Quasilinear approach to ray tracing in weakly turbulent, randomly fluctuating media

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Ray propagation in weakly turbulent media is described by means of a quasilinear (QL) approach in which the dispersion relation and the ray equations are expanded up to, and including, second-order terms in the medium and ray fluctuations, leading to equations for the ensemble-averaged ray and its root-mean-square (rms) spreading. An important feature of the QL formalism is that the average ray does not coincide with the zero-order, unperturbed ray but may exhibit a drift with respect to the latter that is governed by the mean squared fluctuations. The theory is complete in that equations can be set for all quantities necessary to compute the ray trajectory and the rms spreading along its path, yet they obey an infinite downward recurrence in which equations involving lower-order derivatives of the medium fluctuations are recursively generated by the subsequent higher-order derivative, and which must thus be truncated for practical purposes. Using as examples the propagation of rays in homogeneous media with fluctuations arising from the presence of either a single random mode or a multimode isotropic turbulent spectrum, the QL formalism is validated against Monte Carlo (MC) calculations and, whenever possible, its numerical implementation is verified by comparison with analytical predictions. Choosing 4% both for the level of fluctuations and for the maximum ratio between the wavelengths of the propagating ray and of the turbulent modes, so as to remain within the validity of the second-order expansion in the random perturbations and of the eikonal approximation, the overall agreement between QL and MC results is fairly good, particularly for quantities such as the distance traveled by the average ray, its perpendicular rms spread, and the averages of the wave-vector components.

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I. INTRODUCTION

The propagation of electromagnetic and sound waves in turbulent media, where environmental quantities (such as the density) fluctuate randomly, has attracted considerable interest for more than half a century (at least since attempts to understand scintillation and the twinkling of stars) and has always counted on the geometrical optics (or acoustics) approximation, also known as ray tracing, as one of its most helpful and dependable workhorses [1–14]. Indeed, the use of geometrical ray (or eikonal) theory in random media has spanned a wide variety of fields and applications, ranging from the general theory of light, sound, or radio-wave propagation in the atmosphere or in the ocean [1–5,10–13] to more practical problems such as interpreting acoustical travel times and amplitudes in the framework of harmonic sound transmission, noise control, and sonic detection schemes and how they may be affected by sound dispersion in the ocean or in the atmosphere [15], of seismic analysis and ocean or solid-earth tomography [16–18], or of ultrasonic techniques for nondestructive defect evaluation in solids or telemetry of immersed objects in liquids [19]; understanding pulsar or GPS scintillation as radio signals propagate across thin phase-changing layers (or screens) in the interstellar medium or in the ionosphere [13,20,21]; comparing the intensity distributions of rays and waves in the semiclassical limit when studying the formation of so-called rogue (or freak) waves [22,23]; and describing the scattering of radio-frequency or millimeter waves by density or magnetic fluctuations as they propagate in a plasma [24–28]. In most of these works, when a numerical treatment is performed, the effect of the fluctuating field (say, the density, velocity, or index of refraction) is simulated by randomly sampling some probability distribution that is supposed to mimic the properties of the turbulent medium in which propagation is taking place [basically, by employing Monte Carlo (MC) or similar techniques] [15–28].

As is well known, and if the outcome of the calculation is not to be plagued by noise, any approach relying on the numerical mimicking of a fluctuating medium (or of its consequences on wave propagation) demands the realization of very large sets of data. Hence, an alternative, so-called statistical approach has been proposed to trace rays in random media by writing the fluctuating field (in this case, the density) as the sum of a nonfluctuating, unperturbed background plus a randomly varying component with a known spectrum, the ray equations then being properly averaged over the fluctuations to yield equations describing both the propagation of an ensemble-averaged ray and its root-mean-square (rms) angular spreading [26]. The purpose of this article is to present a different, although similar approach based on the Hamiltonian form of the ray equations [29–31], making it eventually better suited to treat more complex dispersion relations (other than simple optical or laser dispersion [26]) and geometries (as with ray tracing in tokamaks [24,25,27,28]). So, in Sec. II the general formalism is developed and discussed, in Sec. III test cases

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and benchmarks are presented, and in Sec. IV results are summarized and conclusions drawn. Also, and in order not to burden the text, some of the lengthier equations (but which are not part of the core formalism) are reported in the appendixes. As somewhat suggested by the titles of the two founding textbooks in the field [1,2], which are mirrored in the title of this paper, one uses the terms *turbulent* and *random* almost indistinctly (with the former clearly implying the latter), but keeping in mind that there is always a *spectrum* associated with turbulence (hence the care not to qualify as turbulent the case in Sec. III B of a single random mode, saving that qualifier for the case in Sec. III C of an isotropic spectrum of random modes).

II. GENERAL QUASILINEAR FORMALISM FOR RAY TRACING IN RANDOM MEDIA

A. Dispersion relation and ray equations as power series in the fluctuations

The starting point, for waves with frequency ω propagating in a given medium, is to write the local dispersion relation in the form

$$D(\omega, \mathbf{r}, \mathbf{k}) \equiv \omega - \omega(\mathbf{r}, \mathbf{k}) = 0, \quad (1)$$

where $\mathbf{r} \equiv (r_1, r_2, r_3)$ and $\mathbf{k} \equiv (k_1, k_2, k_3)$ are the canonically conjugate coordinate and wave vectors, respectively, so the ray equations [29–31]

$$\frac{dr_i}{dt} = -\frac{\partial D(\omega, \mathbf{r}, \mathbf{k})/\partial k_i}{\partial D(\omega, \mathbf{r}, \mathbf{k})/\partial \omega} \quad \text{and} \quad \frac{dk_i}{dt} = \frac{\partial D(\omega, \mathbf{r}, \mathbf{k})/\partial r_i}{\partial D(\omega, \mathbf{r}, \mathbf{k})/\partial \omega} \quad (2)$$

take the explicitly Hamiltonian form

$$\frac{dr_i}{dt} = \frac{\partial \omega(\mathbf{r}, \mathbf{k})}{\partial k_i} \quad \text{and} \quad \frac{dk_i}{dt} = -\frac{\partial \omega(\mathbf{r}, \mathbf{k})}{\partial r_i}, \quad (3)$$

with t some timelike integration variable along the ray. Assuming that one of the ways $\omega(\mathbf{r}, \mathbf{k})$ depends on $\mathbf{r}$ is via some medium property, say, its density $n_e(\mathbf{r})$, and the latter can be split into an ensemble average $\langle n_e(\mathbf{r})\rangle$ plus a randomly fluctuating term $\delta n_e(\mathbf{r})$, whence

$$n_e(\mathbf{r}) = \langle n_e(\mathbf{r})\rangle + \delta n_e(\mathbf{r}), \quad (4)$$

with $|\delta n_e(\mathbf{r})| \ll \langle n_e(\mathbf{r})\rangle$ and $\langle \delta n_e(\mathbf{r})\rangle = 0$ by construction, one may write

$$\omega(\mathbf{r}, \mathbf{k}) \simeq \omega_0(\mathbf{r}, \mathbf{k}) + \omega_1(\mathbf{r}, \mathbf{k})\delta n_e(\mathbf{r}) + \omega_2(\mathbf{r}, \mathbf{k})\delta n_e(\mathbf{r})\delta n_e(\mathbf{r}), \quad (5)$$

where

$$\omega_m(\mathbf{r}, \mathbf{k}) \equiv \frac{1}{m!}\frac{\partial^m \omega(\mathbf{r}, \mathbf{k})}{\partial n_e^m}\bigg|_{\langle n_e(\mathbf{r})\rangle}. \quad (6)$$

The ray equations (3) then become

$$\frac{dr_i}{dt} \simeq \frac{\partial \omega_0(\mathbf{r}, \mathbf{k})}{\partial k_i} + \frac{\partial \omega_1(\mathbf{r}, \mathbf{k})}{\partial k_i}\delta n_e(\mathbf{r}) + \frac{\partial \omega_2(\mathbf{r}, \mathbf{k})}{\partial k_i}\delta n_e(\mathbf{r})\delta n_e(\mathbf{r}) \quad (7)$$

and

$$\frac{dk_i}{dt} \simeq -\frac{\partial \omega_0(\mathbf{r}, \mathbf{k})}{\partial r_i} - \frac{\partial \omega_1(\mathbf{r}, \mathbf{k})}{\partial r_i}\delta n_e(\mathbf{r}) - \omega_1(\mathbf{r}, \mathbf{k})\frac{\partial \delta n_e(\mathbf{r})}{\partial r_i} - \frac{\partial \omega_2(\mathbf{r}, \mathbf{k})}{\partial r_i}\delta n_e(\mathbf{r})\delta n_e(\mathbf{r}) - \omega_2(\mathbf{r}, \mathbf{k})\frac{\partial \delta n_e(\mathbf{r})\delta n_e(\mathbf{r})}{\partial r_i}. \quad (8)$$

Parenthetically, the subscript e in n_e has no special meaning, having been imported from plasma physics (where it is used to identify the electron density) and being used here simply to distinguish density from index of refraction.

The next step is to also split a ray into ensemble-averaged plus fluctuating terms,

$$\mathbf{r} \equiv \langle \mathbf{r}\rangle + \delta \mathbf{r} \quad \text{and} \quad \mathbf{k} \equiv \langle \mathbf{k}\rangle + \delta \mathbf{k}, \quad (9)$$

plug the latter into (7) and (8), and expand up to, and including, second-order terms in the fluctuating quantities. Therefore,

$$\begin{aligned}\frac{dr_i}{dt} \simeq{}& \frac{\partial \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i} + \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial r_j}\delta r_j + \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial k_j}\delta k_j + \frac{1}{2}\frac{\partial^3 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial r_j \partial r_l}\delta r_j \delta r_l + \frac{1}{2}\frac{\partial^3 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial k_j \partial k_l}\delta k_j \delta k_l\\ &+ \frac{\partial^3 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial r_j \partial k_l}\delta r_j \delta k_l + \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i}\delta n_e(\langle \mathbf{r}\rangle) + \frac{\partial^2 \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial r_j}\delta r_j \delta n_e(\langle \mathbf{r}\rangle) + \frac{\partial^2 \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial k_j}\delta k_j \delta n_e(\langle \mathbf{r}\rangle)\\ &+ \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i}\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\delta r_j + \frac{\partial \omega_2(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i}\delta n_e(\langle \mathbf{r}\rangle)\delta n_e(\langle \mathbf{r}\rangle)\end{aligned} \quad (10)$$

and

$$\begin{aligned}\frac{dk_i}{dt} \simeq{}& -\frac{\partial \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i} - \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial r_j}\delta r_j - \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial k_j}\delta k_j - \frac{1}{2}\frac{\partial^3 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial r_j \partial r_l}\delta r_j \delta r_l - \frac{1}{2}\frac{\partial^3 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial k_j \partial k_l}\delta k_j \delta k_l\\ &- \frac{\partial^3 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial r_j \partial k_l}\delta r_j \delta k_l - \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i}\delta n_e(\langle \mathbf{r}\rangle) - \frac{\partial^2 \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial r_j}\delta r_j \delta n_e(\langle \mathbf{r}\rangle) - \frac{\partial^2 \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial k_j}\delta k_j \delta n_e(\langle \mathbf{r}\rangle)\\ &- \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i}\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\delta r_j - \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_i} - \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_j}\delta r_j\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_i} - \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_j}\delta k_j\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_i}\\ &- \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)\frac{\partial^2 \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_i \partial r_j}\delta r_j - \frac{\partial \omega_2(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i}\delta n_e(\langle \mathbf{r}\rangle)\delta n_e(\langle \mathbf{r}\rangle) - 2\omega_2(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)\delta n_e(\langle \mathbf{r}\rangle)\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_i},\end{aligned} \quad (11)$$

where summation over repeated indices has been assumed and it has been made explicit that all functions are calculated taking the ray quantities at their average values.

In the presence of turbulence, the fluctuation $\delta n_e(\mathbf{r})$ in the density of the medium should, in principle, depend on the time t, which would thus imply that the dispersion relation $D(\omega, \mathbf{r}, \mathbf{k})$ should also be a function of t. However, assuming that the transit time of the ray across the medium is much less than the characteristic period, or time scale, of the fluctuations, which amounts to considering that the turbulent medium is frozen while a ray in the ensemble is traced [12,15,19,26,27], one is allowed to drop any explicit time dependency in the ray equations.

