_2^\alpha)}{2|\mathbf{c}|} \\ \frac{(-c_1-ic_2)(\lambda_1^\alpha-\lambda_2^\alpha)}{2|\mathbf{c}|} & \frac{\lambda_1^\alpha+\lambda_2^\alpha}{2} - \frac{c_3(\lambda_2^\alpha-\lambda_1^\alpha)}{2|\mathbf{c}|} \end{pmatrix}.$$

Therefore, the corresponding α affinity of coherence is

$$C_a^{(\alpha)}(\rho) = 1 - (A+B)^{1/\alpha} - (A-B)^{1/\alpha}, \tag{20}$$

where

$$A = \frac{\left(\frac{1-|\mathbf{c}|}{2}\right)^\alpha + \left(\frac{1+|\mathbf{c}|}{2}\right)^\alpha}{2}, \quad \text{and}$$
$$B = \frac{c_3\left[\left(\frac{1+|\mathbf{c}|}{2}\right)^\alpha - \left(\frac{1-|\mathbf{c}|}{2}\right)^\alpha\right]}{2|\mathbf{c}|}.$$

III. 1/2 AFFINITY OF COHERENCE AND LEAST SQUARE MEASUREMENT

Spehner and Orszag [56] first revealed the connection between quantum correlation (Hellinger-distance-based quantum discord) and QSD with least square measurements. In coherence theory, there is a very close relationship between geometric coherence and QSD. The authors of Ref. [57] have recently shown that geometric coherence of ρ is equal to the minimum error probability to discriminate a set of linearly independent pure states $\{|\psi_i\rangle,\eta_i\}_{i=1}^d$ with von Neumann measurement, where $|\psi_i\rangle = \eta_i^{-1/2}\sqrt{\rho}|i\rangle$, $\eta_i = \rho_{ii}$, and $d = \mathrm{rank}(\rho)$. Since the optimal measurement is not easy to find, we consider the least square measurement for $\{|\psi_i\rangle,\eta_i\}_{i=1}^d$.

For $\{|\psi_i\rangle,\eta_i\}_{i=1}^d$, there are two cases. If $\eta_i \neq 0$ $(i = 1,\dots,d)$, then the ensemble contains d states. Since $\sum_i \eta_i|\psi_i\rangle\langle\psi_i| = \rho$, the least square measurement is

$$M_i^{lsm} = \eta_i\rho^{-1/2}|\psi_i\rangle\langle\psi_i|\rho^{-1/2} = |i\rangle\langle i|, \tag{21}$$

where $\rho^{-1/2} := \sum_i \lambda^{-1/2}|a_i\rangle\langle a_i|$ if $\rho = \sum_i \lambda|a_i\rangle\langle a_i|$ is the spectral decomposition. Thus, $\sum_i M_i^{lsm} = I$ and the successful probability to discriminate the ensemble $\{|\psi_i\rangle,\eta_i\}_{i=1}^d$ with $\{M_i^{lsm}\}$ is

$$\begin{aligned} P_S^{lsm}\left(\{|\psi_i\rangle,\eta_i\}_{i=1}^d\right) &= \sum_i \eta_i \mathrm{Tr}\left(M_i^{lsm}|\psi_i\rangle\langle\psi_i|\right) \\ &= \sum_i \langle i|\sqrt{\rho}|i\rangle^2 \\ &= [A^{(1/2)}(\rho)]^2. \end{aligned} \tag{22}$$

If $\eta_i = 0$ for some $i = i_1, i_2, \dots, i_s$, then the ensemble $\{|\psi_i\rangle,\eta_i\}_{i=1}^d$ reduces to $\{|\psi_{i'}\rangle,\eta_{i'}\}_{i'=1}^{d-s}$. In fact, as $\eta_i = \langle i|\rho|i\rangle = |\sqrt{\rho}|i\rangle|^2$, $\eta_i = 0$ implies that $|\psi_i\rangle$ is a zero vector. If $\mathcal{S}$ is the subspace spanned by $\{|\psi_{i'}\rangle\}_{i'=1}^{d-s}$, then

$$M_{i'}^{lsm} = \eta_{i'}\rho^{-1/2}|\psi_{i'}\rangle\langle\psi_{i'}|\rho^{-1/2} = |i'\rangle\langle i'|$$

for all i', and $\sum_{i'}^{d-s} M_{i'}^{lsm} = I_{\mathcal{S}}$. Moreover, the successful probability to discriminate the ensemble $\{|\psi_{i'}\rangle,\eta_{i'}\}_{i'=1}^{d-s}$ with $\{M_{i'}^{lsm}\}$ is

$$\begin{aligned} P_S^{lsm}\left(\{|\psi_{i'}\rangle,\eta_{i'}\}_{i'=1}^{d-s}\right) &= \sum_{i'=1}^{d-s} \eta_{i'} \mathrm{Tr}\left(M_{i'}^{lsm}|\psi_{i'}\rangle\langle\psi_{i'}|\right) \\ &= \sum_{i'=1}^{d-s} \langle i'|\sqrt{\rho}|i'\rangle^2 \\ &= \sum_{i=1}^{d} \langle i|\sqrt{\rho}|i\rangle^2 \\ &= [A^{(1/2)}(\rho)]^2. \end{aligned}$$

