Bell measurement ruling out supraquantum correlations

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(Received 12 April 2018; published 24 September 2018)

The so-called bipartite nonsignaling boxes are systems whose statistics is constrained solely by the principle of no instantaneous signaling between distant locations. Such systems can exhibit much stronger correlations than those admitted by quantum mechanics. Inspired by the quantum logic approach of Tylec and Kuś [J. Phys. A: Math. Theor. **48**, 505303 (2015)], we consider nonsignaling boxes with three inputs per party and extend the set of measurements with just a single *global* measurement—one that mimics a quantum two-party Bell measurement. We then show that this seemingly mild extension completely rules out supraquantum correlations: the resulting system admits precisely quantum-mechanical correlations of two qubits. We also consider nonmaximally entangled measurements, obtaining interpolation between quantum and full nonsignaling theory. Our study paves the way to a general program of amending nonsignaling theories with some measurements inherited from quantum mechanics, leading to various interpolations between nonsignaling boxes and quantum mechanics.

DOI: 10.1103/PhysRevA.98.032117

I. INTRODUCTION

The idea of nonsignaling boxes introduced in Ref. [1] has become very fruitful for several purposes. First, the nonsignaling boxes have found an application in device-independent cryptography [2], based solely on the assumption of no instantaneous signaling and violation of Bell inequalities [3]. In particular, randomness amplification and expansion protocol have been proposed [4,5], whose verification does not require any knowledge of quantum mechanics but can be done based solely on the statistical behavior of the devices. On the other hand, analysis of nonsignaling boxes leads to a better understanding of the capabilities and limitations of quantum theory itself: the set of quantum-mechanical states forms a convex body situated between two polytopes: the classical polytope of Kolmogorovian probability distributions and the larger polytope of all nonsignaling boxes. The concept of nonsignaling boxes has lead to a vast field of so-called general probabilistic theories (GPTs) [6–8].

Recently, the relations between GPTs and quantum logic were analyzed [9]. The authors use the framework of quantum logic to construct a logic of propositions of two-party nonsignaling boxes. They build the logic from propositions describing a single party and prove that the logic indeed describes spatially separated subsystems.

So far within the subject of GPT, not much has been done regarding joint measurements on composite systems. Boxes with bipartite measurements were considered in Ref. [10] and were used to construct examples of theories violating the "no-hypersignaling principle" formulated therein. These measurements were based on extremal points of the nonsignaling polytope. The resulting models either exhibit solely classical correlations (when measurements corresponding to all extremal points are added) or exhibit maximally nonlocal correlations—those violating the Tsirelson bound [11]. In this context, an important challenge is to build models that interpolate between the two extremes. For a single system, an important example of such interpolation is a family of polygonic models [12], some of them violating the Holevo bound. A bipartite models based on polygonic local systems was also considered [13], some of them violating the Tsirelson bound. Yet, the joint measurements possible for those systems have not been analyzed.

In this paper we want to to avoid the binary situation: classical or full nonsignaling, so we need more sophisticated measurements than ones used in Ref. [10]. To this end we take inspiration from the quantum logic approach to nonsignaling boxes of Ref. [9]. We aim to analyze the effect of enriching the initial model—a standard nonsignaling box, which admits just product measurements—with an *entangled measurement inherited from quantum mechanics*.

The basic global measurement in quantum mechanics one may think of is clearly the Bell measurement [14]. Surprisingly, we obtain that adding just this single measurement severely constrains the set of possible states. Namely, we show that the *nonsignaling box with a Bell measurement exhibits no supraquantum correlations*. It actually reproduces exactly all quantum correlations. We do it by showing that existence of Bell measurement, combined with natural assumption, that product of local states is a legitimate joint state, imposes that local states form a ball, i.e., it is the same as the set of states of qubit. Then we use the result of Ref. [15] where it is shown that bipartite systems which are locally quantum and nonsignaling admit only quantum correlations.

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We also consider nonmaximally entangled measurements, and obtain interpolation between local systems being balls (like in quantum mechanics) and cubes (i.e., completely unrestricted local systems).

II. MODEL

We consider system $\mathcal{S}_{AB}$ composed of two elementary subsystems $\mathcal{S}_A$, $\mathcal{S}_B$. State spaces of elementary systems $\mathcal{S}_A$ and $\mathcal{S}_B$ are identical.