B. The ensemble-averaged ray and fluctuations about it

Keeping in mind that $\langle\delta\mathbf{r}\rangle = \langle\delta\mathbf{k}\rangle = \langle\delta n_e(\mathbf{r})\rangle = \langle\partial\delta n_e(\mathbf{r})/\partial r_i\rangle = 0$ by construction, one can now ensemble average (10) and (11) to obtain the equations for the average ray, which results in

$$\begin{aligned}\frac{d\langle r_i\rangle}{dt} \simeq\; & \frac{\partial\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i} + \frac{1}{2}\frac{\partial^3\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial r_j\partial r_l}\langle\delta r_j\delta r_l\rangle + \frac{1}{2}\frac{\partial^3\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial k_j\partial k_l}\langle\delta k_j\delta k_l\rangle + \frac{\partial^3\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial r_j\partial k_l}\langle\delta r_j\delta k_l\rangle\\ &+ \frac{\partial^2\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial r_j}\langle\delta r_j\delta n_e(\langle\mathbf{r}\rangle)\rangle + \frac{\partial^2\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial k_j}\langle\delta k_j\delta n_e(\langle\mathbf{r}\rangle)\rangle + \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i}\left\langle\delta r_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle\\ &+ \frac{\partial\omega_2(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i}\langle\delta n_e(\langle\mathbf{r}\rangle)\delta n_e(\langle\mathbf{r}\rangle)\rangle \end{aligned}\tag{12}$$

and

$$\begin{aligned}\frac{d\langle k_i\rangle}{dt} \simeq\; & -\frac{\partial\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i} - \frac{1}{2}\frac{\partial^3\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial r_j\partial r_l}\langle\delta r_j\delta r_l\rangle - \frac{1}{2}\frac{\partial^3\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial k_j\partial k_l}\langle\delta k_j\delta k_l\rangle - \frac{\partial^3\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial r_j\partial k_l}\langle\delta r_j\delta k_l\rangle\\ &- \frac{\partial^2\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial r_j}\langle\delta r_j\delta n_e(\langle\mathbf{r}\rangle)\rangle - \frac{\partial^2\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial k_j}\langle\delta k_j\delta n_e(\langle\mathbf{r}\rangle)\rangle - \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i}\left\langle\delta r_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle\\ &- \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_j}\left\langle\delta r_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle - \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_j}\left\langle\delta k_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle - \omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\left\langle\delta r_j\frac{\partial^2\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j\partial r_i}\right\rangle\\ &- \frac{\partial\omega_2(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i}\langle\delta n_e(\langle\mathbf{r}\rangle)\delta n_e(\langle\mathbf{r}\rangle)\rangle - 2\omega_2(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\left\langle\delta n_e(\langle\mathbf{r}\rangle)\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle. \end{aligned}\tag{13}$$

To be noted is that the average ray is not the same as the ray one would have if there were no fluctuations, which would be simply given by the first term in each of equations (12) and (13). Crucial to this result was the retention in expansions (5), (7), (8), (10), and (11) of terms that are of second order in the fluctuations, in what may be viewed as a quasilinear (QL) approach, similar to the one used in the treatment of waves in a weakly turbulent plasma [31,32]. More precisely, the slow deviation of the average ray from its unperturbed trajectory is induced by the ensemble averages of the squared fluctuations in the linearized quantities (4) and (9). Putting it differently, these terms in the squares of the fluctuations are the lowest-order nontrivial terms that survive after the averaging operation is carried out. A drift of the average ray with respect to the unperturbed, zero-fluctuation ray has also been found in previous statistical treatments of ray tracing in media with random density fluctuations [26].

The evolution along the ray of its fluctuating components can be retrieved, according to (9), by taking the difference between (10) and (12), and between (11) and (13). This gives, to lowest order,

$$\begin{aligned}\frac{d\delta r_i}{dt} \simeq\; & \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial r_j}\delta r_j + \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial k_j}\delta k_j\\ &+ \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i}\delta n_e(\langle\mathbf{r}\rangle)\end{aligned}\tag{14}$$

and

$$\begin{aligned}\frac{d\delta k_i}{dt} \simeq\; & -\frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial r_j}\delta r_j - \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial k_j}\delta k_j\\ &- \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i}\delta n_e(\langle\mathbf{r}\rangle) - \omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i},\end{aligned}\tag{15}$$

which amounts to neglecting the difference between the actual squares of the fluctuations and their ensemble averages. Here, it is noteworthy that, seen as stochastic differential equations, (10) and (11), as well as the equivalent set (12)–(15), possess the characteristic structure found in Itô's stochastic calculus: the sum of drift-induced, deterministic terms with so-called martingale terms (proportional to the fluctuations and averaging out to 0) plus terms identified with Itô's correction (proportional to the covariances of the fluctuating quantities) [33].

It is also instructive to see that, using (9) to rewrite (5) according to

$$\omega(\mathbf{r},\mathbf{k}) \simeq \langle\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\rangle + \delta\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle),\tag{16}$$

with the ensemble-averaged and fluctuating components of the dispersion relation given, respectively, by

$$\langle\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\rangle \simeq \omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle) + \frac{1}{2}\frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial r_j}\langle\delta r_i\delta r_j\rangle + \frac{1}{2}\frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial k_j}\langle\delta k_i\delta k_j\rangle + \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial k_j}\langle\delta r_i\delta k_j\rangle$$
$$+ \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i}\langle\delta r_i\delta n_e(\langle\mathbf{r}\rangle)\rangle + \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i}\langle\delta k_i\delta n_e(\langle\mathbf{r}\rangle)\rangle + \omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\left\langle\delta r_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle$$
$$+ \omega_2(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\langle\delta n_e(\langle\mathbf{r}\rangle)\delta n_e(\langle\mathbf{r}\rangle)\rangle \tag{17}$$

and

$$\delta\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle) \simeq \frac{\partial\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i}\delta r_i + \frac{\partial\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i}\delta k_i + \omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\delta n_e(\langle\mathbf{r}\rangle), \tag{18}$$

(12)–(15) can be recast in the rather appealing form

$$\frac{d\langle r_i\rangle}{dt} = \frac{\partial\langle\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\rangle}{\partial\langle k_i\rangle} \quad\text{and}\quad \frac{d\langle k_i\rangle}{dt} = -\frac{\partial\langle\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\rangle}{\partial\langle r_i\rangle} \tag{19}$$

and

$$\frac{d\delta r_i}{dt} = \frac{\partial\delta\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial\langle k_i\rangle} \quad\text{and}\quad \frac{d\delta k_i}{dt} = -\frac{\partial\delta\omega(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial\langle r_i\rangle}. \tag{20}$$

C. Mean-square spreading about the ensemble-averaged ray

To integrate (12) and (13) for the average ray, one must know all the quantities appearing on their right-hand sides (RHSs), so to start with one has to derive equations for the evolution of averages of the types $\langle\delta r_i\delta r_j\rangle$, $\langle\delta k_i\delta k_j\rangle$, and $\langle\delta r_i\delta k_j\rangle$. Accordingly, with δu and δv designating one of δr_i or δk_i, and $d\delta u/dt$ and $d\delta v/dt$ governed by (14) or (15), combining

$$\frac{d\langle\delta u\delta v\rangle}{dt} = \left\langle\delta u\frac{d\delta v}{dt}\right\rangle + \left\langle\delta v\frac{d\delta u}{dt}\right\rangle \tag{21}$$

with (14) and (15) yields

$$\frac{d\langle\delta r_i\delta r_j\rangle}{dt} \simeq \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_j\partial r_l}\langle\delta r_l\delta r_i\rangle + \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_j\partial k_l}\langle\delta k_l\delta r_i\rangle + \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial r_l}\langle\delta r_l\delta r_j\rangle + \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial k_l}\langle\delta k_l\delta r_j\rangle$$
$$+ \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_j}\langle\delta r_i\delta n_e(\langle\mathbf{r}\rangle)\rangle + \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i}\langle\delta r_j\delta n_e(\langle\mathbf{r}\rangle)\rangle, \tag{22}$$

$$\frac{d\langle\delta k_i\delta k_j\rangle}{dt} \simeq -\frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_j\partial r_l}\langle\delta r_l\delta k_i\rangle - \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_j\partial k_l}\langle\delta k_l\delta k_i\rangle - \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial r_l}\langle\delta r_l\delta k_j\rangle - \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i\partial k_l}\langle\delta k_l\delta k_j\rangle$$
$$- \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_j}\langle\delta k_i\delta n_e(\langle\mathbf{r}\rangle)\rangle - \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_i}\langle\delta k_j\delta n_e(\langle\mathbf{r}\rangle)\rangle - \omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\left\langle\delta k_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle$$
$$- \omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\left\langle\delta k_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle, \tag{23}$$

and

$$\frac{d\langle\delta r_i\delta k_j\rangle}{dt} \simeq -\frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_j\partial r_l}\langle\delta r_l\delta r_i\rangle - \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_j\partial k_l}\langle\delta k_l\delta r_i\rangle + \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial r_l}\langle\delta r_l\delta k_j\rangle + \frac{\partial^2\omega_0(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i\partial k_l}\langle\delta k_l\delta k_j\rangle$$
$$- \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial r_j}\langle\delta r_i\delta n_e(\langle\mathbf{r}\rangle)\rangle + \frac{\partial\omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)}{\partial k_i}\langle\delta k_j\delta n_e(\langle\mathbf{r}\rangle)\rangle - \omega_1(\langle\mathbf{r}\rangle,\langle\mathbf{k}\rangle)\left\langle\delta r_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle. \tag{24}$$

Besides being necessary for integrating (12) and (13), (22)–(24) are useful also because they provide the rms spreads in the ray coordinate and wave vectors. Indeed, one can represent the rms spread about the average ray in a fluctuating medium by the vector $\boldsymbol{\sigma}_\mathbf{r} = (\sigma_{r_1}, \sigma_{r_2}, \sigma_{r_3})$, with $\sigma_{r_i} \equiv \sqrt{\langle\delta r_i\delta r_i\rangle}$. In the case of ray propagation in a plane, for instance, putting $r_1 = x$ and $r_2 = y$ while $r_3 = 0$, the rms spreading $\sigma_\perp$ taking place perpendicularly to the average ray can be calculated by projecting $\boldsymbol{\sigma}_\mathbf{r}$ in the direction $(-d\langle y\rangle/dt, d\langle x\rangle/dt, 0)$ perpendicular to the group velocity $(d\langle x\rangle/dt, d\langle y\rangle/dt, 0)$, whence

$$\sigma_\perp = \frac{\sigma_x|d\langle y\rangle/dt| + \sigma_y|d\langle x\rangle/dt|}{\sqrt{(d\langle x\rangle/dt)^2 + (d\langle y\rangle/dt)^2}}. \tag{25}$$

Note that the four possible combinations between $\pm\sigma_x$ and $\pm\sigma_y$ lead to two possible projections in the direction perpendicular to the group velocity, the choice for $\sigma_\perp$ in (25) corresponding to that which is largest. In vector form, and putting

$\boldsymbol{\sigma}_\perp \equiv (\sigma_{\perp x}, \sigma_{\perp y}, 0)$ with

$$\sigma_{\perp x} = \frac{-\sigma_\perp d\langle y\rangle/dt}{\sqrt{(d\langle x\rangle/dt)^2 + (d\langle y\rangle/dt)^2}} \quad \text{and}$$
$$\sigma_{\perp y} = \frac{\sigma_\perp d\langle x\rangle/dt}{\sqrt{(d\langle x\rangle/dt)^2 + (d\langle y\rangle/dt)^2}}, \tag{26}$$

one can then say that the ray path lies within an envelope $\langle \mathbf{r}\rangle \pm \boldsymbol{\sigma}_\perp$.