In other words, the corresponding error probability to discriminate linearly independent pure states $\{|\psi_i\rangle,\eta_i\}_{i=1}^d$ is

$$P_E^{lsm}(\{|\psi_i\rangle,\eta_i\}) = 1 - P_S^{lsm}\left(\{|\psi_i\rangle,\eta_i\}_{i=1}^d\right) = C_a^{(1/2)}(\rho).$$

Thus, we have the following theorem:

Theorem 2. If quantum state ρ describes a quantum system in d-dimensional Hilbert space $\mathcal{H}$ with $\{|i\rangle\}_{i=1}^d$ being a reference basis, then the α affinity of coherence of ρ is equal to the error probability to discriminate $\{|\psi_i\rangle, \eta_i\}_{i=1}^d$ with a least square measurement. That is,

$$C_a^{(1/2)}(\rho) = P_E^{lsm}\big(\{|\psi_i\rangle, \eta_i\}_{i=1}^d\big), \tag{23}$$

where $\eta_i = \langle i|\rho|i\rangle$ and $|\psi_i\rangle = \eta_i^{-1/2}\sqrt{\rho}|i\rangle$.

Remark 2.1. If ρ is an incoherent state, then $C_a^{1/2}(\rho) = 0$ which means that $\{|\psi_i\rangle, \eta_i\}_i$ can be perfectly discriminated by the least square measurement. In other words, the LSM is actually the optimal measurement.

IV. LEAST SQUARE MEASUREMENT AND OPTIMAL MEASUREMENT

A. Quantum state discrimination with least square measurement and 1/2 affinity of coherence

In this section, we review a connection between the least square measurement (as a suboptimal choice) and the optimal measurement in QSD protocol. The authors in Ref. [57] have linked quantum state discrimination to geometric coherence.

Let us consider QSD of a set of pure states $\{|\psi_i\rangle, \eta_i\}_{i=1}^d$. Denote a matrix M with $M_{ij} = \sqrt{\eta_i\eta_j}\langle\psi_i|\psi_j\rangle$, $1 \leqslant i, j \leqslant d$; that is,

$$M = \begin{pmatrix} \eta_1 & \sqrt{\eta_1\eta_2}\langle\psi_1|\psi_2\rangle & \dots & \sqrt{\eta_1\eta_d}\langle\psi_1|\psi_d\rangle \\ \sqrt{\eta_2\eta_1}\langle\psi_2|\psi_1\rangle & \eta_2 & \dots & \sqrt{\eta_2\eta_d}\langle\psi_2|\psi_d\rangle \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ \sqrt{\eta_d\eta_1}\langle\psi_d|\psi_1\rangle & \sqrt{\eta_d\eta_2}\langle\psi_d|\psi_2\rangle & \dots & \eta_d \end{pmatrix}. \tag{24}$$

Then, M is a density matrix and we call it the QSD state of $\{|\psi_i\rangle, \eta_i\}_{i=1}^d$.

Theorem 3 [57]. Let $\mathcal{H}$ be a d-dimensional Hilbert space and $\{|i\rangle\}_{i=1}^d$ be the computable basis; that is, $|i\rangle = (0, \dots, 0, 1, 0, \dots, 0)^t$, the ith entry is 1 for each i. For $|\psi_i\rangle \in \mathcal{H}$, the minimal error probability to discriminate the collection of linearly independent pure states $\{|\psi_i\rangle, \eta_i\}_{i=1}^d$ is equal to the geometric coherence of the corresponding QSD-state M; that is,

$$P_E^{\text{opt}}\big(\{|\psi_i\rangle, \eta_i\}_{i=1}^d\big) = C_g(M). \tag{25}$$

For 1/2 affinity and the least square measurement, there exists a similar relationship. If we denote the corresponding QSD state by M; namely, $\nu_i = M_{ii} = \eta_i$, $|\varphi_i\rangle = \nu_i^{-1/2}\sqrt{M}|i\rangle$ for each i, then $\langle\varphi_i|\varphi_j\rangle = (\nu_i\nu_j)^{-1/2}\langle i|M|j\rangle = \langle\psi_i|\psi_j\rangle$, $1 \leqslant i, j \leqslant d$. With Lemma 8 from Ref. [57], there exists a unitary V such that $|\varphi_i\rangle = V|\psi_i\rangle$ for each i.

