

Repulsive Casimir forceJohan S. Høye¹ and Iver Brevik²¹*Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway*²*Department of Energy and Process Engineering, Norwegian University of Science and Technology, N-7491 Trondheim, Norway*

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The Casimir force between two parallel thick plates, one perfectly dielectric, the other purely magnetic, was calculated long ago by Boyer [T. H. Boyer, *Phys. Rev. A* **9**, 2078 (1974)]. Its most characteristic property is that it is repulsive. The problem of this repulsive behavior is actually delicate and counterintuitive. In the present paper we analyze the problem by first considering the simple harmonic oscillator model introduced by us earlier [J. S. Høye *et al.*, *Phys. Rev. E* **67**, 056116 (2003); *Phys. Rev. A* **94**, 032113 (2016)]. Extension of this model shows how the repulsive behavior can be understood on a microscopic basis, due to the duality between canonical and mechanical momenta in the presence of the electromagnetic vector potential. This duality corresponds to the transverse magnetic and transverse electric modes in electrodynamics. We analyze the generalized Boyer case where the permittivities and permeabilities of the parallel plates are arbitrary. In this respect we first find the induced interaction between a pair of particles with given electric and magnetic polarizabilities and then find it for a pair of parallel plates. The method used for our evaluations is the statistical mechanical one that we introduced and applied earlier. Whether the pair of particles or plates attract or repel each other depends on their polarizabilities or permittivities and permeabilities respectively. For equal particles or equal plates there is always attraction.

DOI: [10.1103/PhysRevA.98.022503](https://doi.org/10.1103/PhysRevA.98.022503)**I. INTRODUCTION**

The Casimir force between two equal parallel isotropic non-magnetic materials separated by a vacuum (air) gap a is under usual circumstances known to be attractive (for reviews on the Casimir effect, see, for instance, Refs. [1–3]). It turns out to be possible, however, in some cases to make the Casimir force switch sign so that it becomes repulsive. Typically, this occurs if a special liquid like bromobenzene is immersed between gold and silica surfaces as shown by Munday *et al.* [4]. Then Casimir repulsion takes place at distances above approximately 5 nm since the dielectric constant of bromobenzene lies between those of gold and silica for imaginary frequencies below a certain value [5–7].

The analog case of two equal purely magnetic plates behaves in the same way; the Casimir force is attractive under usual circumstances. The case of two unequal plates—one purely dielectric and the other purely magnetic—is, however, different in the sense that the Casimir force becomes *repulsive*. This peculiar effect has been known for several years. A well-known reference in this area is the paper of Boyer [8] (the present problem is sometimes called the Boyer problem). Based upon earlier investigations of Feinberg and Sucher [9] on the van der Waals force between two electrically and magnetically particles, he calculated the repulsive force between a conducting and a permeable plate in the limit $\epsilon \rightarrow \infty$, $\mu \rightarrow \infty$. Boyer's formalism was based upon so-called random electrodynamics.

We ought to point out here that this change of force direction is actually quite nontrivial. Consider for definiteness the left plate with boundary $z = 0$ to be purely dielectric, with finite permittivity $\epsilon > 1$, while the right plate with boundary $z = a$ is purely magnetic with permeability $\mu > 1$. In the

boundary region of the dielectric plate around $z = 0$ the only electromagnetic volume force density is $-\frac{1}{2}\epsilon_0 E^2 \nabla \epsilon$. Here the E is the magnitude of the varying electric field from the outside to the inside of the dielectric boundary. This should be expected to give a surface force density acting in the positive z direction, i.e. toward the vacuum region, in accordance with the general property of classical electrodynamics saying that the surface force acts in the direction of the optically thinner medium. In our case the force direction is, however, reversed. Taken literally, this has to be interpreted to imply that the square E^2 becomes negative, thus a counterintuitive result whose physical significance is not obvious.

We mention that the problem has been considered in the literature from a wider viewpoint also, not restricted to the idealized case of Boyer where one plate was electric and the other magnetic. Some papers dealing with these topics are Refs. [10–14]. Most of them, such as Refs. [11–13], analyze Casimir repulsive forces in magnetodielectric configurations, while the very recent Ref. [14] analyzes the repulsive force on a magnetic point particle near a surface. In general, whether the force is found to be repulsive or attractive depends on the relative strengths of ϵ_i and μ_i ($i = 1, 2$). The delicate point mentioned above, that a Casimir calculation leads to an effective square $E^2 < 0$ while always $E^2 \geq 0$ for E real, will not be investigated further in this work. In the present paper our purpose is instead to analyze the Boyer problem in a generalized sense, i.e., take the polarizabilities of the two particles or the permittivities and permeabilities to be general.

