Derivation of Maxwell-type equations for open systems

A. V. Khugaev,[1,2] G. G. Adamian,[1] and N. V. Antonenko[1,3]
[1]*Joint Institute for Nuclear Research, 141980 Dubna, Russia*
[2]*Institute of Nuclear Physics, 100214 Tashkent, Uzbekistan*
[3]*Tomsk Polytechnic University, 634050 Tomsk, Russia*

(Received 7 August 2018; published 28 September 2018)

Using equations of motion with the anisotropic dissipative term for quantum particle and quantum-mechanical commutation rules, the general Maxwell-type differential equations are derived. The direct modifications of the well-known Maxwell equations due to the medium effects (openness of the system) are discussed.

DOI: 10.1103/PhysRevA.98.032122

I. INTRODUCTION

Maxwell equations have been derived as a mathematical reflection of the already established empirical facts. In macroscopic quantum electrodynamics, they have been expressed as a relationship between electric $\mathbf{E}^0$ and magnetic $\mathbf{H}^0$ field vectors, describing an electromagnetic field. The properties of the medium (matter) are taken into account only by means of material equations. Since the latter is not always possible, such a description is not universal [1].

Using the equations of motion

$$m\ddot{x}_j = F_j^0(\mathbf{x}, \dot{\mathbf{x}}, t) \quad (1)$$

of the nonrelativistic point particle of mass m with the Cartesian position operator x_j, $j = 1, 2, 3$, and the quantum-mechanical commutation relations

$$[x_j, x_k] = 0, \quad (2)$$

$$m[x_j, \dot{x}_k] = i\hbar\delta_{jk}, \quad (3)$$

the Lorentz force law (ϵ_{jkl} is the Levi-Cività antisymmetric tensor)

$$F_j^0 = E_j^0 + \epsilon_{jkl}\dot{x}_k H_l^0 \quad (4)$$

and two Maxwell equations

$$\nabla_j H_j^0 = 0, \quad (5)$$

$$\dot{H}_j^0 + \epsilon_{jkl}\nabla_k E_l^0 = 0 \quad (6)$$

were derived in Ref. [2]. As explicitly demonstrated, $\mathbf{E}^0(\mathbf{x}, t)$ and $\mathbf{H}^0(\mathbf{x}, t)$ are the vector functions of the coordinate $\mathbf{x}$ and time t but not the velocities. As shown in Ref. [3], the particle motion in the noninertial frame with the Coriolis-like forces or in the weak gravitational field also satisfies the constraints (1)–(3). The velocity-dependent forces are not limited to electromagnetic ones, and the commutation relations (2) and (3) are also determined by the equations of motion (1) [3]. The generalization of this approach [2,3] to the relativistic case was done in Ref. [4]. The connection of the classical equations of motion and Maxwell electromagnetic equations in an elegant manner was explicitly shown in Ref. [5]. Note also the derivation of the Lorentz force law in Ref. [6] based on the commutation rules.

References [2–6] did not taken into account that systems by their nature are open systems [1,7–11]. In this case, it is necessary to explicitly take into consideration the environmental or medium effects in the equations of motion. In the present paper, we try to solve this problem. In addition to the Lorenz force F_j^0, the motion of the test particle (open system) is under the influence of an additional force f_j induced by the physical medium. This force effectively describes the direct interaction of the moving particle with the medium and has experimental justification.

II. GENERAL CONSIDERATION

Let us consider the general equations of motion

$$m\ddot{x}_j = F_j(\mathbf{x}, \dot{\mathbf{x}}, t) - \lambda_{jk}(\mathbf{x})\dot{x}_k, \quad (7)$$

where the motion of the test particle is under the influence of an additional dissipative type force $f_j = -\lambda_{jk}(\mathbf{x})\dot{x}_k$ induced by physical medium. Here $\lambda_{jk}(\mathbf{x}) = \lambda_{(jk)}(\mathbf{x}) + \lambda_{[jk]}(\mathbf{x})$ is the general tensor field, where $\lambda_{(jk)}$ and $\lambda_{[jk]}$ are the symmetrized and antisymmetrized tensors, respectively. Note that these tensors depend on the coordinate $\mathbf{x}$. In general, Eq. (7) should also include the fluctuating forces, leading to a stochastic dynamics. For simplicity, we consider the average dynamics of the test particle.