As a result, the least square measurement for $\{|\psi_i\rangle, \eta_i\}_{i=1}^d$ is

$$M_i = \eta_i\rho_{\text{out}}^{-1/2}|\psi_i\rangle\langle\psi_i|\rho_{\text{out}}^{-1/2}, \quad i = 1, \dots, d, \tag{26}$$

with $\rho_{\text{out}} = \sum_i \eta_i|\psi_i\rangle\langle\psi_i|$. Since $\sigma_{\text{out}} = \sum_i \eta_i|\varphi_i\rangle\langle\varphi_i| = V\rho_{\text{out}}V^\dagger$, the LSM for $\{|\varphi_i\rangle, \eta_i\}_{i=1}^d$ is

$$N_i = \eta_i\sigma_{\text{out}}^{-1/2}|\varphi_i\rangle\langle\varphi_i|\sigma_{\text{out}}^{-1/2} = VM_iV^\dagger. \tag{27}$$

In addition, one has

$$\begin{aligned} P_E^{lsm}(\{|\psi_i\rangle, \eta_i\}) &= \sum_i \eta_i \text{tr}(M_i|\psi_i\rangle\langle\psi_i|) \\ &= \sum_i \eta_i \text{tr}(N_i|\varphi_i\rangle\langle\varphi_i|) \\ &= P_S^{lsm}(\{|\varphi_i\rangle, \nu_i\}) \\ &= C_a^{(1/2)}(M). \end{aligned}$$

In conclusion, we have the following result:

Theorem 4. Let $\mathcal{H}$ be a d-dimensional Hilbert space and $\{|i\rangle\}_{i=1}^d$ be the computable basis; that is, $|i\rangle = (0, \dots, 0, 1, 0, \dots, 0)^t$, the ith entry is 1 for $i = 1, \dots, d$. For $|\psi_i\rangle \in \mathcal{H}$, $i = 1, \dots, d$, the error probability to discriminate the collection of pure states $\{|\psi_i\rangle, \eta_i\}_{i=1}^d$ with a least square measurement is equal to the 1/2 affinity of coherence of the corresponding QSD-state M; that is,

$$P_E^{lsm}\big(\{|\psi_i\rangle, \eta_i\}_{i=1}^d\big) = C_a^{(1/2)}(M), \tag{28}$$

where the incoherent pure states are $\{|i\rangle\}_{i=1}^d$.

B. Least square measurement and optimal measurement

First, we recall the following result:

Theorem 5 [42,58]. Let $\{\rho_i, \mu_i\}_{i=1}^m$ to be an ensemble of m states of a system in an n-dimensional Hilbert space $\mathcal{H}$ ($m \leqslant n$), then

$$P_S^{\text{opt}}\big(\{\rho_i, \mu_i\}_{i=1}^m\big) \leqslant \sqrt{P_S^{lsm}\big(\{\rho_i, \mu_i\}_{i=1}^m\big)}. \tag{29}$$

Because $P_E^{\text{opt}} = 1 - P_S^{\text{opt}}$ is the minimal error probability of QSD, the error probability with LSM is

$$P_E^{lsm} = 1 - P_S^{lsm} \leqslant 1 - \big(P_S^{\text{opt}}\big)^2 \leqslant 2P_E^{\text{opt}}. \tag{30}$$

Therefore, if P_E^{opt} is very close to 0, so is P_E^{lsm}. In fact, LSM is very close to the optimal measurement for almost orthogonal states.

Because the LSM to discriminate a set of pure states is actually a von Neumann measurement and the result of Theorem 3, one has

$$2C_g(\rho) \geqslant C_a^{(1/2)}(\rho) \geqslant C_g(\rho) \geqslant \widetilde{C}_a(\rho),$$

for any ρ. The last inequality is due to Theorem 5 above and Theorem 1 of Ref. [57] as follows:

$C_g(\rho) \geqslant 1 - P_S^{\text{opt}}(\{\rho_i, \mu_i\}_{i=1}^m) \geqslant 1 - \sqrt{P_S^{lsm}(\{\rho_i, \mu_i\}_{i=1}^m)} = \tilde{C}_a(\rho)$. In addition, since $C_g(\rho) \leqslant \frac{C_{l_1}(\rho)}{d-1}$ for any $\rho > 0$ (that is, ρ is invertible) [57], where $C_{l_1}(\rho)$ is the l_1 norm of coherence defined as $C_{l_1}(\rho) := \sum_{i\neq j} |\langle i|\rho|j\rangle|$, we have that, for any $\rho > 0$, the following inequality holds:

$$\frac{2}{d-1}C_{l_1}(\rho) \geqslant 2C_g(\rho) \geqslant C_a^{(1/2)}(\rho) \geqslant C_g(\rho) \geqslant \widetilde{C}_a(\rho).$$

On the other hand, we consider the connection between least square measurement and optimal measurement through coherence.