We start from a microscopic harmonic oscillator model where two oscillators interact via a third one. We have employed this model before [15]. The model was later extended to cover the case where the third oscillator describes a set of

oscillators [16]. The two oscillators of the model correspond to two polarizable particles that interact via the harmonic oscillator modes of the electromagnetic field. For the latter there are two types of waves that are clearly noticed when a pair of parallel polarizable plates are considered. They are the transverse magnetic (TM) and transverse electric (TE) waves or modes. The former is the one most easily understood in view of the induced attractive interaction. Also the induced interaction increases in magnitude with temperature to reach the classical limit. For the TE mode, however, the situation is less obvious as the force decreases to zero in the classical limit since its Matsubara frequency $\zeta = 0$ does not contribute. To cope with this situation, interaction terms similar to the interaction of a particle with the electromagnetic vector potential were used.

To obtain the situation corresponding to a dielectric plate interacting with a magnetic one, we find that our model can be extended in a straightforward way to encompass such a situation. During computations two quantities D_1 and D_2 appear [defined in Eq. (9) below], where their product $D_1 D_2 > 0$ determines an induced attractive force. Now one notes that for the analog of the TM case both quantities are positive while for the TE case both are negative by which $D_1 D_2 > 0$ remains.

We will show how the model can be extended such that D_1 and D_2 have opposite signs. This will be the analog of a pair of dielectric and magnetic plates that repel each other.

In Sec. II we consider the harmonic oscillator model, showing how the transition from attractive to repulsive force between dipoles can be traced back to the fundamental distinction between mechanical momentum and canonical momentum in classical electrodynamics. In Sec. III we derive the induced interaction between a pair of particles where there are couplings between both electric and magnetic dipole moments. In Sec. IV we generalize the statistical mechanical derivations of Ref. [18] to two dielectric half-planes with different dielectric constants. From this the well-known Lifshitz result is recovered. Finally, in Sec. V this is generalized further to obtain the Casimir force with magnetic properties included. Section VI summarizes the obtained results.

II. HARMONIC OSCILLATOR MODEL FOR REPULSIVE CASIMIR FORCE

Now consider the harmonic oscillator model treated in Refs. [15] and [16]. We will here only sketch the previous derivations while more details can be found in the mentioned references. Let the three oscillators have eigenfrequencies ω_i ($i = 1, 2, 3$). Their classical partition function is then proportional to the inverse of \sqrt{Q} where

$$Q = a_1 a_2 a_3, \quad a_i = \omega_i^2 \quad (i = 1, 2, 3). \quad (1)$$

For simplicity all three oscillators of the model are one dimensional.

By quantization using the path integral method [17,19] the classical system turns out to be split into a set of classical harmonic oscillator systems where the Matsubara frequencies are added to the eigenfrequencies. This replaces expression (1) by

$$Q = A_1 A_2 A_3, \quad A_i = a_i + \zeta^2 = \omega_i^2 + \zeta^2, \quad (2)$$

where $\zeta = i\omega$ (the convention $\zeta = -i\omega$ may also be used).

When the oscillators interact (via bilinear terms) the Q can be expressed through a determinant. With interaction parameter c for the interactions via oscillator 3, one obtains the result in Eq. (9) of Ref. [16] [or Eq. (4.3a) of [15]]:

$$\begin{aligned} Q &= \begin{vmatrix} A_1 & 0 & c \\ 0 & A_2 & c \\ c & c & A_3 \end{vmatrix} = A_1 A_2 A_3 - c^2 (A_1 + A_2) \\ &= A_1 A_2 A_3 (1 - D_1 - D_2) \\ &= A_1 A_2 A_3 (1 - D_1)(1 - D_2) \left(1 - \frac{D_1 D_2}{(1 - D_1)(1 - D_2)} \right), \end{aligned} \quad (3)$$

$$D_j = \frac{c^2}{A_j A_3} \quad (j = 1, 2). \quad (4)$$

Here the $A_j(1 - D_j)$ terms represent each of the two oscillators and their radiation reaction with the third oscillator while the last factor corresponds to the Casimir free energy. With D_j given by Eq. (4) this factor is larger than zero which means that the induced force between the two oscillators is attractive. This situation is the analog of the TM mode.