D. Ensemble averages involving fluctuations of ray quantities and derivatives of medium fluctuations

Whereas ensemble averages such as $\langle \delta n_e(\langle \mathbf{r}\rangle)\delta n_e(\langle \mathbf{r}\rangle)\rangle$ and $\langle \delta n_e(\langle \mathbf{r}\rangle)\partial \delta n_e(\langle \mathbf{r}\rangle)/\partial r_i\rangle$ can be directly computed from the fluctuation model and are evaluated at the average coordinate $\langle \mathbf{r}\rangle$, which is traced along the ray according to (12), quadratic means such as $\langle \delta r_i \delta n_e(\langle \mathbf{r}\rangle)\rangle$, $\langle \delta k_i \delta n_e(\langle \mathbf{r}\rangle)\rangle$, $\langle \delta r_i \partial \delta n_e(\langle \mathbf{r}\rangle)/\partial r_j\rangle$, and $\langle \delta k_i \partial \delta n_e(\langle \mathbf{r}\rangle)/\partial r_j\rangle$ have to be traced simultaneously with the other ray quantities, which means that (12), (13), and (22)–(24) do not yet form a closed system. Thus, if δu stands for one of δr_i or δk_i, and $\delta f(\langle \mathbf{r}\rangle)$ represents $\delta n_e(\langle \mathbf{r}\rangle)$ or any of its derivatives, then

$$\frac{d\langle \delta u \delta f(\langle \mathbf{r}\rangle)\rangle}{dt} = \left\langle \delta u \frac{d\delta f(\langle \mathbf{r}\rangle)}{dt}\right\rangle + \left\langle \delta f(\langle \mathbf{r}\rangle)\frac{d\delta u}{dt}\right\rangle, \tag{27}$$

where

$$\frac{d\delta f(\langle \mathbf{r}\rangle)}{dt} = \frac{\partial \delta f(\langle \mathbf{r}\rangle)}{\partial t} + \frac{\partial \delta f(\langle \mathbf{r}\rangle)}{\partial r_i}\frac{d\langle r_i\rangle}{dt}, \tag{28}$$

$d\delta u/dt$ is given by (14) or (15), and $d\langle r_i\rangle/dt$ by (12). Within the already discussed assumption of frozen turbulence during the transit time of a ray across the medium [12,15,19,26,27], the first term on the RHS of (28) can be neglected, so (27) and (28) give

$$\frac{d\langle \delta u \delta f(\langle \mathbf{r}\rangle)\rangle}{dt} \simeq \left\langle \delta u \frac{\partial \delta f(\langle \mathbf{r}\rangle)}{\partial r_i}\right\rangle \frac{d\langle r_i\rangle}{dt} + \left\langle \delta f(\langle \mathbf{r}\rangle)\frac{d\delta u}{dt}\right\rangle. \tag{29}$$

Hence, putting together (14), or (15), and (29),

$$\frac{d\langle \delta r_i \delta n_e(\langle \mathbf{r}\rangle)\rangle}{dt} \simeq \left\langle \delta r_i \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle \frac{d\langle r_j\rangle}{dt} + \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial r_j}\langle \delta r_j \delta n_e(\langle \mathbf{r}\rangle)\rangle + \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial k_j}\langle \delta k_j \delta n_e(\langle \mathbf{r}\rangle)\rangle$$
$$+ \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i}\langle \delta n_e(\langle \mathbf{r}\rangle)\delta n_e(\langle \mathbf{r}\rangle)\rangle, \tag{30}$$

$$\frac{d\langle \delta k_i \delta n_e(\langle \mathbf{r}\rangle)\rangle}{dt} \simeq \left\langle \delta k_i \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle \frac{d\langle r_j\rangle}{dt} - \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial r_j}\langle \delta r_j \delta n_e(\langle \mathbf{r}\rangle)\rangle - \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial k_j}\langle \delta k_j \delta n_e(\langle \mathbf{r}\rangle)\rangle$$
$$- \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i}\langle \delta n_e(\langle \mathbf{r}\rangle)\delta n_e(\langle \mathbf{r}\rangle)\rangle - \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)\left\langle \delta n_e(\langle \mathbf{r}\rangle)\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_i}\right\rangle, \tag{31}$$

$$\frac{d}{dt}\left\langle \delta r_i \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle \simeq \left\langle \delta r_i \frac{\partial^2 \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j \partial r_l}\right\rangle \frac{d\langle r_l\rangle}{dt} + \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial r_l}\left\langle \delta r_l \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle + \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i \partial k_l}\left\langle \delta k_l \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle$$
$$+ \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial k_i}\left\langle \delta n_e(\langle \mathbf{r}\rangle)\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle, \tag{32}$$

and

$$\frac{d}{dt}\left\langle \delta k_i \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle \simeq \left\langle \delta k_i \frac{\partial^2 \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j \partial r_l}\right\rangle \frac{d\langle r_l\rangle}{dt} - \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial r_l}\left\langle \delta r_l \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle - \frac{\partial^2 \omega_0(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i \partial k_l}\left\langle \delta k_l \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle$$
$$- \frac{\partial \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)}{\partial r_i}\left\langle \delta n_e(\langle \mathbf{r}\rangle)\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle - \omega_1(\langle \mathbf{r}\rangle, \langle \mathbf{k}\rangle)\left\langle \frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_i}\frac{\partial \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_j}\right\rangle. \tag{33}$$

Despite one's effort, the system (12), (13), and (22)–(24) is not yet closed by means of (30)–(33) because of the new quadratic means entering the first terms on their RHSs, which contain the second derivatives of the medium fluctuations and also have to be traced with the average ray described by (12) and (13). One can again use (14), (15), and (29) to track the evolution of such terms but the new equations will contain, in turn, the third derivatives of $\delta n_e(\langle \mathbf{r}\rangle)$, and so forth. In fact, the problem only becomes closed by a downward infinite recurrence, in which the equation allowing one to trace a quadratic mean with a lower-order derivative is recursively generated by the quadratic mean with the next-higher-order derivative, namely,

$$\frac{d}{dt}\left\langle \delta u \frac{\partial^m \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_{i_1}\partial r_{i_2}\ldots \partial r_{i_m}}\right\rangle$$
$$\simeq \left\langle \delta u \frac{\partial^{m+1} \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_{i_1}\partial r_{i_2}\ldots \partial r_{i_m}\partial r_{i_{m+1}}}\right\rangle \frac{d\langle r_{i_{m+1}}\rangle}{dt}$$
$$+ \left\langle \frac{\partial^m \delta n_e(\langle \mathbf{r}\rangle)}{\partial r_{i_1}\partial r_{i_2}\ldots \partial r_{i_m}}\frac{d\delta u}{dt}\right\rangle. \tag{34}$$

For this QL approach to be implemented, the recurrence has to be truncated somewhere, for instance, if the derivatives of the medium fluctuations higher than some given order $m-1$ become arbitrarily small. One is thus left with 6 ordinary differential equations (ODEs) for the average ray in (12) and (13), 21 ODEs for the mean-square spreadings in (22)–(24), 6 ODEs for (30) and (31), 18 ODEs for (32) and (33), and $3(m+1)(m+2)$ ODEs per each additional set (34) in a higher-order derivative m of $\delta n_e(\langle \mathbf{r}\rangle)$ [with $(m+1)(m+2)/2$ the number of partitions of m into 3 parts [34]]. In what follows, if the downward recursion relation (34) is truncated by neglecting the first term on its RHS, the one with the $m+1$ derivative of $\delta n_e(\langle \mathbf{r}\rangle)$, it is said that the QL formalism has been implemented up to order m (or truncated at order $m+1$).

III. RAY TRACING IN HOMOGENEOUS RANDOM MEDIA: A TEST CASE

A. QL ray tracing in homogeneous random media

A model often utilized to describe the propagation of optical or radio-wave rays in the atmosphere, or of acoustical rays in the ocean, or even of seismic rays, and which is particularly suited for verification and validation of the QL approach [35], is that of a constant index of refraction n_0 (for the unperturbed medium) on which a random fluctuation is superposed,

$$n(\mathbf{r}) \equiv n_0\left(1+\frac{\delta n_e(\mathbf{r})}{\langle n_e\rangle}\right), \tag{35}$$

where the fluctuations in the medium are here assumed to come from the density field but may also come from other fields such as the temperature, humidity, elastic modulus, or velocity [1–4,6–13,15–17,19]. The dispersion relation $\omega(\mathbf{r},\mathbf{k}) = ck/n(\mathbf{r})$, with c the speed of light in vacuum or the speed of sound in the unperturbed medium, and $k^2 \equiv k_1^2+k_2^2+k_3^2$, becomes

$$\omega(\mathbf{r},\mathbf{k}) \simeq \frac{ck}{n_0}\left[1-\frac{\delta n_e(\mathbf{r})}{\langle n_e\rangle}+\frac{\delta n_e(\mathbf{r})\delta n_e(\mathbf{r})}{\langle n_e\rangle^2}\right], \tag{36}$$

yielding, for (6),

$$\omega_0(k) = \frac{ck}{n_0} \quad\text{and}\quad \omega_m(k) = (-1)^m\frac{\omega_0(k)}{\langle n_e\rangle^m} \tag{37}$$

and, for the derivatives entering (12), (13), (22)–(24), and (30)–(33),

$$\frac{\partial^m\omega_0(k)}{\partial k_i^n\partial k_j^p\partial k_l^{m-n-p}} = \frac{c}{n_0}\frac{\partial^m k}{\partial k_i^n\partial k_j^p\partial k_l^{m-n-p}}, \tag{38}$$

$$\frac{\partial k}{\partial k_i} = \frac{k_i}{k},\quad \frac{\partial^2 k}{\partial k_i\partial k_j} = \frac{\delta_{ij}k^2-k_ik_j}{k^3},\quad\text{and}$$

$$\frac{\partial^3 k}{\partial k_i\partial k_j\partial k_l} = \frac{3k_ik_jk_l}{k^5}-\frac{\delta_{jl}k_i+\delta_{il}k_j+\delta_{ij}k_l}{k^3}, \tag{39}$$

where δ_{ij} is the Kronecker delta. Hence, with the help of (37)–(39), the QL ray equations (12), (13), (22)–(24), and (30)–(33) take the form (A1)–(A9) given in Appendix A.

B. QL ray tracing in homogeneous random media with single-mode fluctuations

The simplest example to consider, and with which to start the verification and validation of the QL approach presented here for ray tracing in random fluctuating media, is that of a single mode propagating with a given wave number q along a given direction, say r_1, and with an amplitude δn_{e0},

$$\delta n_e(\mathbf{r}) \equiv \delta n_e(r_1) \equiv \delta n_{e0}\cos(qr_1+\phi), \tag{40}$$

where ϕ stands for a random phase. In this case,

$$\frac{\partial\delta n_e(\mathbf{r})}{\partial r_i} = -\delta_{1i}q\delta n_{e0}\sin(qr_1+\phi)\quad\text{and}$$

$$\frac{\partial^2\delta n_e(\mathbf{r})}{\partial r_i\partial r_j} = -\delta_{1i}\delta_{1j}q^2\delta n_e(\mathbf{r}), \tag{41}$$

so the recurrence (30)–(34) is naturally closed (with no need to be truncated), and still

$$\langle\delta n_e(\mathbf{r})\delta n_e(\mathbf{r})\rangle = \frac{(\delta n_{e0})^2}{2},\quad \left\langle\delta n_e(\mathbf{r})\frac{\partial\delta n_e(\mathbf{r})}{\partial r_i}\right\rangle = 0,\quad\text{and}$$

$$\left\langle\frac{\partial\delta n_e(\mathbf{r})}{\partial r_i}\frac{\partial\delta n_e(\mathbf{r})}{\partial r_j}\right\rangle = \delta_{1i}\delta_{1j}q^2\frac{(\delta n_{e0})^2}{2}. \tag{42}$$

For ray tracing on the plane defined by r_1 and r_2, with $r_3=0$, changing variables according to $\tau = (ck_0/2\pi n_0)t$, $x[y] = (k_0/2\pi)r_{1[2]}$, and $\kappa_{x[y]} = k_{1[2]}/k_0$ (so time and space are normalized to the unperturbed wave period and wavelength, respectively, and wave numbers, including q, to the initial wave number k_0), and setting $\langle n_e\rangle = 1$ (which corresponds to having δn_{e0} given as, say, a percentage), (A1)–(A9) become, on account of (41) and (42), the set (B1)–(B9) in Appendix B.