In Ref. [59], Zhang *et al.* gave an upper bound for geometric coherence as

$$C_g(\rho) \leqslant \min\{l_1, l_2\}, \tag{31}$$

where $l_1 = 1 - \max_i\{\rho_{ii}\}$ and $l_2 = 1 - \sum_i b_{ii}^2$ with b_{ij} being the (i, j)th entry of $\sqrt{\rho}$. This is interesting to note that l_2 is actually equal to $C_a^{(1/2)}(\rho)$ and, moreover, they also show that l_2 is tight for the maximally coherent mixed states given by

$$\rho_m = p|\psi_d\rangle\langle\psi_d| + \frac{1-p}{d}I_d, \tag{32}$$

where $0 < p < 1$, and $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_i |i\rangle$ is the maximally coherent state.

In other words, one has

$$C_g(\rho_m) = C_a^{(1/2)}(\rho_m). \tag{33}$$

Combining Theorem 3, Theorem 1, and Eq. (33), we recover the following result:

Theorem 6 [22,28]. For the equiprobable quantum state discrimination task $\{|\phi_i\rangle, 1/d\}_{i=1}^d$ with $\langle\phi_i|\phi_j\rangle = p$ for $i \neq j$, the least square measurement is optimal. Moreover, the maximum successful probability is

$$P_S^{\text{opt}}\big(\{|\phi_i\rangle, 1/d\}_{i=1}^d\big) = \left[\frac{d-1}{d}\sqrt{1-p} + \frac{1}{d}\sqrt{1-p+dp}\right]^2.$$

Proof. Note that the QSD-state of the above-mentioned task is ρ_m. Because

$$P_E^{\text{opt}}\big(\{|\phi_i\rangle, 1/d\}_{i=1}^d\big) = C_g(\rho_m) = C_a^{(1/2)}(\rho_m) = P_E^{lsm}\big(\{|\phi_i\rangle, 1/d\}_{i=1}^d\big),$$

then the least square measurement is optimal. The first equality is the result of Theorem 3 and the fact that $\{|\phi_i\rangle\}$ is linearly independent. The last equality is due to Theorem 4. Using the result from Ref. [59],

$$C_g(\rho_m) = 1 - \left[\frac{d-1}{d}\sqrt{1-p} + \frac{1}{d}\sqrt{1-p+dp}\right]^2,$$

the maximum successful probability is

$$P_S^{\text{opt}}\big(\{|\phi_i\rangle, 1/d\}_{i=1}^d\big) = \left[\frac{d-1}{d}\sqrt{1-p} + \frac{1}{d}\sqrt{1-p+dp}\right]^2,$$

and the corresponding optimal measurement is

$$M_i^{\text{opt}} = \frac{1}{d}\rho_{\text{out}}^{-1/2}|\phi_i\rangle\langle\phi_i|\rho_{\text{out}}^{-1/2},$$

where $\rho_{\text{out}} = \frac{1}{d}\sum_i |\phi_i\rangle\langle\phi_i|$ $(i = 1, \dots, d,)$. ■

V. WHEN IS LEAST SQUARE MEASUREMENT OPTIMAL?

Theorem 6 indicates that LSM is optimal for the equiprobable case. However, we find that this is not the only case, as discussed below.

A. Case of two pure states

Since we have the explicit expressions of geometric coherence and 1/2 affinity of coherence for single-qubit states, we can derive the condition for LSM being optimal for an ensemble containing two pure states. Given an ensemble $\{|\psi_i\rangle, \eta_i\}_{i=1}^2$, the corresponding QSD-state is a single-qubit state $\rho = \sum_i c_i\sigma_i$. From Eq. (20), one has

$$[A^{(1/2)}(\rho)]^2 = \frac{1}{2}\left(1 + \sqrt{1-|\mathbf{c}|^2} + \frac{c_3^2}{1+\sqrt{1-|\mathbf{c}|^2}}\right).$$

On the other hand, with fidelity $F(\rho,\sigma) := \text{tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}$,

$$F(\rho) := \max_{\sigma\in\mathcal{I}} F(\rho,\sigma) = \sqrt{\frac{1}{2}\big(1+\sqrt{1-c_1^2-c_2^2}\big)}.$$

The above expressions reduce to simpler forms when ρ is a pure state $[|\mathbf{c}| = (c_1^2+c_2^2+c_3^2)^{1/2} = 1]$. That is, $[A^{(1/2)}(\rho)]^2 = \frac{1}{2}(1+c_3^2)$ and $F^2(\rho) = \frac{1}{2}(1+|c_3|)$. Then, $A^{(1/2)}(\rho) = F(\rho)$ if and only if $c_3 = 0$ or ± 1. The same can be shown to be true for mixed states with some tedious calculations. Hence, the least square measurement is optimal for two pure states case if and only if these states are orthogonal or have equal probabilities.