The second situation is the analog of the TE mode. To analyze this we may start from the situation where oscillator 3 interacts with the momenta of oscillators 1 and 2. Then the interaction with the third oscillator is like the interaction via the electromagnetic vector field. With this field the Hamiltonian for a particle j is $[\mathbf{p}_j - (e/c)\mathbf{A}]^2/2m_j$, where here c is the velocity of light, \mathbf{p}_j the canonical momentum, and $\mathbf{A} = \mathbf{A}(\mathbf{r}_j)$ the vector potential (in Gaussian units). The mechanical momentum is $\mathbf{p}_{Mj} = \mathbf{p}_j - (e/c)\mathbf{A}$. In our model the analogous interaction can be written as $a_j[p_j - (c/a_j)x_3]^2$ ($j = 1, 2$), where again c is the coupling parameter and the coordinate x_3 corresponds formally to the vector potential \mathbf{A} .

Now for harmonic oscillators the roles of the momenta and coordinates can be exchanged to obtain the energy term

$$a_j \left(x_j - \frac{c}{a_j} x_3 \right)^2. \quad (5)$$

With this the A_3 term in the determinant (3) is changed into

$$A_3 \rightarrow A_3 + \frac{c^2}{a_1} + \frac{c^2}{a_2}. \quad (6)$$

The rows $j = 1, 2$ of the determinant can be multiplied with c/a_1 and c/a_2 respectively to be subtracted from row 3. Then rows 1 and 2 can be multiplied with $\zeta/\sqrt{a_j}$, and columns 1 and 2 can be divided with the same factors. These operations do not change the determinant. We obtain

$$\begin{aligned} Q &= \begin{vmatrix} A_1 & 0 & c \\ 0 & A_2 & c \\ cq_1 & cq_2 & A_3 \end{vmatrix} \\ &= \begin{vmatrix} A_1 & 0 & \zeta c/\sqrt{a_1} \\ 0 & A_2 & \zeta c/\sqrt{a_2} \\ -\zeta c/\sqrt{a_1} & -\zeta c/\sqrt{a_2} & A_3 \end{vmatrix}, \end{aligned} \quad (7)$$

$$q_j = 1 - \frac{A_j}{a_j} = -\frac{\zeta^2}{a_j}, \quad (j = 1, 2). \quad (8)$$

With determinant (7) the Q when evaluated can still be written as result (3) but now with

$$D_j = -\frac{\zeta^2 c^2}{a_j A_j A_3} < 0. \quad (9)$$

Again the induced force will be attractive since $D_1 D_2 > 0$, but this time both factors are negative. However, in the classical limit where only $\zeta = 0$ contributes this force will vanish as also is the case for the TE mode.

From the above it is now easily seen that the induced force can be repulsive too. This will be the case if only one of the energies has form (5), say for $j = 2$ but not for $j = 1$. Then instead of Eq. (6) the change of A_3 will be

$$A_3 \rightarrow A_3 + \frac{c^2}{a_2}. \quad (10)$$

The determinant (7) will be modified to

$$Q = \begin{vmatrix} A_1 & 0 & c \\ 0 & A_2 & \zeta c/\sqrt{a_2} \\ c & -\zeta c/\sqrt{a_2} & A_3 \end{vmatrix}, \quad (11)$$

by which relation (9) changes into

$$D_1 = \frac{c^2}{A_1 A_3} > 0 \quad \text{and} \quad D_2 = -\frac{\zeta^2 c^2}{a_j A_j A_3} < 0. \quad (12)$$

With this $D_1 D_2 < 0$ by which the induced force will be repulsive. This force will be the analog of the repulsive force between a dielectric medium and a magnetic one.

III. INDUCED INTERACTION BETWEEN A PAIR OF PARTICLES WITH BOTH DIELECTRIC AND MAGNETIC POLARIZATION

In the static case electric and magnetic properties separate. However, for nonzero frequencies there is a coupling between electric and magnetic dipole moments. The fields are determined by Maxwell's equations which in standard notation can be written (Gaussian units)

$$\begin{aligned} \nabla \mathbf{D} &= 0, & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{c \partial t}, & \mathbf{D} &= \mathbf{E} + 4\pi \mathbf{P} = \varepsilon \mathbf{E}, \\ \nabla \mathbf{B} &= 0, & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{c \partial t}, & \mathbf{B} &= \mathbf{H} + 4\pi \mathbf{M} = \mu \mathbf{H}. \end{aligned} \quad (13)$$

The equations can be Fourier transformed by which

$$\nabla \rightarrow -i\mathbf{k} \quad \text{and} \quad \frac{\partial}{c \partial t} \rightarrow i\frac{\omega}{c} = \zeta. \quad (14)$$

(The signs of Fourier transforms depend upon the convention chosen.) Note, here and below, definition (14) is used for ζ .