Taking the total time derivative of Eq. (3), we obtain

$$[\dot{x}_j, \dot{x}_k] + [x_j, \ddot{x}_k] = 0. \quad (8)$$

Substituting (7) in Eq. (8), we find

$$[x_j, F_k] = \frac{i\hbar}{m}\left(\lambda_{(jk)} + \left\{\frac{im^2}{\hbar}[\dot{x}_j, \dot{x}_k] - \lambda_{[jk]}\right\}\right) = \frac{i\hbar}{m}(\lambda_{(jk)} - \epsilon_{jkl}H_l). \quad (9)$$

Because the term in curly brackets in Eq. (9) is antisymmetrized with respect to $j \rightarrow k$ and $k \rightarrow j$, we introduce, without loss of generality, the vector magnetic field

$$H_l = -\frac{1}{2}\epsilon_{jkl}\left(\frac{im^2}{\hbar}[\dot{x}_j, \dot{x}_k] - \lambda_{[jk]}\right) = H_l^0 + \frac{1}{2}\epsilon_{jkl}\lambda_{[jk]}. \quad (10)$$

Expressing $[\dot{x}_j, \dot{x}_k]$ from Eq. (10) and substituting into the Jacobi identity

$$[x_l, [\dot{x}_j, \dot{x}_k]] = 0, \tag{11}$$

we derive

$$\epsilon_{jkp}[x_l, H_p] = [x_l, \lambda_{[jk]}] = 0 \rightarrow [x_l, H_p] = 0, \tag{12}$$

which means that H_p only depends on $\mathbf{x}$ and t. The Jacobi identity

$$\epsilon_{jkl}[\dot{x}_l, [\dot{x}_j, \dot{x}_k]] = 0 \tag{13}$$

together with Eq. (10) implies

$$[\dot{x}_l, H_l] = \tfrac{1}{2}\epsilon_{jkl}[\dot{x}_l, \lambda_{[jk]}], \tag{14}$$

which is equivalent to

$$\text{div}\mathbf{H} = \tfrac{1}{2}\epsilon_{jkl}\nabla_l\lambda_{[jk]}. \tag{15}$$

One can assign the role of magnetic charge density ρ_m to the term on the right-hand side of Eq. (15). Thus, the magnetic charge density ρ_m is a possible source of the static magnetic field by analogy with the electric charge density ρ_e as a source of the static electric field. Here the magnetic charge (Dirac monopole [12]) is related to the heterogeneous antisymmetric tensor field $\lambda_{[jk]}$. When $\lambda_{[jk]}$ is a homogeneous field and, correspondingly, $\nabla_l\lambda_{[jk]} = 0$, we again obtain Eq. (5). In magnets, the magnetic charge is related to the magnetization $\mathbf{I}$: $\rho_m = -\text{div}\mathbf{I}$ [13]. The well-known Maxwell equations assert that $\rho_m = 0$ and there are no other sources of magnetic fields, except electric currents.

Substituting now the vector field

$$F_j = E_j + \epsilon_{jkl}\dot{x}_k H_l \tag{16}$$

into Eq. (9), we obtain

$$[x_j, E_k] = \frac{i\hbar}{m}\lambda_{(jk)}. \tag{17}$$

The vector electric field

$$E_j = E_j^0 + \lambda_{(jk)}\dot{x}_k \tag{18}$$

follows from Eqs. (2) and (17). As seen from Eqs. (10) and (18), the vector electric and magnetic fields contain the individual property of the medium, resistance, or conductivity. At $\mathbf{E}^0 = 0$, Eq. (18) is reduced to the material-type equation (e.g., the Ohmic law in the case of electromagnetic forces). Using Eq. (18), one can derive the Maxwell type equation

$$\text{div}\mathbf{E} = \rho_e + \nabla_j\lambda_{(jk)}\dot{x}_k, \tag{19}$$

where $\rho_e = \text{div}\mathbf{E^0}$ is an analog of the electric charge density.