B. Multiple copy quantum state discrimination with least square measurement

We consider QSD protocol with multiple copies, as the error probability of a QSD task decreases when we have more copies of states.

For the N-copy case $\{|\psi_i\rangle^{\otimes N}, \eta_i\}_{i=1}^d$, the (i, j)th entry of the corresponding QSD-state is

$$\rho_{ij}^{(N)} = \sqrt{\eta_i\eta_j}\langle\psi_i|\psi_j\rangle^N \quad (1 \leqslant i,\ j \leqslant d).$$

Let $N \to \infty$ and $\rho_{ij}^{(N)} \to 0$ for each $i \neq j$. Since $\{|\psi_i\rangle^{\otimes N}\}_{i=1}^d$ is linearly independent for large N, the QSD-state $\rho^{(N)}$ is invertible. Then,

$$C_a^{(1/2)}(\rho) \leqslant \frac{2}{d-1}C_{l_1}(\rho),$$

and the error probability to discriminate $\{|\psi_i\rangle^{\otimes N}, \eta_i\}_{i=1}^d$ tends to zero. In other words, if we have enough copies of states, pure states $\{|\psi_i\rangle, \eta_i\}_{i=1}^d$ can be almost perfectly distinguished by the LSM. In other words, we prove that LSM is asymptotically optimal for the discrimination of pure states in the sense that the corresponding QSD-state $\rho \to \rho^{\text{diag}} = \sum_i \langle i|\rho|i\rangle|i\rangle\langle i|$.

VI. DUALITY BETWEEN 1/2 AFFINITY OF COHERENCE AND PATH DISTINGUISHABILITY

Wave-particle duality is an intriguing but central concept in quantum physics. In the double-slit interference experiment, a single quantum object can exhibit the wave nature as long as knowledge about the path chosen by the object is uncertain. More knowledge about the path corresponds to poor interference. In this direction, quantitative relations in the form of trade-offs between wave and particle aspects were studied by Greenberger–Yasin [60] and Englert [61], respectively. Englert, in his famous paper [61], derived a path-visibility duality relation for the optimal detector measurement for two paths as follows:

$$\mathcal{V}^2 + \mathcal{D}^2 \leqslant 1, \tag{34}$$

where $\mathcal{V}$ is the visibility of the interference pattern and $\mathcal{D}$ is a measure of path distinguishability or which-way information. Recently, Bera *et al.* [62] obtained a complementarity relation between the l_1 norm of coherence and path distinguishability in the case of Yang's n-slit experiment. Here, although we are unable to provide a general proof for mixed states in arbitrary dimensions, we establish the complementarity between $1/2$ affinity of coherence and path distinguishability for some special cases.

Consider the case of d-slit quantum interference with pure quantons. In Yang's n-slit experiment, if the quanton passes through the ith slit or takes the ith path, then we denote $|i\rangle$ as the possible state. As a result, the state of the quanton can be represented with d basis states $\{|1\rangle, \dots, |d\rangle\}$ as

$$|\Psi\rangle = c_1|1\rangle + \cdots + c_d|d\rangle, \tag{35}$$

where $|i\rangle$ represents the ith slit and c_i is the amplitude of taking the ith slit. To determine through which slit the quanton passes, one needs to perform a quantum measurement. According to quantum measurement theory, the quanton will interact with a detector state and the compound state is given by

$$U(|\Psi\rangle|0_d\rangle) = \sum_i c_i|i\rangle|d_i\rangle, \tag{36}$$

where $\{|d_i\rangle\}$ are normalized but not necessarily orthogonal states of the detector.

To quantify the coherence of quanton, one considers the reduced density matrix of the quanton after tracing out the detector states,

$$\rho_s = \sum_{i,j=1}^{d} c_i\bar{c}_j\langle d_j|d_i\rangle|i\rangle\langle j|. \tag{37}$$

From Theorem 2, the $1/2$ affinity of coherence is

$$C_a^{(1/2)}(\rho_s) = 1 - P_S^{lsm}\big(\{|\psi_i\rangle, \eta_i\}_{i=1}^d\big), \tag{38}$$

where $\eta_i = |c_i|^2$, $|\psi_i\rangle = \exp(\sqrt{-1}\theta_i)\eta_i^{-1/2}\sqrt{\rho_s}|i\rangle$, and θ_i is the argument of c_i.

Now, to know which path the quanton takes, one has to discriminate the detector states $\{|d_i\rangle, |c_i|^2\}_{i=1}^d$. In other words, the path distinguishability is actually equivalent to the discrimination of the corresponding detector states.