For a given polarization one finds the solution [20]

$$\begin{aligned} \mathbf{E} &= -\frac{4\pi}{3} \left\{ \frac{k^2}{k^2 + \zeta^2} [3(\hat{\mathbf{k}} \cdot \mathbf{P})\hat{\mathbf{k}} - \mathbf{P}] + \left[\frac{2\zeta^2}{k^2 + \zeta^2} + 1 \right] \mathbf{P} \right\} \\ &= -\frac{4\pi}{3} \left\{ \frac{k^2}{k^2 + \zeta^2} 3(\hat{\mathbf{k}} \cdot \mathbf{P})\hat{\mathbf{k}} + \frac{3\zeta^2}{k^2 + \zeta^2} \mathbf{P} \right\}, \end{aligned} \quad (15)$$

where the hat is used for unit vectors. For simplicity the same vector notation is used in both r space and k space as the

argument used will specify. In r space Eq. (15) becomes [20]

$$\begin{aligned} \mathbf{E} = \mathbf{E}_1 &= \frac{e^{-x}}{r^3} \left\{ \left(1 + x + \frac{1}{3}x^2 \right) [3(\hat{\mathbf{r}} \cdot \mathbf{s}_1)\hat{\mathbf{r}} - \mathbf{s}_1] - \frac{2}{3}x^2 \mathbf{s}_1 \right\} \\ &\quad - \frac{4\pi}{3} \delta(\mathbf{r}) \mathbf{s}_1, \end{aligned} \quad (16)$$

where $x = \zeta r$. Here the polarization used is a point dipole

$$\mathbf{P} = \mathbf{P}(\mathbf{r}) = \mathbf{s}_1 \delta(\mathbf{r}), \quad [\mathbf{P} = \mathbf{P}(\mathbf{k}) = \mathbf{s}_1]. \quad (17)$$

The interaction with a dipole moment \mathbf{s}_2 at position \mathbf{r} is then $-\mathbf{E}_1 \mathbf{s}_2$ which also is proportional to the correlation function for the two dipoles \mathbf{s}_1 and \mathbf{s}_2 . Thus for (large) r the induced free energy is proportional to $\langle (\mathbf{E}_1 \mathbf{s}_2)^2 \rangle$ where the thermal average is over the fluctuating dipole moments for each value of ζ . With $\langle (\mathbf{a} \cdot \hat{\mathbf{s}})(\mathbf{b} \cdot \hat{\mathbf{s}}) \rangle = \mathbf{a} \cdot \mathbf{b}/3$ one obtains

$$\langle (\mathbf{E}_1 \cdot \mathbf{s}_2)^2 \rangle = \frac{1}{3} \langle s_1^2 \rangle \langle s_2^2 \rangle \frac{1}{r^6} L_{EE}(x), \quad (18)$$

$$L_{EE}(x) = e^{-2x} \left[2 \left(1 + x + \frac{1}{3}x^2 \right)^2 + \left(\frac{2}{3}x^2 \right)^2 \right]. \quad (19)$$

Now the electric polarizability is given by ($i = 1, 2$)

$$\alpha_{iE} = \alpha_{iE}(\zeta) \propto \langle s_i^2 \rangle. \quad (20)$$

With other details that follow from the derivations in Ref. [17], the induced free energy becomes Eq. (5.15) of the reference

$$F_{EE} = -\frac{3}{2\beta r^6} \sum_{n=-\infty}^{\infty} L_{EE}(x) \alpha_{1E} \alpha_{2E}, \quad (21)$$

with $\beta = 1/(k_B T)$ where k_B is Boltzmann's constant. (The factor 2 in the denominator is missing in the reference.) The sum is over the Matsubara frequencies

$$x = \zeta r = \frac{K r}{\hbar c}, \quad K = \frac{2\pi n}{\beta} \quad \text{with } n \text{ integer.} \quad (22)$$

At $T = 0$ the sum can be replaced by an integral with

$$dx = r d\zeta = \frac{2\pi r}{\hbar c \beta} dn \quad \text{or} \quad \sum_{n=-\infty}^{\infty} \rightarrow 2 \frac{\hbar c \beta}{2\pi r} \int_0^{\infty} dx. \quad (23)$$

With constant (i.e., independent of ζ) polarizabilities this further gives

$$F_{EE} = -\frac{3\hbar c \alpha_{1E} \alpha_{2E}}{2\pi r^7} I_{EE}, \quad (24)$$

$$I_{EE} = \int_0^{\infty} L_{EE}(x) dx = \frac{23}{6}, \quad (25)$$

which is the well-established result in Eq. (5.16) of Ref. [17].