The elementary system may be measured by means of one of three dichotomic measurements X, Y, Z. The measurements are not compatible, i.e., they cannot be measured together. In this sense they mimic Pauli measurements for quantum system. However, at this step we do not put any constraints on the measurements outcomes probabilities beside standard positivity and normalization constraints. In particular, there are no uncertainty constraints for elementary system.

The state of the elementary system $\mathcal{S}_A$ is described by probabilities of measurement outcomes: $p(a|x_A)$, where $x_A \in \{X, Y, Z\}$ enumerates measurements and a enumerates outcomes "+," "−" (similarly for $\mathcal{S}_B$).

Now we move to the composed system $\mathcal{S}_{AB}$. Consider first standard nonsignaling bipartite boxes [1]. These boxes are described by probabilities $p(ab|x_A x_B)$ where $a \in \{+,-\}$ denote the output of measurement $x_A \in \{X, Y, Z\}$ performed on subsystem A, analogously for b and x_B. Probabilities $p(ab|x_A x_B)$ fulfill nonsignaling conditions, i.e., the probability of outcomes of measurement performed on subsystem A do not depend on the measurement performed on the subsystem B (analogously for B and A). The condition is expressed by the equation

$$\sum_b p(a, b|x_A, x_B) = \sum_b p(a, b|x_A, x'_B), \qquad (1)$$

which holds for all $a \in \{+,-\}$ and $x_A, x_B, x'_B \in \{X, Y, Z\}$.

So far this is a standard "nonsignaling box." We shall now assume that there is an additional two-party measurement which cannot be represented as a joint measurement of two local measurements. This intrinsically two-party measurement returns one of the four outcomes $k = 1, 2, 3, 4$. We will define probabilities of these outcomes by using parity relations for Bell measurements known from quantum mechanics, hence the probabilities will be denoted by $p(k|\text{Bell})$, and the measurement will be called a "Bell measurement."

In quantum mechanics we have

$$\begin{aligned} |\phi^+\rangle\langle\phi^+| + |\psi^+\rangle\langle\psi^+| &= P_A^{X,+} \otimes P_B^{X,+} + P_A^{X,-} \otimes P_B^{X,-}, \\ |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| &= P_A^{Y,+} \otimes P_B^{Y,+} + P_A^{Y,-} \otimes P_B^{Y,-}, \qquad (2) \\ |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| &= P_A^{Z,+} \otimes P_B^{Z,+} + P_A^{Z,-} \otimes P_B^{Z,-}, \end{aligned}$$

where

$$\begin{aligned} \phi_\pm &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle), \\ \psi_\pm &= \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle), \qquad (3) \end{aligned}$$

and $P_A^{X,\pm}$, $P_A^{Y,\pm}$, $P_A^{Z,\pm}$ are eigenprojectors of Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ respectively (same for B). We now impose the same relations for our joint measurement on the level of statistics:

$$\begin{aligned} p(1|\text{Bell}) + p(3|\text{Bell}) &= p(++|XX) + p(--|XX), \\ p(2|\text{Bell}) + p(3|\text{Bell}) &= p(++|YY) + p(--|YY), \qquad (4) \\ p(1|\text{Bell}) + p(2|\text{Bell}) &= p(++|ZZ) + p(--|ZZ). \end{aligned}$$

Notice that it would not make sense to impose nonsignaling conditions onto a Bell measurement because the latter is performed on the whole system. The system will be now fully described by the set of probabilities $p(ab|x_A x_B)$ and $p(k|\text{Bell})$.

It is worth mentioning that the state space of the composed system without a Bell measurement is a maximal tensor product space [16] and, with the elementary system as described above, the state space of such a composed system is a full nonsignaling polytope [17]. In the remainder of the paper we show that equipping the composed system with a Bell measurement will change this picture a lot.