Since $\langle\psi_i|\psi_j\rangle = \langle d_j|d_i\rangle = \overline{\langle d_i|d_j\rangle}$, there exists a unitary matrix V such that $|d_i\rangle = V|\overline{\psi_i}\rangle$. Therefore, one has

$$\rho_{\text{out}} = \sum_i |c_i|^2|d_i\rangle\langle d_i| = V\sum_i |c_i|^2|\overline{\psi_i}\rangle\langle\overline{\psi_i}|V^\dagger = V\overline{\rho_s}V^\dagger,$$

and then the corresponding LSM for $\{|d_i\rangle, |c_i|^2\}$ is

$$N_i^{lsm} = |c_i|^2\rho_{\text{out}}^{-1/2}|d_i\rangle\langle d_i|\rho_{\text{out}}^{-1/2} = V|i\rangle\langle i|V^\dagger.$$

As a result, one has

$$\begin{aligned} P_S^{lsm}\big(\{|d_i\rangle, |c_i|^2\}_{i=1}^d\big) &= \sum_i |c_i|^2|\langle i|V^\dagger|d_i\rangle|^2 \\ &= \sum_i |\langle i|\sqrt{\overline{\rho_s}}|i\rangle|^2 \\ &= P_S^{lsm}\big(\{|\psi_i\rangle, |c_i|^2\}_{i=1}^d\big). \end{aligned}$$

Even though it is not the optimal choice for quantum state discrimination, LSM is very close to the optimal choice when the states to be distinguished are almost orthogonal, and its construction is also relatively simple. Moreover, the complementarity between coherence and path distinguishability holds just for linearly independent detector states [57,62]. Therefore, if we define the optimal successful probability to discriminate the detector states with LSM as path distinguishability, $D_q := P_S^{lsm}(\{|d_i\rangle, |c_i|^2\}_{i=1}^d)$, and the $1/2$ affinity of coherence as coherence, $C := C_a^{(1/2)}(\rho_s)$, we obtain the complementarity between $1/2$ affinity of coherence and path distinguishability as

$$C + D_q = 1. \tag{39}$$

Thus, the wave nature of the quanton can also be characterized by $C_a^{(1/2)}(\rho_s)$. If the quantum system is exposed to the environment; that is, the quanton state is a mixed state $\rho = \sum_{i,j}\rho_{ij}|i\rangle\langle j|$, we can obtain a generalized complementarity. The composite system of the quanton and the path detector after the unitary interaction can be given as

$$\rho_{sd} = \sum_{i,j}\rho_{ij}|i\rangle\langle j| \otimes |d_i\rangle\langle d_j|, \tag{40}$$

and the reduced density matrix of the quanton after tracing out the detector states is

$$\rho_s = \sum_{i,j=1}^{d}\rho_{ij}\langle d_j|d_i\rangle|i\rangle\langle j|. \tag{41}$$

As every principal 2×2 submatrix in Eq. (41) is positive semidefinite [63, p. 434], we have

$$\sqrt{\rho_{ii}\rho_{jj}} - |\rho_{ij}| \geqslant 0 \quad (1 \leqslant i,\ j \leqslant d). \tag{42}$$

Assuming that the corresponding ensemble to ρ_s is $\{|\psi_i\rangle, \rho_{ii}\}$, we have

$$|\langle\psi_i|\psi_j\rangle| = \frac{|\langle i|\rho_s|j\rangle|}{\sqrt{\rho_{ii}\rho_{jj}}} = \frac{|\rho_{ij}|}{\sqrt{\rho_{ii}\rho_{jj}}}|\langle d_i|d_j\rangle| \leqslant |\langle d_i|d_j\rangle|,$$

for each i and j. In other words, a pair of states in $\{|d_i\rangle, \rho_{ii}\}_{i=1}^d$ is more difficult to distinguish than the corresponding pair in $\{|\psi_i\rangle, \rho_{ii}\}_{i=1}^d$. Suppose $\rho_s' = \sum_{i,j=1}^d\sqrt{\rho_{ii}\rho_{jj}}\langle d_j|d_i\rangle|i\rangle\langle j|$. Then for certain special cases where there exists an incoherent operation Φ (see Appendix B), we have

$$\Phi(\rho_s') = \rho_s. \tag{43}$$

Because $C_a^{(1/2)}$ is a coherence measure, we have

$$1 - C_a^{(1/2)}(\rho_s) \geqslant 1 - C_a^{(1/2)}(\rho_s') = P_S^{lsm}\big(\{|d_i\rangle, \rho_{ii}\}_{i=1}^d\big) \equiv D_q.$$

Thus, we have the following complementarity relation between coherence and path distinguishability,

$$C + D_q \leqslant 1. \tag{44}$$

VII. CONCLUSION

In this paper, we have introduced a family of coherence measures; namely, the α affinity of coherence for $\alpha \in (0, 1)$. Moreover, we obtained the analytic formulas for these quantifiers and also studied their convex roof extension. In particular, we have offered an operational meaning for the 1/2 affinity of coherence by showing that this equals the error probability to discriminate a set of pure states with least square measurement. Based on the relationship between the LSM and the optimal measurement, we obtained the optimal measurement for the equiprobable quantum state discrimination. Furthermore, we obtained conditions for the LSM to be the optimal measurement for two pure states from the perspective of coherence theory. In addition, we also studied the multiple copy QSD and concluded that LSM is optimal in the asymptotical sense. At last, we established the complementary relationship between the 1/2 affinity of coherence and path distinguishability.