Three cases have been considered, which correspond to three distinct initializations of the wave-vector components, namely, $\kappa_{x0}=1$ and $\kappa_{y0}=0$, $\kappa_{x0}=0$ and $\kappa_{y0}=1$, and $\kappa_{x0}=\kappa_{y0}=1/\sqrt{2}$ (henceforth identified as the parallel, perpendicular, and oblique cases, respectively). In addition, rays are started at $x_0=y_0=0$, while their fluctuating components initially vanish, meaning that $\delta x_0=\delta y_0=\delta\kappa_{x0}=\delta\kappa_{y0}=0$, implying in this manner that all quadratic means containing any of these quantities are also set to 0 initially. Hence, some results can already be predicted by simple inspection of the equations; for instance, from (B2) and (B4) it follows that

$$\langle\kappa_y\rangle = \kappa_{y0}\quad\text{and}\quad\langle\delta\kappa_y\delta\kappa_y\rangle = 0, \tag{43}$$

which was to be expected, as y is an ignorable variable in this example, and indicates that there is no rms spreading in wave number along the direction perpendicular to the wave vector of the turbulent mode. In the parallel case (when $\kappa_{y0}=0$), one has

$$\langle y\rangle = \langle\delta y\delta y\rangle = 0 \tag{44}$$

since, from (B1), (B4), (B7), (B9), and (43), one sees that $d\langle y\rangle/dt$ is proportional to $\langle\delta\kappa_x\delta\kappa_y\rangle$ or $\langle\delta\kappa_y\delta n_e(\langle x\rangle)\rangle$, $d\langle\delta\kappa_x\delta\kappa_y\rangle/dt$ and $d\langle\delta\kappa_y\delta n_e(\langle x\rangle)\rangle/dt$ to $\langle\delta\kappa_y\partial\delta n_e(\langle x\rangle)/\partial x\rangle$, and $d\langle\delta\kappa_y\partial\delta n_e(\langle x\rangle)/\partial x\rangle/dt$ to $\langle\delta\kappa_y\delta n_e(\langle x\rangle)\rangle$, meaning that all these average values vanish along the ray trajectory if they initially do so, while from (B3), (B5), and (43) one concludes that $\langle\delta y\delta y\rangle$ vanishes simultaneously with $\langle\delta\kappa_y\delta\kappa_y\rangle$. In particular, recalling (25), (44) implies that there is no rms spreading about the average ray in the parallel case. As for the perpendicular case (when $\kappa_{x0}=0$), (B1), (B2), (B4), and (B7) yield

$$\langle x\rangle = \langle\kappa_x\rangle = 0, \tag{45}$$

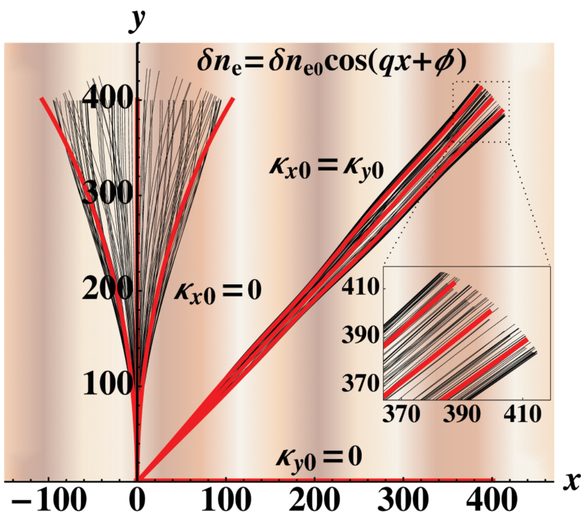

FIG. 1. Average ray trajectories $\langle y\rangle$ vs $\langle x\rangle$ and their perpendicular rms spreadings $\sigma_\perp$ from the QL formalism (red lines) vs an MC calculation with $N = 100$ rays (black lines) for the parallel, oblique, and perpendicular cases, superimposed on a single-mode random background with $\delta n_{e0} = 4\%$ and $q = 0.04$. QL results (red lines) are presented in the form $\langle \mathbf{r}\rangle \pm \boldsymbol{\sigma}_\perp$.

because the terms in $d\langle x\rangle/dt$ are proportional to $\langle\kappa_x\rangle$, $\langle\delta\kappa_x\delta\kappa_y\rangle$, or $\langle\delta\kappa_x\delta n_e(\langle x\rangle)\rangle$, whereas those in $d\langle\kappa_x\rangle/dt$ and $d\langle\delta\kappa_x\delta\kappa_y\rangle/dt$ are proportional to $\langle\kappa_x\rangle$, $\langle\delta\kappa_y\partial\delta n_e(\langle x\rangle)/\partial x\rangle$, or $\langle\delta x\delta n_e(\langle x\rangle)\rangle$, with $d\langle\delta x\delta n_e(\langle x\rangle)\rangle/dt$, $d\langle\delta\kappa_x\delta n_e(\langle x\rangle)\rangle/dt$, and $d\langle\delta\kappa_y\partial\delta n_e(\langle x\rangle)/\partial x\rangle/dt$ being proportional, in turn, to $d\langle x\rangle/dt$, $\langle k_x\rangle$, or $\langle\delta\kappa_x\delta n_e(\langle x\rangle)\rangle$, thus entailing a loop that keeps all these quantities equal to 0 once they are initially set to this value. The results in (44) and (45) can be checked in Figs. 1 and 2 for $q = 0.04$ (so the turbulence wavelength is much greater than $2\pi/k_0$, as required by the eikonal approximation) and $\delta n_{e0} = 4\%$ (within the range of values reported in works on ray tracing in plasmas with density fluctuations [24–28]), where they are gauged (for purposes of validation) against the outcome of an MC calculation with $N = 100$ rays, a different ϕ in (40) being randomly chosen (from a uniform distribution between 0 and 2π) for each ray. The overall agreement is good, with the largest deviations occurring for the perpendicular case; it is noteworthy that, in general, the rms spread $\sigma_\perp$ from the QL formalism provides strikingly good envelopes for the MC rays. Needless to say, the smaller δn_{e0} or q, the better the agreement between the QL approach developed here and the MC calculation.

Furthermore, use of (43) and (45) in (B3)–(B5), (B8), and (B9) yields

$$\frac{d\langle\delta x\delta x\rangle}{d\tau} \simeq \frac{2\langle\delta x\delta\kappa_x\rangle}{\kappa_{y0}}, \tag{46}$$

$$\frac{d\langle\delta\kappa_x\delta\kappa_x\rangle}{d\tau} \simeq 2\kappa_{y0}\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle, \tag{47}$$

$$\frac{d\langle\delta x\delta\kappa_x\rangle}{d\tau} \simeq \frac{\langle\delta\kappa_x\delta\kappa_x\rangle}{\kappa_{y0}} + \kappa_{y0}\left\langle\delta x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle, \tag{48}$$

$$\frac{d}{d\tau}\left\langle\delta x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle \simeq \frac{1}{\kappa_{y0}}\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle, \tag{49}$$

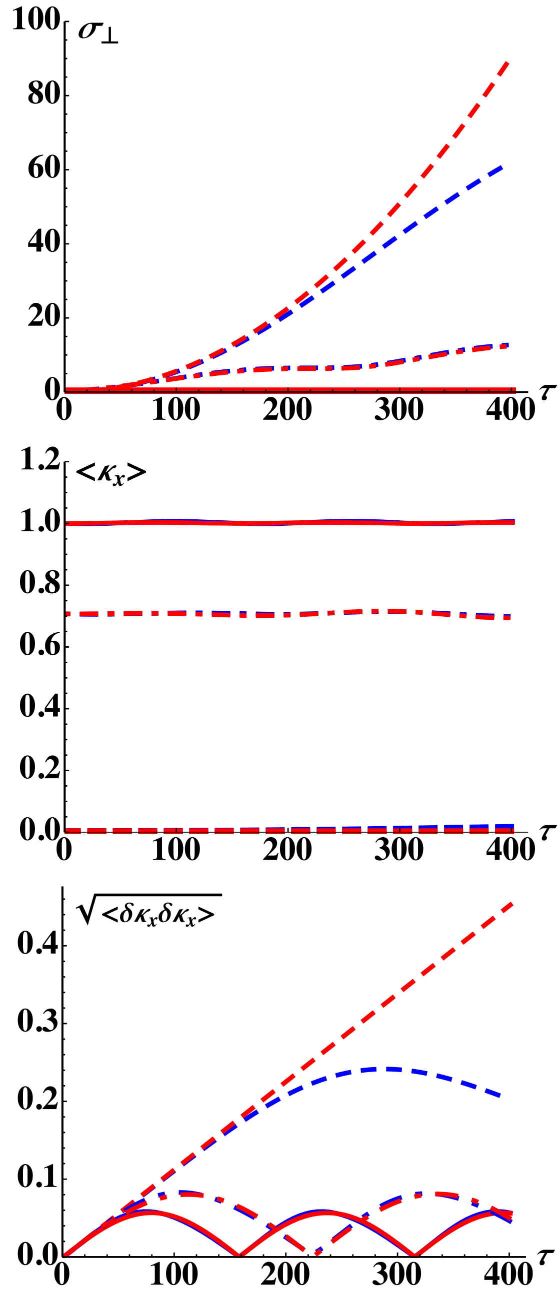

FIG. 2. QL formalism (red, lighter lines) vs an MC calculation with $N = 100$ rays (blue, darker lines) for the parallel (solid lines), oblique (dot-dashed lines), and perpendicular (dashed lines) cases and for a single random mode with $\delta n_{e0} = 4\%$ and $q = 0.04$: perpendicular rms spread $\sigma_\perp$, average wave number $\langle\kappa_x\rangle$ in the direction parallel to the wave vector of the mode, and respective rms spread $\sqrt{\langle\delta\kappa_x\delta\kappa_x\rangle}$ as functions of the time τ.

and

$$\frac{d}{d\tau}\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle \simeq \kappa_{y0}\frac{q^2(\delta n_{e0})^2}{2}, \tag{50}$$

whence, integrating (46)–(50) backwards and recalling (25),

$$\sigma_\perp(\tau) = \sqrt{\langle\delta x\delta x\rangle}(\tau) \simeq \frac{q\delta n_{e0}\tau^2}{2\sqrt{2}} \tag{51}$$

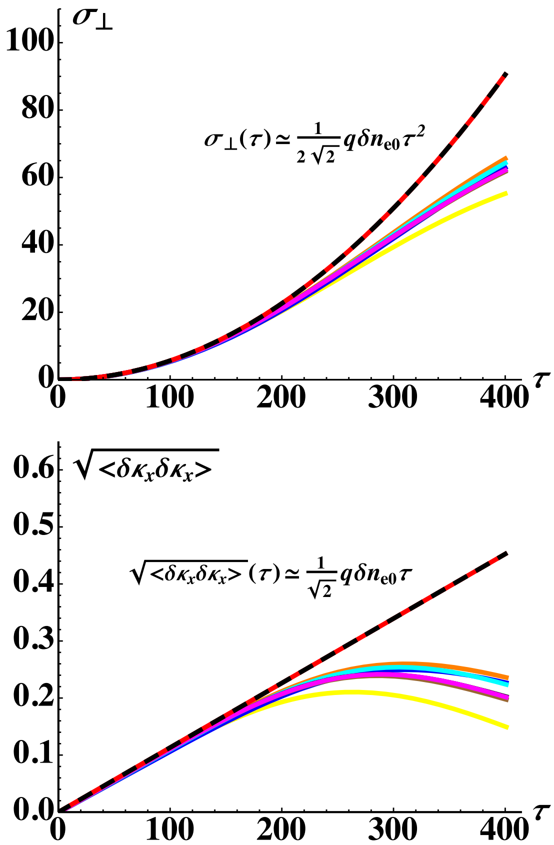

FIG. 3. QL formalism, numerical (red line) vs analytical (dashed black line), vs MC calculations with $N = 20$ (yellow line), $N = 40$ (blue line), $N = 60$ (orange line), $N = 80$ (cyan line), $N = 100$ (black line), $N = 200$ (brown line), and $N = 400$ (magenta line) rays for the perpendicular case and for a single random mode with $\delta n_{e0} = 4\%$ and $q = 0.04$: perpendicular rms spread $\sigma_\perp$ and rms spread $\sqrt{\langle\delta\kappa_x\delta\kappa_x\rangle}$ of the wave number in the direction parallel to the wave vector of the mode as functions of the time τ. The QL red and dashed black lines are coincident.