State representation with probabilities $p(ab|x_A x_B)$ and $p(k|\text{Bell})$ contains 40 parameters. However, they are not independent. By using nonsignaling conditions together with normalization and relation (4), we can express the state of the composed system by using 15 free parameters: probabilities of positive outcomes for every measurement settings $p(++|x_A x_B)$ and marginal probabilities $p(+|x_A)$, $p(+|y_B)$ [due to nonsignaling condition, we can write marginal probability as $p(+|x_A) = p(++|x_A X) + p(+-|x_A X)$]. We can arrange these parameters in the form of a matrix:

$$\begin{pmatrix} p(++|XX) & \cdots & \cdots & p(+|X_B) \\ \vdots & p(++|YY) & \cdots & p(+|Y_B) \\ \vdots & \vdots & p(++|ZZ) & p(+|Z_B) \\ p(+|X_A) & p(+|Y_A) & p(+|Z_A) & 1 \end{pmatrix}. \qquad (5)$$

In particular, the state is fully determined by the statistics of local measurements satisfying therefore the local tomography principle [18,19].

We can treat Eq. (5) as $p(++|x_A, x_B)$ in a extended measurement set, i.e., $x_A, x_B \in \{X, Y, Z, I\}$, where I denotes some trivial measurement which always gets the + result: $p(+|I_A) = p(+|I_A) = p(++|I_A, I_B) = 1$.

Before we move further in analysis of state space Ω_{AB}, let us make a digression. Namely, suppose that the elementary system is equipped only with two measurements X, Y. One then finds that, in such a theory, extending the set of measurements with the Bell measurement leads to an additional free parameter. It follows from fact that we can write only the first two equations from Eq. (4). This theory does not fulfill the local tomography principle [18,19] since the state of the composed system cannot be fully described in terms of joint probabilities of local measurements, i.e., in terms of $p(ab|x_A, x_B)$. This is analogous to the difference between complex and real quantum mechanics where local tomography is a crucial piece [20].

III. CONSTRAINTS FOR CORRELATIONS IMPOSED BY EXISTENCE OF BELL MEASUREMENT

In this section we show that correlations exhibited by boxes admitting Bell measurement are exactly the quantum ones. To this end we study how conditions imposed by the existence of a Bell measurement impacts the state space of the elementary system. We shall assume that two natural conditions hold:

(i) the sets of states of local systems are the same, i.e., $\Omega_A = \Omega_B$;

(ii) all product states are allowed, i.e., $\Omega_A \otimes \Omega_B \subset \Omega_{AB}$.

We now show that non-negativity of $p(k|\text{Bell})$ together with these assumptions leads to the equivalence of elementary-system state space with a Bloch ball. Then, knowing that elementary state space is quantum and the composed system is nonsignaling, we can directly use results from Ref. [15] to obtain that all correlations in bipartite system are quantum correlations.

To proceed, consider the product of two identical states $\omega_{AB} = \omega_A \otimes \omega_B$ which by assumption (ii) is allowed. Denote marginals by

$$\begin{aligned} p(+|x_A = X) &= p(+|x_B = X) = p_X,\\ p(+|x_A = Y) &= p(+|x_B = Y) = p_Y,\\ p(+|x_A = Z) &= p(+|x_B = Z) = p_Z, \end{aligned} \tag{6}$$

and consider the probability $p(4|\text{Bell})$ for that state. Then, from simple algebra, we get

$$\begin{aligned} &p(4|\text{Bell})\\ &= \tfrac{1}{2}[p(+|x_A = X) + p(+|x_A = Y) + p_A(+|x_A = Z)\\ &\quad + p(+|x_B = X) + p(+|x_B = Y) + p_B(+|x_B = Z)\\ &\quad - p(++|XX) - p(++|YY) - p(++|ZZ) - 1], \end{aligned} \tag{7}$$

and for state ω_{AB}:

$$p(4|\text{Bell}) = p_X + p_Y + p_Z - p_X^2 - p_Y^2 - p_Z^2 - \tfrac{1}{2}. \tag{8}$$

We can rewrite the above expression together with the positivity condition for $p(4|\text{Bell})$ as

$$\left(p_X - \tfrac{1}{2}\right)^2 + \left(p_Y - \tfrac{1}{2}\right)^2 + \left(p_Z - \tfrac{1}{2}\right)^2 \leqslant \left(\tfrac{1}{2}\right)^2. \tag{9}$$

This formula constrains the state space of elementary system, and we see that the condition is equivalent to the Bloch ball for averages of observables X, Y, X. In particular, the constraints can be interpreted as an uncertainty relation expressed in terms of probability of measurement outcome.