Our results not only offer a class of bona fide coherence quantifiers, but also reveal a close link between the quantification of coherence and quantum state discrimination. However, the operational interpretation of general α-affinity coherence needs further investigation.

Note added. Recently, we have been informed by Hyukjoon Kwon that the 1/2 affinity of coherence has been computed and proven to be a coherence measure independently in Refs. [64,65] by different methods, yielding the same result.

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APPENDIX A: $A^{(\alpha)}$ is Bounded

Proposition 7. $0 \leqslant A^{(\alpha)}(\rho,\sigma) \leqslant 1$, with $A^{(\alpha)}(\rho,\sigma) = 1$ if and only if $\rho = \sigma$.

Proof. Because $\rho^{\frac{\alpha}{2}}\sigma^{1-\alpha}\rho^{\frac{\alpha}{2}}$ is a positive matrix, one has

$$\mathrm{Tr}(\rho^\alpha\sigma^{1-\alpha}) = \mathrm{Tr}\big(\rho^{\frac{\alpha}{2}}\sigma^{1-\alpha}\rho^{\frac{\alpha}{2}}\big) \geqslant 0.$$

The other part can be proved as in Ref. [66]. Let $\{|x\rangle\}_x$ be a basis of $\mathcal{H}$, then $M = \{M_x | M_x = |x\rangle\langle x|\}$ is an informationally complete measurement. Denoting $\Phi(\rho) = \sum_x \langle x|\rho|x\rangle |x\rangle\langle x|$, we have from the monotonicity of $A^{(\alpha)}(\rho,\sigma)$ and Jensen's inequality

$$A^{(\alpha)}(\rho,\sigma) \leqslant A^{(\alpha)}(\Phi(\rho),\Phi(\sigma)) = \sum_x \left(\frac{\langle x|\rho|x\rangle}{\langle x|\sigma|x\rangle}\right)^\alpha \langle x|\sigma|x\rangle \leqslant \left(\sum_x \langle x|\rho|x\rangle\right)^\alpha = 1.$$

Because the equality holds if and only if $\langle x|\rho|x\rangle = \langle x|\sigma|x\rangle$ for any informationally complete measurement, one has $A^{(\alpha)}(\rho,\sigma) = 1$ if and only if $\rho = \sigma$. ■

APPENDIX B: $C_a^{(1/2)}(\rho_s) \leqslant C_a^{(1/2)}(\rho_s')$

1. $d = 2$ case

For $d = 2$, let $\Phi = \{K_{12}, K_{11}, K_{22}\}$ with

$$K_{12} = \begin{pmatrix} \frac{\sqrt{\rho_{12}}}{(\rho_{11}\rho_{22})^{1/4}} & 0 \\ 0 & \frac{\sqrt{\rho_{21}}}{(\rho_{11}\rho_{22})^{1/4}} \end{pmatrix},$$

$$K_{11} = \begin{pmatrix} \sqrt{1 - \frac{|\rho_{12}|}{\sqrt{\rho_{11}\rho_{22}}}} & 0 \\ 0 & 0 \end{pmatrix},$$

$$K_{22} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - \frac{|\rho_{12}|}{\sqrt{\rho_{11}\rho_{22}}}} \end{pmatrix}.$$

Since $\frac{|\rho_{12}|}{\sqrt{\rho_{11}\rho_{22}}} \leqslant 1$ we have

$$\left|\frac{\sqrt{\rho_{12}}}{(\rho_{11}\rho_{22})^{1/4}}\right|^2 + \left|\sqrt{1 - \frac{|\rho_{12}|}{\sqrt{\rho_{11}\rho_{22}}}}\right|^2 = 1,$$

and Φ is an incoherent operation such that $\Phi(\rho_s') = \rho_s$. Thus, $C_a^{(1/2)}(\rho_s') \geqslant C_a^{(1/2)}(\rho_s)$.