Another situation is the induced interaction between the magnetic moments of the two particles. But due to the symmetry of Maxwell's equations (13) this will be precisely like results (21) and (24) with the subscript E replaced by H .

What remains is the induced interaction between electric and magnetic moments. To do so we need to obtain the L_{EH} function to replace L_{EE} and replace an electric polarizability with a magnetic one. So consider the electric field created by a

polarization. For nonzero frequencies it also creates a magnetic field. From Eqs. (13) and (14) one has

$$-i(\mathbf{k} \times \mathbf{E}) = -i\frac{\omega}{c}\mathbf{B} = -\zeta\mathbf{H}, \quad \mathbf{B} = \mathbf{H} \quad (\mathbf{M} = 0). \quad (26)$$

With the last equality of Eq. (15) this simplifies to

$$\mathbf{H} = \frac{i}{\zeta} \left[\mathbf{k} \times \left(\frac{-4\pi\zeta^2}{k^2 + \zeta^2} \mathbf{P} \right) \right] \quad (27)$$

(since $\hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$). From this follows $[\nabla \times (\psi \mathbf{P}) = (\nabla \psi) \times \mathbf{P} + \psi \nabla \times \mathbf{P}]$

$$\mathbf{H} = -\frac{1}{\zeta} \nabla \times \left(\frac{-\zeta^2 e^{-\zeta r}}{r} \mathbf{P} \right) = -\zeta(1 + \zeta r) \frac{e^{-\zeta r}}{r^2} (\hat{\mathbf{r}} \times \mathbf{P}). \quad (28)$$

So the interaction of an electric dipole moment \mathbf{s}_1 with a magnetic one \mathbf{s}_2 is

$$-\mathbf{H}_1 \mathbf{s}_2 = -\zeta(1 + \zeta r) \frac{e^{-\zeta r}}{r^2} (\hat{\mathbf{r}} \times \mathbf{s}_1) \cdot \mathbf{s}_2. \quad (29)$$

The average of interest is the square of Eq. (29), but now with a minus sign when compared with expression (18). This change of sign is due to a property of Fourier transforms. For two functions $f(x)$ and $g(x)$ and their Fourier transforms $\tilde{f}(k)$ and $\tilde{g}(k)$ one has the relation

$$\int f(x)g(x)dx = \frac{1}{2\pi} \int \tilde{f}(k)\tilde{g}(-k)dk. \quad (30)$$

This relation is similar for situations when x and/or k have discrete values too. In the present case the x corresponds to imaginary time with ζ the Fourier variable. In the free energy expression (21) the sum over ζ corresponds to the k integration in relation (30). The transform (29) changes sign when $\zeta \rightarrow -\zeta$ (as its $\zeta r = |\zeta|r$).

In standard notation one can write $(\mathbf{r} \times \mathbf{s}_1) \mathbf{s}_2 = \varepsilon_{ijk} x_i s_{1j} s_{2k}$ to obtain the average

$$\begin{aligned} \langle [(\mathbf{r} \times \mathbf{s}_1) \mathbf{s}_2]^2 \rangle &= \varepsilon_{ijk} \varepsilon_{npq} x_i x_n \langle s_{1j} s_{1p} \rangle \langle s_{2k} s_{2q} \rangle \\ &= \frac{2}{9} r^2 \langle s_1^2 \rangle \langle s_2^2 \rangle. \end{aligned} \quad (31)$$

Thus

$$\langle (\mathbf{H}_1 \cdot \mathbf{s}_2)^2 \rangle = \frac{1}{3} \langle s_1^2 \rangle \langle s_2^2 \rangle \frac{1}{r^6} L_{EH}(x), \quad (32)$$

$$L_{EH}(x) = \frac{2}{3} e^{-2x} [x(1+x)]^2. \quad (33)$$

Like Eq. (21), but now with opposite sign, the induced free energy expression becomes

$$F_{EH} = \frac{3}{2\beta r^6} \sum_{n=-\infty}^{\infty} L_{EH}(x) \alpha_{1E} \alpha_{2H} \quad (34)$$

plus the contribution F_{HE} which will be the same except that the product of polarizabilities is modified into $\alpha_{1H} \alpha_{2E}$. With constant polarizabilities this, similar to Eq. (24), gives

$$F_{EH} = \frac{3\hbar c \alpha_{1E} \alpha_{2H}}{2\pi r^7} I_{EH}, \quad (35)$$

$$I_{EH} = \int_0^{\infty} L_{EH}(x) dx = \frac{7}{6}. \quad (36)$$

This part of the free energy gives a repulsive force in agreement with earlier derivations [8].