Taking the total time derivative of the vector magnetic field $\mathbf{H}$ (10),

$$\frac{\partial H_l}{\partial t} + \dot{x}_m\frac{\partial H_l}{\partial x_m} = \frac{1}{2}\epsilon_{jkl}\dot{x}_m\nabla_m\lambda_{[jk]} - \frac{im^2}{\hbar}\epsilon_{jkl}[\ddot{x}_j, \dot{x}_k], \tag{20}$$

and making some tedious but simple algebra, we obtain the Maxwell-type equation

$$\frac{\partial H_l}{\partial t} = -\epsilon_{lkj}\frac{\partial E_j}{\partial x_k} - \frac{1}{m}\{\lambda_{jj}H_l - \lambda_{ql}H_q - \lambda_{[lk]}H_k - \epsilon_{jkl}\lambda_{jp}\lambda_{[pk]}\} - \epsilon_{jkl}\dot{x}_p\nabla_k\lambda_{jp} + \frac{1}{2}\{\epsilon_{jkl}\dot{x}_m\nabla_m + \epsilon_{jkp}\dot{x}_l\nabla_p\}\lambda_{[jk]}. \tag{21}$$

This equation is the generalized law of electromagnetic induction and more complicated than the corresponding Maxwell equation. In the case of $\lambda_{jk} \equiv 0$, we rederive all the results of Refs. [2,3]. If λ_{jk} are constants, Eq. (21) is transformed into

$$\frac{\partial H_l}{\partial t} = -\epsilon_{lkj}\frac{\partial E_j}{\partial x_k} - \frac{1}{m}\{\lambda_{jj}H_l - \lambda_{ql}H_q - \lambda_{[lk]}H_k - \epsilon_{jkl}\lambda_{jp}\lambda_{[pk]}\}. \tag{22}$$

In the particular case of the symmetric tensor field $\lambda_{jk} = \lambda(\mathbf{x})\delta_{jk}$, Eq. (21) is simplified as

$$\frac{\partial H_l}{\partial t} = -\epsilon_{lkj}\frac{\partial E_j}{\partial x_k} - \frac{2\lambda}{m}H_l - \epsilon_{ljk}\dot{x}_j\nabla_k\lambda \tag{23}$$

or

$$\frac{\partial \mathbf{H}}{\partial t} = -\text{rot}\mathbf{E} - \frac{2\lambda}{m}\mathbf{H} - \dot{\mathbf{x}} \times \nabla\lambda. \tag{24}$$

At constant λ, we have

$$\frac{\partial \mathbf{H}}{\partial t} = -\text{rot}\mathbf{E} - \frac{2\lambda}{m}\mathbf{H}, \tag{25}$$

which is similar to one for the ferromagnetic materials, in which λ is proportional to the magnetic conductivity (inverse to the magnetic viscosity) [13].

Taking total time derivative of vector electric field $\mathbf{E}$ (18),

$$\frac{\partial E_l}{\partial t} + \dot{x}_m\frac{\partial E_l}{\partial x_m} = \frac{\partial E_l^0}{\partial t} + \dot{x}_m\frac{\partial E_l^0}{\partial x_m} + \dot{x}_i\nabla_i\lambda_{(lk)} + \lambda_{(lk)}\ddot{x}_k, \tag{26}$$

we obtain the Maxwell-type equation

$$\frac{\partial E_l}{\partial t} = \epsilon_{lkj}\frac{\partial H_j}{\partial x_k} - j_l + \dot{x}_i\nabla_i\lambda_{[lk]}\dot{x}_k + \frac{\lambda_{(lk)}}{m}\{E_k + \epsilon_{lkj}\dot{x}_l H_j - \lambda_{kl}\dot{x}_l\}. \tag{27}$$

Here we employ that

$$\frac{\partial E_l^0}{\partial t} = \epsilon_{lkj}\frac{\partial H_j^0}{\partial x_k} - j_l, \tag{28}$$

where $\mathbf{j}$ is an analog of the electric current density. Equation (27) is the generalized Ampère law.