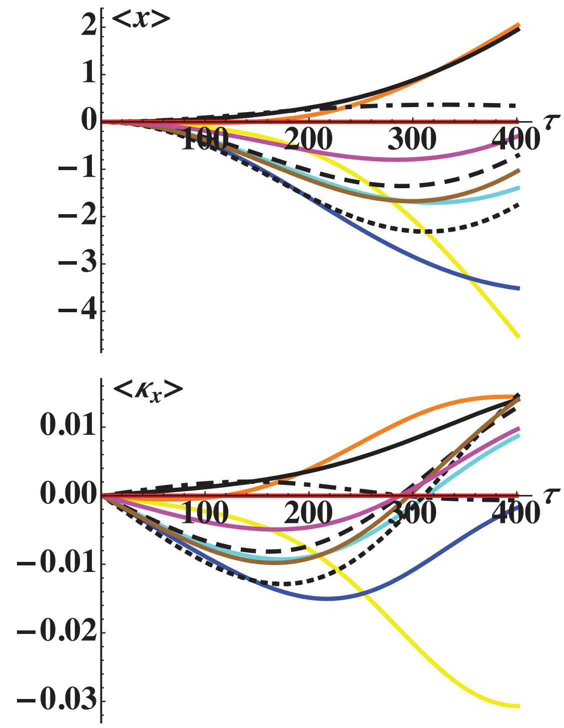

FIG. 4. QL formalism (red line) vs MC calculations with $N = 20$ (yellow line), $N = 40$ (blue line), $N = 60$ (orange line), $N = 80$ (cyan line), $N = 100$ (solid, dashed, dotted, and dot-dashed black lines, corresponding to different MC realizations), $N = 200$ (brown line), and $N = 400$ (magenta line) rays for the perpendicular case and for a single random mode with $\delta n_{e0} = 4\%$ and $q = 0.04$: average distance $\langle x\rangle$ traveled along the direction parallel to the wave vector of the mode and respective average wave-vector component $\langle\kappa_x\rangle$ as functions of the time τ.

and

$$\sqrt{\langle\delta\kappa_x\delta\kappa_x\rangle}(\tau) \simeq \frac{q\delta n_{e0}\tau}{\sqrt{2}} \tag{52}$$

for the perpendicular case. Equations (51) and (52) can be used for verification of the numerical implementation of the QL formalism, as demonstrated in Fig. 3, where the outcomes of several MC calculations for various N's are also plotted, showing that the MC convergence in the number of rays is fast and $N = 100$ is an appropriate choice. Regarding MC convergence and statistical noise, and for completeness, it is shown in Fig. 2 (although only barely visible because of the compressed scale) that the MC calculated $\langle\kappa_x\rangle$ for the perpendicular case is not strictly 0, as it should be. This difficulty, that MC calculations have to converge to a quantity that strictly vanishes, is illustrated in Fig. 4, where several MC results are plotted for different numbers of rays used; it is apparent that neither $\langle x\rangle$ nor $\langle\kappa_x\rangle$ for the perpendicular case has already fully converged to 0 for $N = 400$ and, further, that different realizations for the same $N = 100$ yield distinct outcomes. An additional test that can be conducted with the single-mode example has to do with the rate of convergence of the QL approach as one goes higher in the order at which the recursion relation (34) is truncated. To do so, instead of closing (A8) and (A9) making use of (41), thus entailing (B8) and (B9), the latter have been replaced, going through (34), (40), (A8), and (A9), with (B10)–(B12) in Appendix B. Equations (B1)–(B7) and (B10)–(B12) have thus been implemented, for the oblique case, going up through various values of m (corresponding to truncating the QL formalism at successive orders $m + 1$); the results are shown in Fig. 5, which can be compared with the outcome of the nontruncated QL calculation given in Fig. 2. It is clear that the QL approach converges to the MC benchmark as one truncates at higher and higher $m + 1$, its having been verified that the more oscillatory the modeled quantity is (which, in the single-mode case under analysis, means closer to the parallel case), the harder the convergence. Although in this particular case of a single mode the convergence is rather slow (from $\tau \approx 100$ onwards, one needs roughly to go up $\Delta m \approx 4$ in the order at which the QL formalism is truncated to see good convergence every $\Delta\tau \approx 50$), it is shown in the forthcoming example that convergence becomes much quicker when a realistic spectrum of many modes is utilized.

Attention has already been called to the fact that the QL average ray is different from the unperturbed ray one would get if tracing in the absence of fluctuations, which, in the present

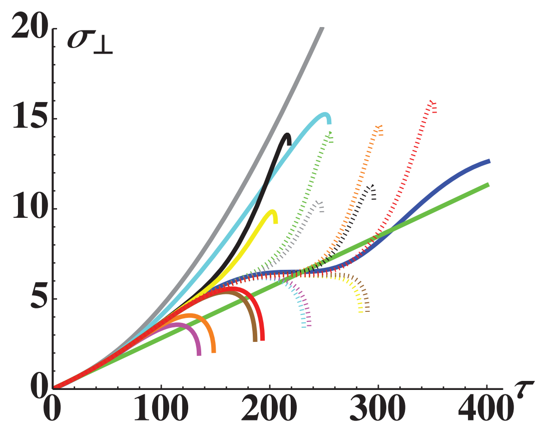

FIG. 5. QL formalism implemented up to orders $m = 0$ (solid green line), $m = 1$ (solid gray line), $m = 2$ (solid cyan line), $m = 3$ (solid magenta line), $m = 4$ (solid orange line), $m = 5$ (solid black line), $m = 6$ (solid yellow line), $m = 7$ (solid brown line), $m = 8$ (solid red line), $m = 9$ (dotted green line), $m = 10$ (dotted gray line), $m = 11$ (dotted cyan line), $m = 12$ (dotted magenta line), $m = 13$ (dotted orange line), $m = 14$ (dotted black line), $m = 15$ (dotted yellow line), $m = 16$ (dotted brown line), and $m = 17$ (dotted red line) vs an MC calculation with $N = 100$ rays (solid blue line) for the oblique case and for a single random mode with $\delta n_{e0} = 4\%$ and $q = 0.04$: perpendicular rms spread $\sigma_\perp$ as a function of the time τ. Curves that terminate abruptly while still within the range of the plot do so because one of the quantities σ_x or σ_y, which appear in (25), becomes purely imaginary.

single-mode example, is easy to check by simple inspection of (B1) and (B2). Indeed, neglecting in the latter all terms in the squares of the fluctuations, immediate integration yields, for the unperturbed ray,

$$x[y](\tau) = \kappa_{x0[y0]}\tau \quad \text{and} \quad \kappa_{x[y]}(\tau) = \kappa_{x0[y0]}, \qquad (53)$$

which is clearly different from the average ray $\langle x[y]\rangle(\tau)$ and $\langle\kappa_{x[y]}\rangle(\tau)$, as can be verified in Fig. 6, where the fluctuation level has been increased to $\delta n_{e0} = 7\%$ (so as to make the distinction more pronounced).

C. QL ray tracing in homogeneous random media with isotropic spectra of fluctuations

Although the single-mode case was useful for assessing and testing some features of the QL approach, namely, concerning the verification and validation of the model, it does not correspond to a full spectrum of turbulent modes. So, instead of the single mode (40), an isotropic turbulence spectrum is now considered, which corresponds to a flat distribution in wave number, with a prescribed cutoff $q_{\max}$:

$$\delta n_e(\mathbf{r}) \equiv \delta n_e(x, y)$$
$$\equiv \frac{\delta n_{e0}}{\sqrt{N_q N_\theta}} \sum_{r=1}^{N_q}\sum_{s=1}^{N_\theta} \cos(q_r \cos\theta_s x + q_r \sin\theta_s y + \phi_{rs}), \qquad (54)$$

where $q_1 = 0$, $q_{N_q} = q_{\max}$, $\theta_1 = 0$, $\theta_{N_\theta} = 2\pi$, and the ϕ_{rs} are random phases (distributed uniformly between 0 and 2π).

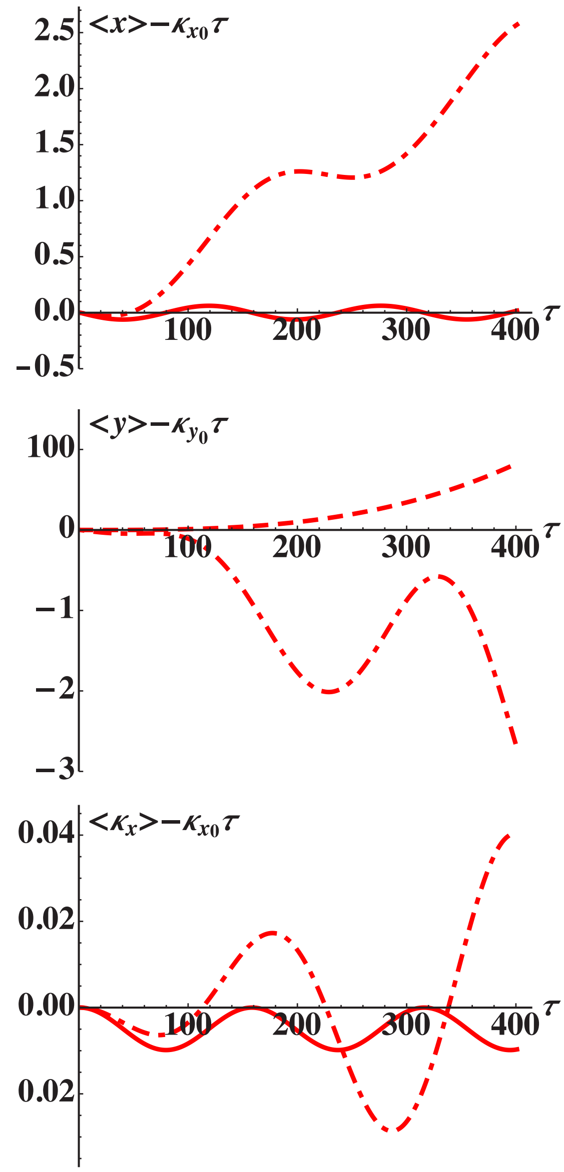

FIG. 6. Difference between the QL average ray and the unperturbed ray for the parallel (solid line), oblique (dot-dashed line), and perpendicular (dashed line) cases and for a single random mode with $\delta n_{e0} = 7\%$ and $q = 0.04$: differences in the coordinates and in the wave-vector component in the direction parallel to the wave vector of the mode, respectively, $\langle x\rangle - \kappa_{x0}\tau$, $\langle y\rangle - \kappa_{y0}\tau$, and $\langle\kappa_x\rangle - \kappa_{x0}$, as functions of the time τ.

Its having been checked that there are no visible differences in results when going from $m = 2$ to $m = 3$, the infinite recurrence (34) has been truncated at $m + 1 = 4$, meaning that the quadratic-mean quantities

$$\left\langle \frac{\partial^n \delta n_e(\langle x\rangle)}{\partial x^l \partial y^{n-l}} \frac{\partial^m \delta n_e(\langle x\rangle)}{\partial x^p \partial y^{m-p})} \right\rangle$$
$$= i^{n+m}\frac{(-1)^n + (-1)^m}{2}\frac{1}{N_q}\sum_{r=1}^{N_q} q_r^{n+m}\frac{1}{N_\theta}$$
$$\times \sum_{s=1}^{N_\theta}(\cos\theta_s)^{l+p}(\sin\theta_s)^{n+m-l-p}\frac{(\delta n_{e0})^2}{2}, \qquad (55)$$

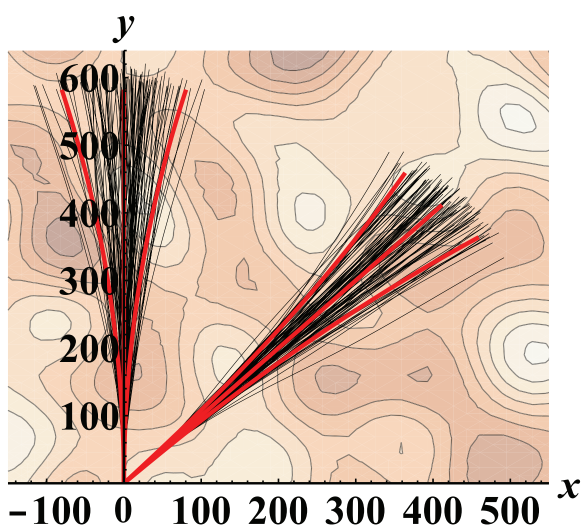

FIG. 7. Average ray trajectories $\langle y \rangle$ vs $\langle x \rangle$ and their perpendicular rms spreadings $\sigma_\perp$ from the QL formalism (red lines) vs an MC calculation with $N = 100$ rays (black lines) for the oblique and perpendicular cases, superimposed on an $N_q \times N_\theta = 100 \times 100$ multimode isotropic random background with $\delta n_{e0} = 4\%$ and $q_{\max} = 0.04$. QL results (red lines) are presented in the form $\langle \mathbf{r} \rangle \pm \boldsymbol{\sigma}_\perp$.