IV. INDUCED INTERACTION BETWEEN A PAIR OF DIELECTRIC HALF-PLANES

For a pair of half-planes one can generalize the statistical mechanical derivations of Høye and Brevik where two dielectric half-planes with the same dielectric constant were considered [18]. First we will reconsider this situation and modify it somewhat to include extension to half-planes with different dielectric constants. With this modification the method becomes more suitable for the extension to include different magnetic permeabilities too. Since equations of Ref. [18] will be used repeatedly, we will designate them with the numeral I when referred to.

The half-planes with separation a are parallel to the xy plane. With this the dipole-dipole interaction can be Fourier transformed along the x and y directions to obtain Eq. (I6.4) for the radiating interaction in vacuum between electric dipoles \mathbf{s}_1 and \mathbf{s}_2 ($z \neq 0$)

$$\hat{\phi}(12) = -2\pi s_1 s_2 \frac{e^{-q|z|}}{q} [(\mathbf{h} \cdot \hat{\mathbf{s}}_1)(\mathbf{h} \cdot \hat{\mathbf{s}}_2) - \zeta^2 \hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2], \quad (37)$$

$$\mathbf{h} = \{ik_x, ik_y, \pm q\}, \quad q^2 = k_{\perp}^2 + \zeta^2, \quad k_{\perp}^2 = k_x^2 + k_y^2. \quad (38)$$

Vectors with hats mean unit vectors. Without loss of generality the \mathbf{k} can be directed along the x axis by which $\mathbf{h} = \{ik_{\perp}, 0, \pm q\}$. With Eq. (I6.5) the interaction can be split in two parts:

$$\hat{\phi}(12) = -2\pi s_1 s_2 \frac{e^{-q|z|}}{q} (H_1 + H_2), \quad (39)$$

$$H_1 = (\mathbf{h} \cdot \hat{\mathbf{s}}_1)(\mathbf{h} \cdot \hat{\mathbf{s}}_2) - \zeta^2 \hat{\mathbf{s}}_1^{\parallel} \cdot \hat{\mathbf{s}}_2^{\parallel}, \quad H_2 = -\zeta^2 \hat{\mathbf{s}}_1^{\perp} \cdot \hat{\mathbf{s}}_2^{\perp}, \quad (40)$$

where \mathbf{s}^{\parallel} is the component of \mathbf{s} in the \mathbf{h} plane while \mathbf{s}^{\perp} is transverse to it. These two cases are the separation into the TM and TE modes, respectively.

For a medium with dielectric constant ε and magnetic permeability μ the interaction (39) is modified into

$$\hat{\phi}_{\varepsilon}(12) = -2\pi s_1 s_2 \frac{e^{-q_{\varepsilon}|z|}}{q_{\varepsilon}} \left(\frac{1}{\varepsilon} H_{\varepsilon 1} + \mu H_2 \right), \quad (41)$$

$$H_{\varepsilon 1} = (\mathbf{h}_{\varepsilon} \cdot \hat{\mathbf{s}}_1)(\mathbf{h}_{\varepsilon} \cdot \hat{\mathbf{s}}_2) - \varepsilon \mu \zeta^2 \hat{\mathbf{s}}_1^{\parallel} \cdot \hat{\mathbf{s}}_2^{\parallel}, \quad (42)$$

$$\mathbf{h}_{\varepsilon} = \{ik_{\perp}, 0, \pm q_{\varepsilon}\}, \quad q_{\varepsilon}^2 = k_{\perp}^2 + \varepsilon \mu \zeta^2, \quad (43)$$

while H_2 is unchanged.

The interaction is due to the electric field from \mathbf{s}_1 that acts upon \mathbf{s}_2 , or vice versa. Therefore it should be symmetric in the two dipole moments also when the dielectric constants are different. Thus $H_{\varepsilon 1}$ must be modified. With use of Eq. (43) one finds Eq. (I6.10) which can be written as

$$H_{\varepsilon 1} = \mathbf{G}_{\varepsilon 1} \cdot \hat{\mathbf{s}}_2 = g_{\varepsilon 1} g_{\varepsilon 2}, \quad \mathbf{G}_{\varepsilon 1} = g_{\varepsilon 1} \frac{q_{\varepsilon}}{k_{\perp}} \mathbf{u}_{\varepsilon \pm},$$

$$\mathbf{u}_{\varepsilon \pm} = \left\{ ik_{\perp}, 0, \pm \frac{k_{\perp}^2}{q_{\varepsilon}} \right\}, \quad (44)$$

$$\begin{aligned} g_{\varepsilon i} &= |\mathbf{h}_{\varepsilon} \times \hat{\mathbf{s}}_i| = \frac{q_{\varepsilon}}{k_{\perp}} \left(ik_{\perp} \hat{s}_{\perp i} \pm \frac{k_{\perp}^2}{q_{\varepsilon}} \hat{s}_{\parallel i} \right) \\ &= (iq_{\varepsilon} \hat{s}_{\perp i} \pm k_{\perp} \hat{s}_{\parallel i}), \quad (i = 1, 2). \end{aligned} \quad (45)$$