2. $d = 3$ case

For $d = 3$, we denote $\sigma_{ij} = \frac{\rho_{ij}}{\sqrt{\rho_{ii}\rho_{jj}}}$ and $\rho_{ij} = |\rho_{ij}|e^{i\theta_{ij}}$. Without any loss of generality, we can assume that $|\sigma_{12}| \geqslant |\sigma_{13}| \geqslant |\sigma_{23}|$. Then the quantum operation $\Phi = \{K_{12}, K_{13}, K_{11}, K_{22}, K_{33}\}$ is

$$K_{12} = \begin{pmatrix} \sqrt{\sigma_{12}} & 0 & 0 \\ 0 & \sqrt{\sigma_{12}} & 0 \\ 0 & 0 & \frac{\sigma_{23}}{\sqrt{\sigma_{12}}} \end{pmatrix},$$

$$K_{13} = \begin{pmatrix} \sqrt{\sigma_{13} - \sigma_{23}e^{i\theta_{12}}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\sigma_{31} - \sigma_{32}e^{-i\theta_{12}}} \end{pmatrix},$$

$$K_{11} = \begin{pmatrix} \sqrt{1 - |\sigma_{12}| - |\sigma_{13} - \sigma_{23}e^{i\theta_{12}}|} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{1 - |\sigma_{12}|} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{1 - \frac{|\sigma_{23}|^2}{|\sigma_{12}|} - |\sigma_{13} - \sigma_{23}e^{i\theta_{12}}|} \end{pmatrix}.$$

If $|\sigma_{12}| + |\sigma_{13} - \sigma_{23}e^{i\theta_{12}}| \leqslant 1$ and thus $|\sigma_{13} - \sigma_{23}e^{i\theta_{12}}| \leqslant 1$, then Φ is an incoherent operation. But, this may not be true for all states because these conditions may not be satisfied. Moreover,

$$K_{12}\rho_s' K_{12}^\dagger = \begin{pmatrix} |\sigma_{12}|\rho_{11} & \rho_{12}\langle d_2|d_1\rangle & \frac{\sqrt{\rho_{11}}\rho_{23}e^{i\theta_{12}}}{\sqrt{\rho_{22}}}\langle d_3|d_1\rangle \\ \rho_{21}\langle d_1|d_2\rangle & |\sigma_{12}|\rho_{22} & \rho_{23}\langle d_3|d_2\rangle \\ \frac{\sqrt{\rho_{11}}\rho_{32}e^{-i\theta_{12}}}{\sqrt{\rho_{22}}}\langle d_1|d_3\rangle & \rho_{32}\langle d_2|d_3\rangle & \frac{|\sigma_{23}|^2}{|\sigma_{12}|}\rho_{33} \end{pmatrix}$$

and

$$K_{13}\rho_s' K_{13}^\dagger = \begin{pmatrix} |\sigma_{13} - \sigma_{23}e^{i\theta_{12}}|\rho_{11} & 0 & \left(\rho_{13} - \frac{\sqrt{\rho_{11}}\rho_{23}e^{i\theta_{12}}}{\sqrt{\rho_{22}}}\right)\langle d_3|d_1\rangle \\ 0 & 0 & 0 \\ \left(\rho_{31} - \frac{\sqrt{\rho_{11}}\rho_{32}e^{-i\theta_{12}}}{\sqrt{\rho_{22}}}\right)\langle d_3|d_1\rangle & 0 & |\sigma_{13} - \sigma_{23}e^{i\theta_{12}}|\rho_{33} \end{pmatrix}.$$

As a result, $\Phi(\rho_s') = \rho_s$ and $C_a^{(1/2)}(\rho_s) \leqslant C_a^{(1/2)}(\rho_s')$.

3. Finite-dimensional case

If $\sum_{j\neq i} \frac{|\rho_{ij}|}{\sqrt{\rho_{ii}\rho_{jj}}} \leqslant 1$ (for each i, j), then the duality relation is true. We denote the Kraus operators of quantum operation $\Phi \equiv \{K_{ij}\}$ $(1 \leqslant i \leqslant j \leqslant d)$ as follows:

$$K_{ij}(i<j) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & \frac{\sqrt{\rho_{ij}}}{(\rho_{ii}\rho_{jj})^{1/4}} & \dots & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \dots & \frac{\sqrt{\rho_{ji}}}{(\rho_{ii}\rho_{jj})^{1/4}} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix}_{d\times d},$$

$$K_{ii} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & \sqrt{1 - \sum_{j\neq i}\frac{|\rho_{ij}|}{\sqrt{\rho_{ii}\rho_{jj}}}} & \dots & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix}_{d\times d}.$$

Then, it is not difficult to check that Φ is an incoherent operation and, for $\rho_s' = \sum_{i,j=1}^d \sqrt{\rho_{ii}\rho_{jj}}\langle d_j|d_i\rangle|i\rangle\langle j|$, we have $\Phi(\rho_s') = \sum_{i,j=1}^d \rho_{ij}\langle d_j|d_i\rangle|i\rangle\langle j| = \rho_s$. Since $C_a^{(1/2)}$ is a coherence measure, we have

$$C_a^{(1/2)}(\rho_s) = C_a^{(1/2)}[\Phi(\rho_s')] \leqslant C_a^{(1/2)}(\rho_s'). \quad \text{(B1)}$$

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