which are obtained directly from the spectrum (54), are needed for $n = 0, 1,\ l = 0, n,\ m = 0, 1, 2, 3,$ and $p = 0, 1, \ldots, m$. To great computational advantage, and for $N_\theta \gg 1$ ($N_\theta \gtrsim 10$ being already more than enough), the sum over the angles can be replaced with an integral according to

$$\frac{1}{N_\theta} \sum_{s=1}^{N_\theta} (\cos\theta_s)^{l+p} (\sin\theta_s)^{n+m-l-p} \simeq \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta)^{l+p} (\sin\theta)^{n+m-l-p} d\theta, \qquad (56)$$

the integral (56) surviving only if both $l + p$ and $n + m$ are even; more precisely [36],

$$\frac{1}{2\pi} \int_0^{2\pi} (\cos\theta)^{l+p} (\sin\theta)^{n+m-l-p} d\theta = \frac{[1 + (-1)^{l+p}][1 + (-1)^{n+m}]}{2^{n+m+2}} \times \frac{(l+p)!(n+m-l-p)!}{\left(\frac{l+p}{2}\right)!\left(\frac{n+m-l-p}{2}\right)!\left(\frac{n+m}{2}\right)!}. \qquad (57)$$

So, the QL formalism has been applied to the fluctuation spectrum (54), by putting together (34), (A1)–(A9), and (55)–(57), with $N_q = 100$, $N_\theta = 100$, $q_{\max} = 0.04$, and $\delta n_{e0} = 4\%$, and the results have been compared with an MC calculation with $N = 100$ rays. The ray trajectories, with their perpendicular rms spreadings, are plotted in Fig. 7, where, once more, one can confirm how well the QL calculated $\sigma_\perp$ follows the unfolding of the MC ray pencils, with no apparent distinction between the oblique and the perpendicular cases (as expected for an isotropic turbulent background). The relevant ensemble-averaged ray quantities are given in Figs. 8 and 9, for several launching angles $\theta_0 \equiv \tan^{-1}(\kappa_{y0}/\kappa_{x0})$ (always with $\kappa_{x0}^2 + \kappa_{y0}^2 = 1$), where one can verify that the system

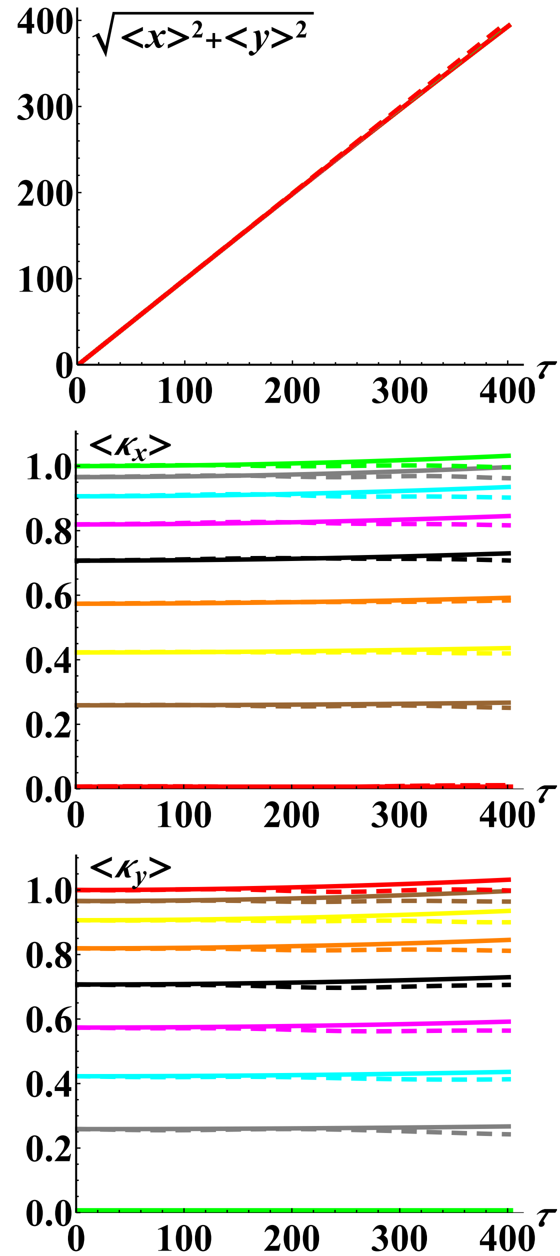

FIG. 8. QL formalism (solid lines) vs an MC calculation with $N = 100$ rays (dashed lines) for $\theta_0 = 0^\circ$ (green lines), $\theta_0 = 15^\circ$ (gray lines), $\theta_0 = 25^\circ$ (cyan lines), $\theta_0 = 35^\circ$ (magenta lines), $\theta_0 = 45^\circ$ (black lines), $\theta_0 = 55^\circ$ (orange lines), $\theta_0 = 65^\circ$ (yellow lines), $\theta_0 = 75^\circ$ (brown lines), and $\theta_0 = 90^\circ$ (red lines) and for an $N_q \times N_\theta = 100 \times 100$ multimode isotropic turbulent spectrum with $\delta n_{e0} = 4\%$ and $q_{\max} = 0.04$: rms traveled distance $\sqrt{\langle x^2 \rangle + \langle y^2 \rangle}$ and ensemble-averaged wave-vector components $\langle \kappa_x \rangle$ and $\langle \kappa_y \rangle$ as functions of the time τ. The launching angles θ_0 increase monotonically downwards (upwards) in the middle (bottom) frame.

indeed behaves as essentially isotropic and has a symmetry around $\theta = 45^\circ$ upon exchange of x with y. Actually, this system has four axes of symmetry with respect to reflection, which are $\theta = 0^\circ, \theta = 45^\circ, \theta = 90^\circ$, and $\theta = 135^\circ$, and which correspond, respectively, to exchanging y with $-y$, x with y, x with $-x$, and x with $-y$. One can also check that the QL and MC results are almost coincident for the most important

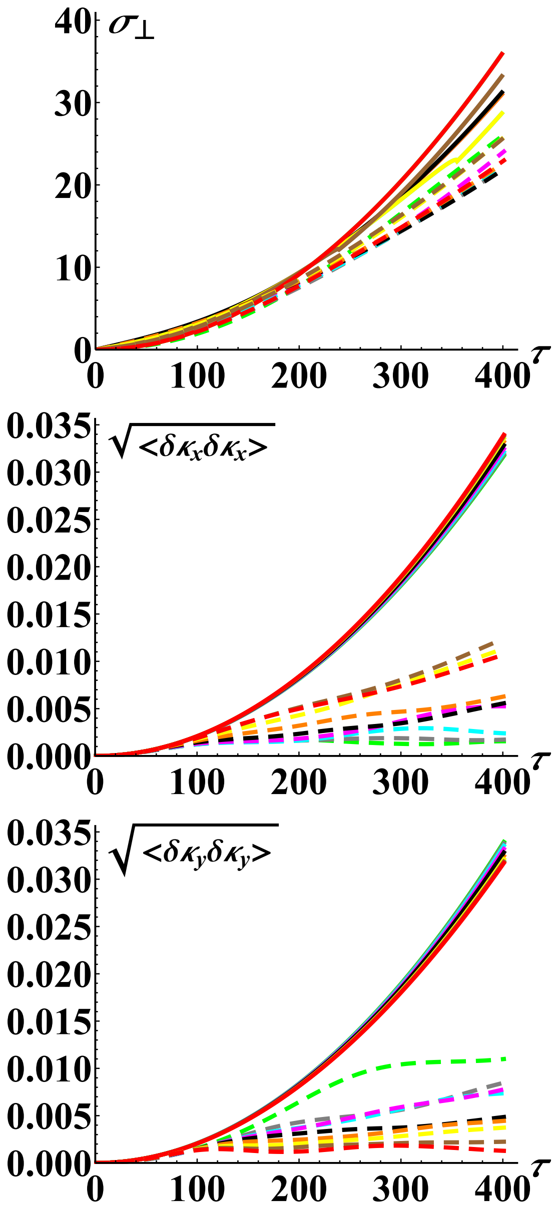

FIG. 9. QL formalism (solid lines) vs an MC calculation with $N = 100$ rays (dashed lines) for $\theta_0 = 0°$ (green lines), $\theta_0 = 15°$ (gray lines), $\theta_0 = 25°$ (cyan lines), $\theta_0 = 35°$ (magenta lines), $\theta_0 = 45°$ (black lines), $\theta_0 = 55°$ (orange lines), $\theta_0 = 65°$ (yellow lines), $\theta_0 = 75°$ (brown lines), and $\theta_0 = 90°$ (red lines) and for an $N_q \times N_\theta = 100 \times 100$ multimode isotropic turbulent spectrum with $\delta n_{e0} = 4\%$ and $q_{\max} = 0.04$: perpendicular rms spread $\sigma_\perp$ and rms spreads $\sqrt{\langle \delta\kappa_x \delta\kappa_x \rangle}$ and $\sqrt{\langle \delta\kappa_y \delta\kappa_y \rangle}$ in the wave-vector components as functions of the time τ.

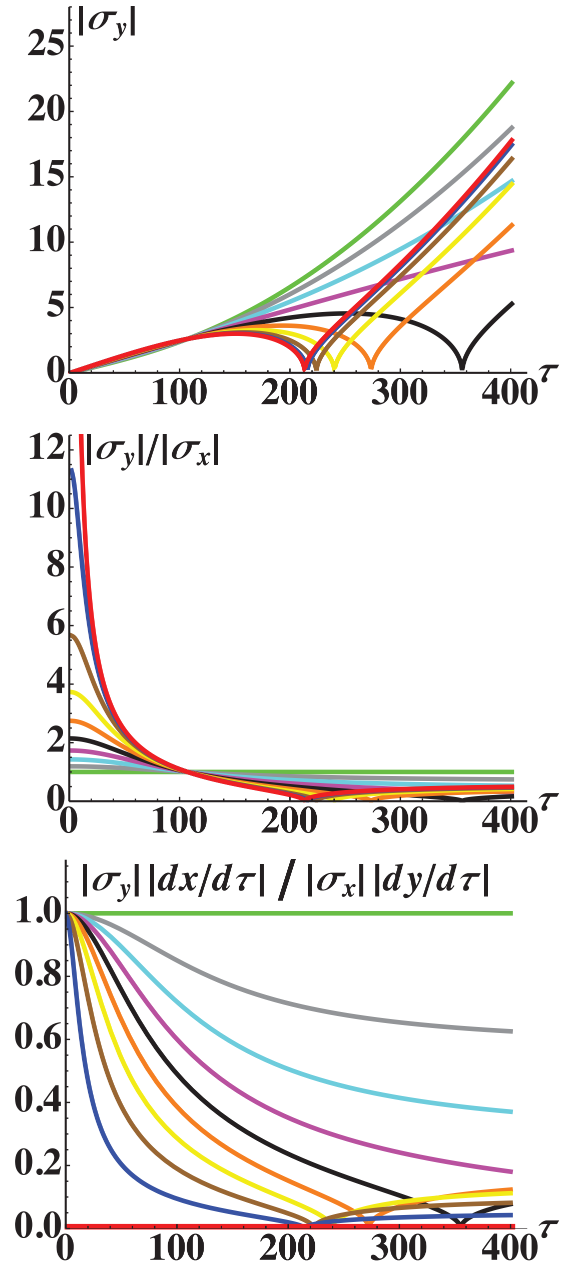

FIG. 10. QL formalism for $\theta_0 = 45°$ (green line), $\theta_0 = 50°$ (gray line), $\theta_0 = 55°$ (cyan line), $\theta_0 = 60°$ (magenta line), $\theta_0 = 65°$ (black line), $\theta_0 = 70°$ (orange line), $\theta_0 = 75°$ (yellow line), $\theta_0 = 80°$ (brown line), $\theta_0 = 85°$ (blue line), and $\theta_0 = 90°$ (red line) and for an $N_q \times N_\theta = 100 \times 100$ multimode isotropic turbulent spectrum with $\delta n_{e0} = 4\%$ and $q_{\max} = 0.04$: rms spread $|\sigma_y|$ in the y direction, ratio $|\sigma_y|/|\sigma_x|$ between the rms spreads in the y and x directions, and ratio $|\sigma_y||dx/d\tau|/|\sigma_x||dy/d\tau|$ between the two terms contributing to the perpendicular rms spread $\sigma_\perp$ as functions of the time τ. Cusps visible when σ_y goes through 0 identify curves for which σ_y^2 becomes negative.

quantities characterizing the rays, such as the average traveled distance, the spatial rms perpendicular spreading, and the average wave-vector components. An interesting feature to note is that, somewhere in the QL calculation, $\sigma_x^2 \equiv \langle \delta x \delta x \rangle$ or $\sigma_y^2 \equiv \langle \delta y \delta y \rangle$ can take unphysical negative values, turning $\sigma_\perp$ in (25) into a complex number, which can be detected by slight discontinuities in some of the solid curves in Fig. 9 (namely, for $\theta_0 = 65°$ and $\theta_0 = 75°$). This problem can nonetheless be alleviated and effectively circumvented since, as illustrated in Fig. 10 for launching angles in the upper half of the first quadrant, when σ_y^2 becomes negative the contribution to $\sigma_\perp$ of the term that becomes imaginary is negligible (roughly 12% at most), so it can simply be discarded without practically affecting the final outcome of (25). This behavior is the combined result of the smallness of $|\sigma_y|$, compared with $|\sigma_x|$, when the anomalous imaginary values of σ_y set in and of the fact that, when sweeping different values of θ_0 in this half-quadrant, $|dx/d\tau|$ approaches 0 as θ_0 approaches $\pi/2$. Similar explanations apply, *mutatis mutandi*, to the behaviors observed in the other half-quadrants, taking into account the reflection symmetries around the four axes already mentioned.