Now we show that these constraints are tight, i.e., that all the products of states fulfilling Eq. (9) give positive values for the outcome probabilities of measurement. Of course, for products of local measurements, the product states give positive probabilities by definition. So we need to check whether they give positive values of probabilities of outcomes just for the Bell measurement.

To this end we rewrite the positivity conditions for $p(k|\text{Bell})$ in moments representation [i.e., mean values of local measurement, e.g., $m_X^A = \frac{1}{2}[p(+|x_A = X) - p(-|x_A = X)] = p(+|x_A = X) - \frac{1}{2}]$. When we arrange moments in the form of the vector $m = (m_X, m_Y, m_Z, 1)$, then the positivity condition take the form

$$0 \leqslant \langle m^A|T_k|m^B\rangle, \tag{10}$$

where T_k are diagonal matrices representing outcomes k given by

$$\begin{aligned} T_1 &= \tfrac{1}{4}\text{diag}(1, -1, 1, 1),\\ T_2 &= \tfrac{1}{4}\text{diag}(-1, 1, 1, 1),\\ T_3 &= \tfrac{1}{4}\text{diag}(1, 1, -1, 1),\\ T_4 &= \tfrac{1}{4}\text{diag}(-1, -1, -1, 1), \end{aligned} \tag{11}$$

where diag$(\dots)$ denotes diagonal matrix with the given entries.

The relation (10) comes solely from assumption (4) and the definition of moments, and does not use quantum formalism. To see this, first observe that, for a product state, we have $p(++|x_A, x_B) = p(+|x_A)p(+|x_B)$ [here we use the extended measurements set $\{X, Y, Z, I\}$, cf. Eq. (5)]. Then express $p(k|\text{Bell})$ as a linear combination of $p(++|x_A, x_B)$ [cf. (4)]:

$$\begin{aligned} p(k|\text{Bell}) &= \sum_{i,j\in\{X,Y,Z,I\}} M_k^{i,j}\, p(++|x_A = i, x_B = j)\\ &= \sum_{i,j\in\{X,Y,Z,I\}} M_k^{i,j}\, p(+|x_A = i)p(+|x_B = j), \end{aligned} \tag{12}$$

where M_k represents $p(k|\text{Bell})$ in terms of $p(++|x_A, x_B)$. Putting probabilities as vectors $p(+|x_A) = v_A$, $p(+|x_B) = v_B$, we can write the above equation in a matrix form: $p(k|\text{Bell}) = v_A^T M_k v_B$. Notice that vectors v_A and v_B have form (.,.,.,1). Moments representation m is related with probability representation v by the relation $v = Cm$, where matrix C has the form

$$\begin{pmatrix} 1 & 0 & 0 & 1/2\\ 0 & 1 & 0 & 1/2\\ 0 & 0 & 1 & 1/2\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

This leads to

$$\begin{aligned} p(k|\text{Bell}) &= v_A^T M_k v_B\\ &= m_A^T C^T M_k C m_B\\ &= m_A T_k m_B, \end{aligned} \tag{14}$$

where $T_k = C^T M_k C$.

Formula (10) can be unwind to

$$k = 1: -m_X^A m_X^B + m_Y^A m_Y^B - m_Z^A m_Z^B \leqslant r^2, \tag{15}$$

$$k = 2: m_X^A m_X^B - m_Y^A m_Y^B - m_Z^A m_Z^B \leqslant r^2, \tag{16}$$

$$k = 3: -m_X^A m_X^B - m_Y^A m_Y^B + m_Z^A m_Z^B \leqslant r^2, \tag{17}$$

$$k = 4: m_X^A m_X^B + m_Y^A m_Y^B + m_Z^A m_Z^B \leqslant r^2, \tag{18}$$

where $r = \frac{1}{2}$. First observe that the left-hand side of Eq. (18) has the form of a scalar product between vectors (m_X^A, m_Y^A, m_Z^A) and (m_X^B, m_Y^B, m_Z^B). We know from Eq. (9) that the norm of these vectors is bounded by r, therefore Eq. (18) holds for all states from the ball given by Eq. (9). The other inequalities can be easily translated to the form of a scalar product: because of symmetry of state space we can always replace the state on

$\mathcal{S}_B$ by the state with the appropriate observable flipped. We can conclude that all states given by Eq. (9) fulfill positivity constraints, therefore Eq. (9) defines the state space of the local system and in fact is a Bloch ball.