Here the \hat{s}_\perp and \hat{s}_\parallel are the components of $\hat{\mathbf{s}}^\parallel$ along the direction of \mathbf{k}_\perp (in the xy plane) and the z axis, respectively. The advantage of this form is that it can be extended to half-planes with different dielectric constants with energy symmetric in \hat{s}_1 and \hat{s}_2 . Thus Eq. (44) can be modified to

$$H_{\varepsilon_1} = \mathbf{G}_{\varepsilon_1} \cdot \hat{\mathbf{s}}_2 = g_{\varepsilon_1} g_{\varepsilon_2}, \quad \mathbf{G}_{\varepsilon_1} = g_{\varepsilon_1} \frac{q_{\varepsilon_2}}{k_\perp} \mathbf{u}_{\varepsilon_2\pm}. \quad (46)$$

For the interaction in vacuum ($\varepsilon, \mu \rightarrow 1$) one has similar expressions where $q_\varepsilon \rightarrow q$. [Note that with vector $\mathbf{u}_{\varepsilon_2\pm}$ the $\nabla \mathbf{D} = 0$ of Eq. (13) is fulfilled.] The resulting Green's function (electric interaction) for the two half-planes can now still be written in the form of Eq. (I6.20):

$$\begin{aligned} \hat{\phi}_g(12) &= J_\varepsilon (D_E^\parallel H_{\varepsilon_1} + D_E^\perp H_2), \\ J_\varepsilon &= -2\pi s_1 s_2 \frac{e^{-q_{\varepsilon_1} u_1 - q_{\varepsilon_2} (u_2 + a)}}{q_{\varepsilon_1}}, \end{aligned} \quad (47)$$

where $u_1 = -z_1 > 0$ and $u_2 = z_2 - a > 0$. The sought coefficients D_E^\parallel and D_E^\perp will make the resulting expression fully symmetric. [Within one medium this becomes interaction (41).]

With several parallel layers the $\mathbf{u}_{\varepsilon\pm}$ and some other factors will change along with the change of the dielectric constant for various layers. But the dielectric constant for the half-plane $z < 0$ stays fixed at ε_1 . Thus for two half-planes the factors not containing ε_2 or μ_2 can be separated out to be put into a quantity L when determining the electric field through the layers. With this the electric field for the TM mode ($\sim \mathbf{G}_{\varepsilon_1} \sim e^{\mp q_{\varepsilon_1} z} q_{\varepsilon_1} \mathbf{u}_{\varepsilon_1\pm}$) will be a slight modification of Eq. (I6.14):

$$\mathbf{E}/L = \begin{cases} q_{\varepsilon_1} \left(\frac{1}{\varepsilon_1} e^{-q_{\varepsilon_1} z} q_{\varepsilon_1} \mathbf{u}_{\varepsilon_1+} + B e^{q_{\varepsilon_1} z} \mathbf{u}_{\varepsilon_1-} \right), & z_0 < z < 0, \\ q(C e^{-qz} \mathbf{u}_+ + C_1 e^{qz} \mathbf{u}_-), & 0 < z < a, \\ q_{\varepsilon_2} D e^{-q_{\varepsilon_2} z} \mathbf{u}_{\varepsilon_2+}, & a < z, \end{cases} \quad (48)$$

where B, C, C_1 , and D are coefficients to be determined [relative to the $1/\varepsilon_1$ term of Eq. (41)].

One condition to determine the coefficients is that the component of \mathbf{E} along the xy plane is continuous. However, for the other condition we here find it more convenient to turn to the magnetic field (instead of the continuous \mathbf{D} field normal to the interfaces) due to later extension to magnetic media. Then from Eq. (13) we will find Eq. (I6.17)

$$\mathbf{h}_\varepsilon \times \mathbf{E} = \zeta \mathbf{B} = \zeta \mu \mathbf{H}. \quad (49)$$

With the form $\mathbf{E}/L = e^{\mp q_\varepsilon z} q_\varepsilon \mathbf{u}_{\varepsilon\pm}$ one finds

$$\begin{aligned} \zeta \mathbf{B}/L &= \{ik_\perp, 0, \pm q_\varepsilon\} \times \left\{ ik_\perp, 0, \pm \frac{k_\perp^2}{q_\varepsilon} \right\} q_\varepsilon e^{\mp q_\varepsilon z} \\ &= \pm ik_\perp (q_\varepsilon^2 - k_\perp^2) e^{\mp q_\varepsilon z} \hat{\mathbf{e}}_y = \pm ik_\perp \varepsilon \mu \zeta^2 e^{\mp q_\varepsilon z} \hat{\mathbf{e}}_y, \end{aligned} \quad (50)$$

where $\hat{\mathbf{e}}_y$ is the unit vector in the y direction.