IV. DISCUSSION AND CONCLUSIONS

In this paper, a new QL formalism has been developed to describe the propagation of optical or acoustic (or other) rays in random media that relies on consistently expanding the dispersion relation and the ray equations up to (and including) second-order terms in the fluctuations, while simultaneously splitting the ray into average plus fluctuating terms and subsequently carrying out an ensemble-average operation that yields equations governing the average ray and the rms spreading about it. The QL designation is imported from the kinetic theory of waves in weakly turbulent plasmas [31,32] and means that one keeps second-order terms in the perturbations, thus implying that there is a slow drift of the average ray with respect to its unperturbed trajectory that is induced by the ensemble averages of the squared fluctuations. This approach comes as an efficient alternative to MC calculations and is not without similarity to previously proposed methods for so-called statistical ray tracing [26], although it appears to be much easier to implement in the case of more complex geometries or dispersion relations (as happens when tracing rays in tokamaks [24,25,27,28]), since it makes use of the Hamiltonian framework [29–31]. The method presented here assumes that the dispersion relation can be solved explicitly for the frequency variable, but if that is not possible analytically (as is generally the case in fusion-grade plasmas [24,25,28,31]), there is apparently no difficulty in going back from Hamilton's equations to the equations having in the denominator the frequency derivative of the dispersion relation and proceeding straightforwardly with a similar (albeit lengthier) expansion in the fluctuating quantities. The only conundrum of the QL formalism is the existence of a downward recursion relation by means of which the evolution of quadratic means involving a given-order derivative of medium fluctuations depends on the derivatives immediately one order higher, so that such a recurrence needs to be truncated when applying the method to actual problems. It has been shown, for the case of a single random mode (in which the recursion relation becomes exactly closed), that results do converge to the correct outcome when going up in the order at which such a recurrence is truncated and, in addition, it has been verified, for the realistic case of a multimode turbulent spectrum, that practical convergence becomes effective already at the lowest orders (as early as when neglecting the contribution of fourth- and higher-order derivatives of the medium fluctuations).

The QL approach has been benchmarked against MC calculations for rays propagating in homogeneous media on which single-mode or isotropic multimode turbulent spectra are superposed, which has enabled some exercises of validation and verification, respectively, of the model and of its numerical implementation [35]. Overall, the QL and MC results compare pretty well, the former being particularly robust regarding the ensemble average of ray trajectories and the rms width resulting from spreading of the ray pencil in the direction perpendicular to the average ray, being also very good in reproducing the ensemble averages of the wave-vector components. Deviations between the outcomes of QL and MC calculations can be more pronounced in the rms spreads of wave-vector components, but these are usually not the most sought quantities in ray tracing (although they need to be traced simultaneously with the other ray quantities for completeness of the whole QL system of equations), and they correspond to rather small normalized values. The tests of the QL formalism have ben carried out using, for the amplitude of the medium fluctuations, a value commensurate to those reported in related works for the levels of density fluctuations in plasmas [24–28] and deemed to be still within the validity of the perturbative expansion up to second order. As for the wave numbers of the turbulent modes, their maximum value has been chosen so that the eikonal approximation remains effective, its being clear that the agreement between QL and MC worsens if the level of fluctuations or the maximum wave number in the turbulent spectrum are increased. The QL method that has been presented is obviously not limited to density fluctuations and can be straightforwardly extended and applied to ray tracing in any random media when the impact of fluctuations on ray propagation enters the dispersion relation (or index of refraction) via any physical quantity that can be split into an average plus a fluctuating term.

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APPENDIX A: QL RAY EQUATIONS IN HOMOGENEOUS RANDOM MEDIA

The QL set of ray equations derived under the conditions in Sec. III A reads

$$\frac{n_0}{c}\frac{d\langle r_i\rangle}{dt} \simeq \frac{\langle k_i\rangle}{(\langle k_j\rangle\langle k_j\rangle)^{\frac{1}{2}}} + \frac{3\langle k_i\rangle\langle k_j\rangle\langle k_l\rangle\langle\delta k_j\delta k_l\rangle}{2(\langle k_m\rangle\langle k_m\rangle)^{\frac{5}{2}}} - \frac{\langle k_i\rangle\langle\delta k_j\delta k_j\rangle + 2\langle k_j\rangle\langle\delta k_j\delta k_i\rangle}{2(\langle k_l\rangle\langle k_l\rangle)^{\frac{3}{2}}} - \frac{\langle\delta k_i\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_j\rangle\langle k_j\rangle)^{\frac{1}{2}}\langle n_e\rangle}$$
$$+ \frac{\langle k_i\rangle\langle k_j\rangle\langle\delta k_j\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{3}{2}}\langle n_e\rangle} - \frac{\langle k_i\rangle}{(\langle k_j\rangle\langle k_j\rangle)^{\frac{1}{2}}\langle n_e\rangle}\left\langle\delta r_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle + \frac{\langle k_i\rangle\langle\delta n_e(\langle\mathbf{r}\rangle)\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_j\rangle\langle k_j\rangle)^{\frac{1}{2}}\langle n_e\rangle^2}, \tag{A1}$$

$$\frac{n_0}{c}\frac{d\langle k_i\rangle}{dt}\simeq\frac{\langle k_j\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}\langle n_e\rangle}\left\langle\delta k_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle+\frac{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}{\langle n_e\rangle}\left\langle\delta r_j\frac{\partial^2\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j\partial r_i}\right\rangle-\frac{2(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}{\langle n_e\rangle^2}\left\langle\delta n_e(\langle\mathbf{r}\rangle)\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle, \tag{A2}$$

$$\frac{n_0}{c}\frac{d\langle\delta r_i\delta r_j\rangle}{dt}\simeq\frac{\langle\delta r_i\delta k_j\rangle+\langle\delta r_j\delta k_i\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}-\frac{\langle k_i\rangle\langle k_l\rangle\langle\delta k_l\delta r_j\rangle+\langle k_j\rangle\langle k_l\rangle\langle\delta k_l\delta r_i\rangle}{(\langle k_m\rangle\langle k_m\rangle)^{\frac{3}{2}}}-\frac{\langle k_i\rangle\langle\delta r_j\delta n_e(\langle\mathbf{r}\rangle)\rangle+\langle k_j\rangle\langle\delta r_i\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}\langle n_e\rangle}, \tag{A3}$$

$$\frac{n_0}{c}\frac{d\langle\delta k_i\delta k_j\rangle}{dt}\simeq\frac{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}{\langle n_e\rangle}\left[\left\langle\delta k_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle+\left\langle\delta k_j\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle\right], \tag{A4}$$

$$\frac{n_0}{c}\frac{d\langle\delta r_i\delta k_j\rangle}{dt}\simeq\frac{\langle\delta k_i\delta k_j\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}-\frac{\langle k_i\rangle\langle k_l\rangle\langle\delta k_l\delta k_j\rangle}{(\langle k_m\rangle\langle k_m\rangle)^{\frac{3}{2}}}-\frac{\langle k_i\rangle\langle\delta k_j\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}\langle n_e\rangle}+\frac{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}{\langle n_e\rangle}\left\langle\delta r_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle, \tag{A5}$$

$$\frac{n_0}{c}\frac{d\langle\delta r_i\delta n_e(\langle\mathbf{r}\rangle)\rangle}{dt}\simeq\left\langle\delta r_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle\frac{n_0}{c}\frac{d\langle r_j\rangle}{dt}+\frac{\langle\delta k_i\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}-\frac{\langle k_i\rangle\langle k_j\rangle\langle\delta k_j\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{3}{2}}}-\frac{\langle k_i\rangle\langle\delta n_e(\langle\mathbf{r}\rangle)\delta n_e(\langle\mathbf{r}\rangle)\rangle}{(\langle k_j\rangle\langle k_j\rangle)^{\frac{1}{2}}\langle n_e\rangle}, \tag{A6}$$

$$\frac{n_0}{c}\frac{d\langle\delta k_i\delta n_e(\langle\mathbf{r}\rangle)\rangle}{dt}\simeq\left\langle\delta k_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle\frac{n_0}{c}\frac{d\langle r_j\rangle}{dt}+\frac{(\langle k_j\rangle\langle k_j\rangle)^{\frac{1}{2}}}{\langle n_e\rangle}\left\langle\delta n_e(\langle\mathbf{r}\rangle)\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\right\rangle, \tag{A7}$$

$$\begin{aligned}\frac{n_0}{c}\frac{d}{dt}\left\langle\delta r_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle&\simeq\left\langle\delta r_i\frac{\partial^2\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j\partial r_l}\right\rangle\frac{n_0}{c}\frac{d\langle r_l\rangle}{dt}+\frac{1}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}\left\langle\delta k_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle-\frac{\langle k_i\rangle\langle k_l\rangle}{(\langle k_m\rangle\langle k_m\rangle)^{\frac{3}{2}}}\left\langle\delta k_l\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle\\&\quad-\frac{\langle k_i\rangle}{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}\langle\delta n_e\rangle}\left\langle\delta n_e(\langle\mathbf{r}\rangle)\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle,\end{aligned} \tag{A8}$$

and

$$\frac{n_0}{c}\frac{d}{dt}\left\langle\delta k_i\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle\simeq\left\langle\delta k_i\frac{\partial^2\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j\partial r_l}\right\rangle\frac{n_0}{c}\frac{d\langle r_l\rangle}{dt}+\frac{(\langle k_l\rangle\langle k_l\rangle)^{\frac{1}{2}}}{\langle n_e\rangle}\left\langle\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_i}\frac{\partial\delta n_e(\langle\mathbf{r}\rangle)}{\partial r_j}\right\rangle. \tag{A9}$$