As said, knowing that an elementary state space is quantum and a composed system is nonsignaling, we can directly use results from Ref. [15] to say that all correlations in bipartite systems are quantum correlations.

IV. ADDING NEW MEASUREMENTS: QUANTUM LOGIC APPROACH VERSUS LOCAL TOMOGRAPHIC APPROACH

The way we approached the definition of a Bell measurement for a nonsignaling box was to enforce the relation between the statistics of the new measurement and the statistics of the local measurements to be the same as the relation between the statistics of Pauli measurements and Bell measurements in quantum mechanics. The inspiration was taken from quantum logic approach to nonsignaling boxes.

Bell measurement and quantum logic approach. One starts from the logic structure of nonsignaling boxes. The paper [9] provides the set of valid propositions for nonsignaling boxes. An example of a valid proposition is "the system is in state $++$ of measurement XX." Moreover we know that the proposition "the system is in state $++$ or $--$ of measurement XX" is also valid. These two propositions refers to probabilities $p(++|XX)$ and $p(++|XX)+p(--|XX)$ for the given boxes. In contrast, the proposition "the system is in state $++$ of measurement XX or $++$ of measurement ZZ" is not valid. This works in analogy to the algebra of orthogonal projectors in quantum mechanics. Now one observes that some propositions in quantum mechanics may be expressed in several ways: e.g., parity XX may be expressed as $P_A^{X,+}\otimes P_B^{X,+}+P_A^{X,-}\otimes P_B^{X,-}$ or $|\phi^+\rangle\langle\phi^+|+|\psi^+\rangle\langle\psi^+|$. We require the same type of relations to hold in our model. That leads to Eq. (4).

Adding measurements via the local tomographic approach. The considered definition of a Bell measurement can be seen as an instance of a more general way of inheriting joint measurements from quantum theory that is not covered by the quantum logic approach—the one based on local tomography. Namely, suppose that we consider some measurement from quantum mechanics and want to impose it onto a nonsignaling box. In quantum mechanics, due to local tomography, the statistics of the local observables determines the statistics of all measurements. We can thus define a new measurement by requiring that the statistics of its outcomes be determined by the statistics of local observables through the quantum-mechanical relation. It is then possible to extend nonsignaling boxes with the analog of quantum measurement in a nonmaximally entangled basis.

V. NOISY BELL MEASUREMENT AND NONMAXIMALLY ENTANGLED MEASUREMENT

Here we present how a noisy Bell measurement as well as a measurement in a nonmaximally entangled basis modify the local state space. We use the local tomography approach to define these measurements for nonsignaling boxes.