Thus, in the general case for electromagnetic forces, instead of the Maxwell and material equations, more complicated equations should be used: a closed system of coupled equations of motion (7) (or, more generally, the quantum Langevin equations or the corresponding quantum diffusion equation by also taking into account the fluctuations) for the charge particles and field equations (15), (19), (21), and (27). Note that our definitions for H and E are different from the standard definitions. For example, such an approach can be used to describe electromagnetic processes in a fully ionized plasma.

III. CONCLUSION

Employing equations of motion for the test quantum particle and quantum-mechanical commutation rules, we derived the Maxwell-type differential equations for forces F_j of any nature. Because these equations contain the influence of the medium (openness of the system), they are more complicated than the usual Maxwell equations for electromagnetic forces. As shown, for the strongly inhomogeneous anisotropic systems, the effective magnetic charge appears in Eq. (15). Thus, the obtained equations acquire a more symmetrical form, where there are magnetic and electric charges. The magnetic charge is related to the inhomogeneous antisymmetric tensor field $\lambda_{[jk]}$. It should be noted that a magnetic monopole (magnetic charge) was sought in some materials possessing strongly anisotropic crystal structure (e.g., the nematic materials) and, accordingly, possessing strongly anisotropic dissipative properties. We also found that the influence of a homogeneous and isotropic medium leads to the field equation (25) for the ferromagnetic materials with magnetic conductivity.

Using the nonrelativistic classical equations of motion, corresponding to Eq. (7), and the usual commutator–Poisson bracket correspondence, one can deduce the same results for the open classical systems. It is also possible to assume that the forces in nature are united only by the general form of the equations of motion which contain the "electric" and "magnetic" components.

ACKNOWLEDGMENTS

A.V.K. is grateful to Bogoliubov Laboratory of Theoretical Physics (BLTP) of the Joint Institute for Nuclear Research (Dubna). This work was partially supported by the Russian Foundation for Basic Research (Moscow) and DFG (Bonn). The IN2P3 (France)–JINR (Dubna) Cooperation Programme is gratefully acknowledged. The work of N.V.A. was supported by Tomsk Polytechnic University Competitiveness Enhancement Program grant.

[1] Y. L. Klimontovich, *Statistical Theory of Open Systems* (Kluwer, Dordrecht, 1995).
[2] F. J. Dyson, Am. J. Phys. **58**, 209 (1990).
[3] R. J. Hughes, Am. J. Phys. **60**, 301 (1991).
[4] S. Tanimura, Ann. Phys. (NY) **220**, 229 (1992).
[5] P. Singh and N. Dadhich, Int. J. Mod. Phys. A **16**, 1237 (2001); Mod. Phys. Lett. A **16**, 83 (2001).
[6] A. Berard, Y. Grandati, and H. Mohbrach, Phys. Lett. A **254**, 133 (1999).
[7] V. V. Dodonov and V. I. Man'ko, in *Proceedings of the Lebedev Physical Institute of Sciences*, edited by A. A. Komar (Nova Science, Commack, 1987), Vol. 167.
[8] K. Lindenberg and B. J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems* (VCH, New York, 1990).
[9] A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu, and W. Scheid, Int. J. Mod. Phys. E **03**, 635 (1994).
[10] U. Weiss, *Quantum Dissipative Systems* (Wold Scientific, Singapore, 1999).
[11] V. V. Sargsyan, Z. Kanokov, G. G. Adamian, and N. V. Antonenko, Phys. Part. Nuclei **41**, 175 (2010).
[12] P. A. M. Dirac, Proc. R. Soc. London Ser. A **133**, 60 (1931); Phys. Rev. **74**, 817 (1948).
[13] S. V. Vonsowsky, *Magnetism* (Nauka, Moscow, 1971).