APPENDIX B: QL RAY EQUATIONS IN HOMOGENEOUS RANDOM MEDIA WITH SINGLE-MODE FLUCTUATIONS

Under the conditions in Sec. III B, the QL system of ray equations becomes

$$\begin{aligned}\frac{d\langle x[y]\rangle}{d\tau}&\simeq\frac{\langle\kappa_{x[y]}\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\left[1+\frac{(\delta n_{e0})^2}{2}\right]+\frac{3\langle\kappa_{x[y]}\rangle(\langle\kappa_x\rangle^2\langle\delta\kappa_x\delta\kappa_x\rangle+\langle\kappa_y\rangle^2\langle\delta\kappa_y\delta\kappa_y\rangle+2\langle\kappa_x\rangle\langle\kappa_y\rangle\langle\delta\kappa_x\delta\kappa_y\rangle)}{2(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{5}{2}}}\\&\quad-\frac{\langle\kappa_{x[y]}\rangle(\langle\delta\kappa_x\delta\kappa_x\rangle+\langle\delta\kappa_y\delta\kappa_y\rangle)+2(\langle\kappa_x\rangle\langle\delta\kappa_x\delta\kappa_{x[y]}\rangle+\langle\kappa_y\rangle\langle\delta\kappa_y\delta\kappa_{x[y]}\rangle)}{2(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}}-\frac{\langle\delta\kappa_{x[y]}\delta n_e(\langle x\rangle)\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\\&\quad+\frac{\langle\kappa_{x[y]}\rangle[\langle\kappa_x\rangle\langle\delta\kappa_x\delta n_e(\langle x\rangle)\rangle+\langle\kappa_y\rangle\langle\delta\kappa_y\delta n_e(\langle x\rangle)\rangle]}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}}-\frac{\langle\kappa_{x[y]}\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\left\langle\delta x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle,\end{aligned} \tag{B1}$$

$$\begin{aligned}&\frac{d\langle\kappa_x\rangle}{d\tau}\simeq\frac{1}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\left[\langle\kappa_x\rangle\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle+\langle\kappa_y\rangle\left\langle\delta\kappa_y\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle\right]-q^2(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}\langle\delta x\delta n_e(\langle x\rangle)\rangle\quad\text{and}\\&\frac{d\langle\kappa_y\rangle}{d\tau}\simeq 0,\end{aligned} \tag{B2}$$

$$\begin{aligned}&\frac{d\langle\delta x[y]\delta x[y]\rangle}{d\tau}\simeq\frac{2\langle\delta x[y]\delta\kappa_{x[y]}\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}-\frac{2(\langle\kappa_x\rangle\langle\kappa_{x[y]}\rangle\langle\delta x[y]\delta\kappa_x\rangle+\langle\kappa_y\rangle\langle\kappa_{x[y]}\rangle\langle\delta x[y]\delta\kappa_y\rangle)}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}}-\frac{2\langle\kappa_{x[y]}\rangle\langle\delta x[y]\delta n_e(\langle x\rangle)\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\quad\text{and}\\&\frac{d\langle\delta x\delta y\rangle}{d\tau}\simeq\frac{\langle\delta x\delta\kappa_y\rangle+\langle\delta y\delta\kappa_x\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}-\frac{\langle\kappa_x\rangle^2\langle\delta y\delta\kappa_x\rangle+\langle\kappa_y\rangle^2\langle\delta x\delta\kappa_y\rangle+\langle\kappa_x\rangle\langle\kappa_y\rangle(\langle\delta x\delta\kappa_x\rangle+\langle\delta y\delta\kappa_y\rangle)}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}}\\&\qquad-\frac{\langle\kappa_x\rangle\langle\delta y\delta n_e(\langle x\rangle)\rangle+\langle\kappa_y\rangle\langle\delta x\delta n_e(\langle x\rangle)\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}},\end{aligned} \tag{B3}$$

$$\begin{aligned}&\frac{d\langle\delta\kappa_x\delta\kappa_x\rangle}{d\tau}\simeq 2(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle,\quad\frac{d\langle\delta\kappa_x\delta\kappa_y\rangle}{d\tau}\simeq(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}\left\langle\delta\kappa_y\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle,\quad\text{and}\\&\frac{d\langle\delta\kappa_y\delta\kappa_y\rangle}{d\tau}\simeq 0,\end{aligned} \tag{B4}$$

$$\frac{d\langle\delta x[y]\delta\kappa_x\rangle}{d\tau} \simeq \frac{\langle\delta\kappa_{x[y]}\delta\kappa_x\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}} - \frac{\langle\kappa_{x[y]}\rangle(\langle\kappa_x\rangle\langle\delta\kappa_x\delta\kappa_x\rangle+\langle\kappa_y\rangle\langle\delta\kappa_x\delta\kappa_y\rangle)}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}} - \frac{\langle\kappa_{x[y]}\rangle\langle\delta\kappa_x\delta n_e(\langle x\rangle)\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}} + (\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}\left\langle\delta x[y]\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle \quad \text{and}$$

$$\frac{d\langle\delta x[y]\delta\kappa_y\rangle}{d\tau} \simeq \frac{\langle\delta\kappa_{x[y]}\delta\kappa_y\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}} - \frac{\langle\kappa_{x[y]}\rangle(\langle\kappa_x\rangle\langle\delta\kappa_x\delta\kappa_y\rangle+\langle\kappa_y\rangle\langle\delta\kappa_y\delta\kappa_y\rangle)}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}} - \frac{\langle\kappa_{x[y]}\rangle\langle\delta\kappa_y\delta n_e(\langle x\rangle)\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}, \tag{B5}$$

$$\frac{d\langle\delta x[y]\delta n_e(\langle x\rangle)\rangle}{d\tau} \simeq \left\langle\delta x[y]\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle\frac{d\langle x\rangle}{d\tau} + \frac{\langle\delta\kappa_{x[y]}\delta n_e(\langle x\rangle)\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}} - \frac{\langle\kappa_{x[y]}\rangle[\langle\kappa_x\rangle\langle\delta\kappa_x\delta n_e(\langle x\rangle)\rangle+\langle\kappa_y\rangle\langle\delta\kappa_y\delta n_e(\langle x\rangle)\rangle]}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}} - \frac{(\delta n_{e0})^2}{2}\frac{\langle\kappa_{x[y]}\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}, \tag{B6}$$

$$\frac{d\langle\delta\kappa_{x[y]}\delta n_e(\langle x\rangle)\rangle}{d\tau} \simeq \left\langle\delta\kappa_{x[y]}\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle\frac{d\langle x\rangle}{d\tau}, \tag{B7}$$

$$\frac{d}{d\tau}\left\langle\delta x[y]\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle \simeq -q^2\langle\delta x[y]\delta n_e(\langle x\rangle)\rangle\frac{d\langle x\rangle}{d\tau} + \frac{1}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\left\langle\delta\kappa_{x[y]}\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle - \frac{\langle\kappa_{x[y]}\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}}\left[\langle\kappa_x\rangle\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle + \langle\kappa_y\rangle\left\langle\delta\kappa_y\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle\right], \tag{B8}$$

and

$$\frac{d}{d\tau}\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle \simeq -q^2\langle\delta\kappa_x\delta n_e(\langle x\rangle)\rangle\frac{d\langle x\rangle}{d\tau} + q^2\frac{(\delta n_{e0})^2}{2}(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}} \quad \text{and}$$

$$\frac{d}{d\tau}\left\langle\delta\kappa_y\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle \simeq -q^2\langle\delta\kappa_y\delta n_e(\langle x\rangle)\rangle\frac{d\langle x\rangle}{d\tau}, \tag{B9}$$

where x and y can be exchanged consistently in every $x[y]$ throughout the same expression.

If the recurrence (34) is truncated numerically, instead of being closed via (41), (B8) and (B9) are replaced with

$$\frac{d}{d\tau}\left\langle\delta x[y]\frac{\partial^m\delta n_e(\langle x\rangle)}{\partial x^m}\right\rangle \simeq \left\langle\delta x[y]\frac{\partial^{m+1}\delta n_e(\langle x\rangle)}{\partial x^{m+1}}\right\rangle\frac{d\langle x\rangle}{d\tau} + \frac{1}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\left\langle\delta\kappa_{x[y]}\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle - \frac{\langle\kappa_{x[y]}\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{3}{2}}}\left[\langle\kappa_x\rangle\left\langle\delta\kappa_x\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle + \langle\kappa_y\rangle\left\langle\delta\kappa_y\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\right\rangle\right] - \frac{\langle\kappa_{x[y]}\rangle}{(\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}}\left\langle\delta n_e(\langle x\rangle)\frac{\partial^m\delta n_e(\langle x\rangle)}{\partial x^m}\right\rangle \tag{B10}$$

and

$$\frac{d}{d\tau}\left\langle\delta\kappa_{x[y]}\frac{\partial^m\delta n_e(\langle x\rangle)}{\partial x^m}\right\rangle \simeq \left\langle\delta\kappa_{x[y]}\frac{\partial^{m+1}\delta n_e(\langle x\rangle)}{\partial x^{m+1}}\right\rangle\frac{d\langle x\rangle}{d\tau} + (\langle\kappa_x\rangle^2+\langle\kappa_y\rangle^2)^{\frac{1}{2}}\left\langle\frac{\partial\delta n_e(\langle x\rangle)}{\partial x}\frac{\partial^m\delta n_e(\langle x\rangle)}{\partial x^m}\right\rangle, \tag{B11}$$

with

$$\left\langle\frac{\partial^n\delta n_e(\langle x\rangle)}{\partial x^n}\frac{\partial^m\delta n_e(\langle x\rangle)}{\partial x^m}\right\rangle = i^{n+m}\frac{(-1)^n+(-1)^m}{2}q^{n+m}\frac{(\delta n_{e0})^2}{2} \tag{B12}$$

for $n = 0, 1$ and i the imaginary unit.

[1] L. A. Chernov, *Wave Propagation in a Random Medium*, translated by R. A. Silverman (McGraw–Hill, New York, 1960).

[2] V. I. Tatarski, *Wave Propagation in a Turbulent Medium*, translated by R. A. Silverman (McGraw–Hill, New York, 1961).

[3] V. I. Tatarski, *The Effects of the Turbulent Atmosphere on Wave Propagation*, translated by J. W. Strohbehn (Israel Program of Scientific Translations, Jerusalem, 1971).

[4] J. W. Strohbehn, Proc. IEEE **56**, 1301 (1968).

[5] Yu. A. Kravtsov, L. A. Ostrovsky, and N. S. Stepanov, Proc. IEEE **62**, 1492 (1974).

[6] A. M. Prokhorov, F. V. Bunkin, K. S. Gochelashvily, and V. I. Shishov, Proc. IEEE **63**, 790 (1975).

[7] R. L. Fante, Proc. IEEE **63**, 1669 (1975).

[8] A. Ishimaru, Proc. IEEE **65**, 1030 (1977).

[9] S. F. Clifford, in *Laser Beam Propagation in the Atmosphere*, edited by J. W. Strohbehn (Springer-Verlag, Berlin, 1978), p. 9.
[10] J. W. Strohbehn, in *Laser Beam Propagation in the Atmosphere*, edited by J. W. Strohbehn (Springer-Verlag, Berlin, 1978), p. 45.
[11] B. J. West, Phys. Rev. A **18**, 1646 (1978).
[12] Yu. A. Kravtsov, Rep. Prog. Phys. **55**, 39 (1992).
[13] P. M. Kintner, B. M. Ledvina, and E. R. de Paula, Space Weather **5**, S09003 (2007).
[14] F. Wang, X. Liu, and Y. Cai, Prog. Electromagn. Res. **150**, 123 (2015).
[15] M. Karweit, Ph. Blanc-Benon, D. Juvé, and G. Comte-Bellot, J. Acoust. Soc. Am. **89**, 52 (1991).
[16] B. White, B. Nair, and A. Bayliss, Geophysics **53**, 903 (1988).
[17] R. Snieder and A. Lomax, Geophys. J. Int. **125**, 796 (1996).
[18] A. M. Baig and F. A. Dahlen, Geophys. J. Int. **158**, 187 (2004).
[19] B. Lü, M. Darmon, and C. Potel, J. Appl. Phys. **112**, 054902 (2012).
[20] J. M. Cordes, A. Pldwerbetsky, and R. V. E. Lovelace, Astrophys. J. **310**, 737 (1986).
[21] A. Pldwerbetsky and R. V. E. Lovelace, Phys. Lett. A **140**, 411 (1989).
[22] E. J. Heller, L. Kaplan, and A. Dahlen, J. Geophys. Res. **113**, C09023 (2008).
[23] J. J. Metzger, R. Fleischmann, and T. Geisel, Phys. Rev. Lett. **111**, 013901 (2013).
[24] B. Hui, E. Ott, P. T. Bonoli, and P. N. Guzdar, Nucl. Fusion **21**, 339 (1981).
[25] P. T. Bonoli and E. Ott, Phys. Fluids **25**, 359 (1982).
[26] R. Epstein and R. S. Craxton, Phys. Rev. A **33**, 1892 (1986).
[27] F. R. Hansen, J. P. Lynov, P. Michelsen, and H. L. Pécseli, Nucl. Fusion **28**, 769 (1988).
[28] G. Vahala, L. Vahala, and P. T. Bonoli, Phys. Fluids B **4**, 4033 (1992).
[29] S. Weinberg, Phys. Rev. **126**, 1899 (1962).
[30] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 3rd ed. (Pergamon, Oxford, UK, 1971).
[31] T. H. Stix, *Waves in Plasmas* (AIP, New York, 1982).
[32] A. A. Vedenov, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1967), Vol. 3, p. 229.
[33] G. Grimmett and D. Stirzaker, *Probability and Random Processes*, 3rd. ed. (Oxford University Press, Oxford, UK, 2001).
[34] E. B. Rosenthal, Math. Teacher **54**, 336 (1961).
[35] C. Holland, Phys. Plasmas **23**, 060901 (2016).
[36] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., edited by A. Jeffrey (Academic Press, San Diego, CA, 1994).