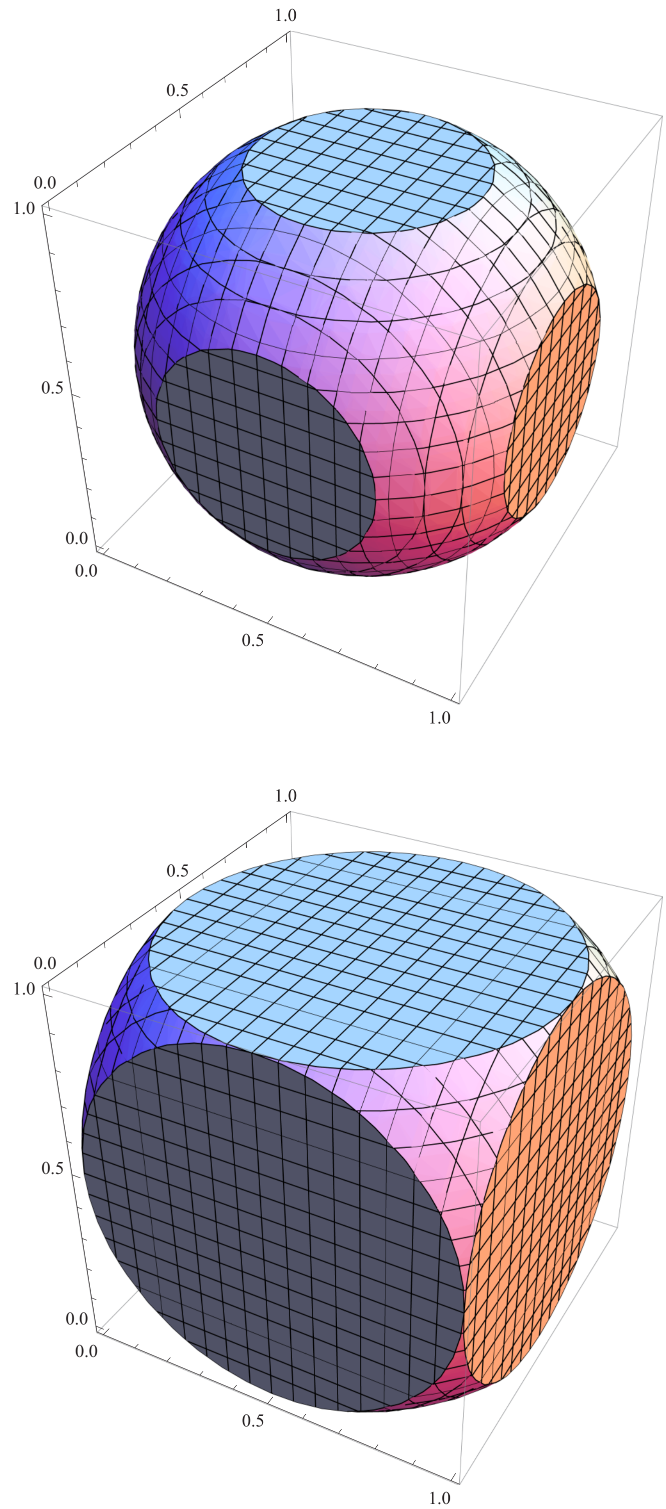

FIG. 1. State space for noisy Bell measurement for $\lambda = 1/4$ (upper) and $\lambda = 1/2$ (lower). The axes OX, OY, OZ refer to p_X, p_Y, p_Z, respectively.

Noisy Bell measurement. We consider here a measurement inherited from a positive-operator-value measure with elements

$$(1-\lambda)|\phi_\pm\rangle\langle\phi_\pm| + \lambda I/4,$$
$$(1-\lambda)|\psi_\pm\rangle\langle\psi_\pm| + \lambda I/4. \tag{19}$$

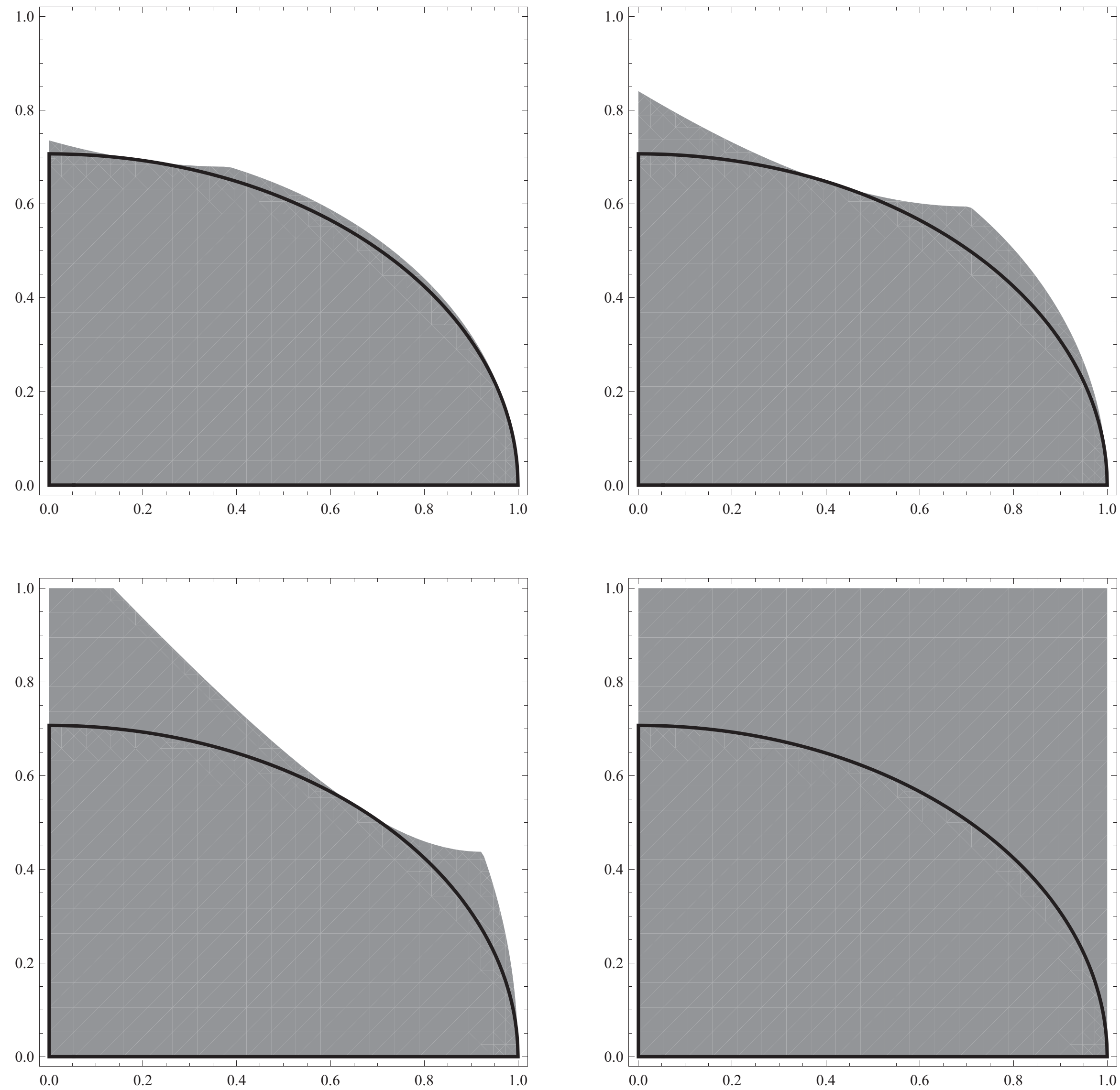

FIG. 2. Allowed values of l (OY) and h (OX) for $\alpha = \pi/16, 2\pi/16, 3\pi/16, 4\pi/16$. The thick black line bounds the region permitted in quantum mechanics. We can observe interpolation between quantum and unrestricted systems.

The probabilities $p(k|\widetilde{\text{Bell}})$ of the measurements are related to the probabilities $p(k|\text{Bell})$ as follows:

$$p(k|\widetilde{\text{Bell}}) = (1-\lambda)p(k|\text{Bell}) + \lambda/4, \qquad (20)$$

and can be expressed in the terms of moments by Eqs. (15)–(18) with $r = \frac{1}{2\sqrt{1-\lambda}}$. The argumentation analogous to the one present in the case of a Bell measurement can be used here to show that the state space of the elementary system (in terms of probabilities) is 1/2 centered ball with $r = \frac{1}{2\sqrt{1-\lambda}}$ restricted to the box of $0 \leqslant p_X, p_Y, p_Z \leqslant 1$ (see Fig. 1).

Nonmaximally entangled measurement. Here we take a nonmaximally entangled basis parametrized by real a and b:

$$\tilde{\phi}_\pm = \frac{1}{\sqrt{2}}(a|0\rangle|0\rangle \pm b|1\rangle|1\rangle),$$
$$\tilde{\psi}_\pm = \frac{1}{\sqrt{2}}(a|0\rangle|1\rangle \pm b|1\rangle|0\rangle). \qquad (21)$$

Taking $a = 0, b = 1$ leads to product basis and $a = b = 1/\sqrt{2}$ leads to the standard Bell basis.

We express positivity conditions in terms of formula (10) (outcome 1 refers to $\tilde{\phi}_+$, 2 to $\tilde{\phi}_-$, etc.; note that a and b are real parameters):

$$T_1 = \frac{1}{4}\begin{pmatrix} 2ab & 0 & 0 & 0 \\ 0 & -2ab & 0 & 0 \\ 0 & 0 & 1 & a^2-b^2 \\ 0 & 0 & a^2-b^2 & 1 \end{pmatrix}, \qquad (22)$$

$$T_2 = \frac{1}{4}\begin{pmatrix} -2ab & 0 & 0 & 0 \\ 0 & 2ab & 0 & 0 \\ 0 & 0 & 1 & a^2-b^2 \\ 0 & 0 & a^2-b^2 & 1 \end{pmatrix}, \qquad (23)$$

$$T_3 = \frac{1}{4}\begin{pmatrix} 2ab & 0 & 0 & 0 \\ 0 & 2ab & 0 & 0 \\ 0 & 0 & -1 & a^2-b^2 \\ 0 & 0 & -a^2+b^2 & 1 \end{pmatrix}, \quad (24)$$

$$T_4 = \frac{1}{4}\begin{pmatrix} -2ab & 0 & 0 & 0 \\ 0 & -2ab & 0 & 0 \\ 0 & 0 & -1 & a^2-b^2 \\ 0 & 0 & -a^2+b^2 & 1 \end{pmatrix}. \quad (25)$$

In the following part we will analyze state space for particular basis with $a = \sin(\pi/4 + \alpha)$, $b = \cos(\pi/4 + \alpha)$. For that parametrization we can write Eq. (10) as [cf. Eqs. (15)–(18)]

$$\begin{aligned} &-m_Z^A m_Z^B + \left(-m_X^A m_X^B + m_Y^A m_Y^B\right)\cos(2\alpha) \\ &\quad - \left(m_Z^A + m_Z^B\right)\sin(2\alpha) \leqslant 1, \\ &-m_Z^A m_Z^B + \left(m_X^A m_X^B - m_Y^A m_Y^B\right)\cos(2\alpha) \\ &\quad - \left(m_Z^A + m_Z^B\right)\sin(2\alpha) \leqslant 1, \\ &m_Z^A m_Z^B - \left(m_X^A m_X^B + m_Y^A m_Y^B\right)\cos(2\alpha) \\ &\quad - \left(m_Z^A - m_Z^B\right)\sin(2\alpha) \leqslant 1, \\ &m_Z^A m_Z^B + \left(m_X^A m_X^B + m_Y^A m_Y^B\right)\cos(2\alpha) \\ &\quad - \left(m_Z^A - m_Z^B\right)\sin(2\alpha) \leqslant 1. \end{aligned} \quad (26)$$

It is hard to obtain the full state space for given positivity conditions. Moreover there may be many inequivalent states spaces. Here we are interested in interpolation between quantum and unrestricted systems. For this reason we bound the state space of a single system from inside by a cube in moments representation with vertices $(\pm l, \pm l, \pm h)$. Because of linearity it is enough to check if the vertices fulfill the positivity conditions. For a Bell basis ($\alpha = 0$ which leads to $a = b = 1/\sqrt{2}$) we get the condition

$$2l^2 + h^2 \leqslant 1. \quad (27)$$

In Fig. 2 we present the permitted values of l and h for different parameters α.

VI. CONCLUDING REMARKS

First of all we would like to stress here that our approach is operational. We assume some relations between statistics, such as Eq. (4). This is an operational relation, which can be tested in experiment: having an ensemble of identically prepared systems, one performs a Bell measurement on one part of the ensemble and estimates $p(k|\text{Bell})$, then one performs product measurements on the other part of the ensemble, obtaining $p(++|XX)$, $p(++|YY)$, and $p(++|ZZ)$.

Our model of nonsignaling boxes admitting Bell measurements or nonmaximally entangled measurements is just an example of constraining the nonsignaling theory by amending it by quantum-inherited joint measurements. The results encourage us to study other amendments, and checking their properties. In particular, it is worth examining multipartite systems or systems with nondichotomic observables where the quantumness of local systems no longer determines the correlations [21]. Another route is to consider more general parity measurements, e.g., with more outcomes than just four, in place of Bell measurements. Finally, it would be interesting to perform a more detailed study of the correlations exhibited by nonsignaling systems with nonmaximally entangled measurements. Such an analysis involves a highly nonlinear problem, which requires further investigation.

ACKNOWLEDGMENTS

Ł.Cz. and M.H. are supported by the John Templeton Foundation, through Grant No. 56033.

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