The components of \mathbf{E} and \mathbf{H} parallel to the the interfaces are both continuous by which one gets the equations

$$\begin{aligned} q_\varepsilon \left(\frac{1}{\varepsilon_1} + B \right) &= q(C + C_1), \\ \varepsilon_1 \left(\frac{1}{\varepsilon_1} - B \right) &= C - C_1, \\ q(C e^{-qa} + C_1 e^{qa}) &= q_{\varepsilon_2} D e^{-q_{\varepsilon_2} a}, \\ C e^{-qa} + C_1 e^{qa} &= \varepsilon_2 D e^{-q_{\varepsilon_2} a}. \end{aligned} \quad (51)$$

The two last equations can be solved for C and C_1 to be inserted in the equation that results from the elimination of B . This extends solution (I6.15) for D to be

$$D = D_E^\parallel = \frac{4\kappa_1 e^{(q_{\varepsilon_2} - q)a}}{(\varepsilon_1 + \kappa_1)(\varepsilon_2 + \kappa_2)[1 - A_n e^{-2qa}]}, \quad (52)$$

$$A_n = \frac{(\varepsilon_1 - \kappa_1)(\varepsilon_2 - \kappa_2)}{(\varepsilon_1 + \kappa_1)(\varepsilon_2 + \kappa_2)}, \quad \kappa_i = \frac{q_{\varepsilon_i}}{q} \quad (i = 1, 2). \quad (53)$$

The subscript $n = 1, 2, 3, \dots$ indicate the Matsubara frequencies (22) $\zeta = \zeta_n = K/(\hbar c)$ and $K = 2\pi n/\beta$.

For the TE mode the electric field lies along the y axis transverse to the \mathbf{h} plane. Thus this component of the field can be written like Eq. (I6.18):

$$E/L = \begin{cases} \mu_1 e^{-q_{\varepsilon_1} z} + B e^{q_{\varepsilon_1} z}, & z_0 < z < 0, \\ C e^{-qz} + C_1 e^{qz}, & 0 < z < a, \\ D e^{-q_{\varepsilon_2} z}, & a < z, \end{cases} \quad (54)$$

where again L is a quantity independent of ε_2 and μ_2 . The corresponding magnetic field will have components along both the x and z directions

$$\begin{aligned} \zeta \mu \mathbf{H}/L &= \zeta \mathbf{B}/L = \mathbf{h}_\varepsilon \times \mathbf{E}/L \\ &= \{ik_\perp, 0, \pm q_\varepsilon\} \times \{0, 1, 0\} e^{\mp q_\varepsilon z} \\ &= (\mp q_\varepsilon \hat{\mathbf{e}}_x + ik_\perp \hat{\mathbf{e}}_z) e^{\mp q_\varepsilon z}. \end{aligned} \quad (55)$$

From this the boundary conditions give the equations (keeping μ although $\mu = 1$ is considered so far)

$$\begin{aligned} \mu_1 + B &= C + C_1, \\ \frac{q_{\varepsilon_1}}{\mu_1} (\mu_1 - B) &= q(C - C_1), \\ C e^{-qa} + C_1 e^{qa} &= D e^{-q_{\varepsilon_2} a}, \\ q(C e^{-qa} + C_1 e^{qa}) &= \frac{q_{\varepsilon_2}}{\mu_2} D e^{-q_{\varepsilon_2} a}. \end{aligned} \quad (56)$$

[The coefficients are relative to the μ_1 term from Eq. (41)]. The solution of these equations is

$$D = D_E^\perp = \frac{4\mu_1 \mu_2 \kappa_1 e^{(q_{\varepsilon_2} - q)a}}{(\kappa_1 + \mu_1)(\kappa_2 + \mu_2)[1 - B_n e^{-2qa}]}, \quad (57)$$

$$B_n = \frac{(\kappa_1 - \mu_1)(\kappa_2 - \mu_2)}{(\kappa_1 + \mu_1)(\kappa_2 + \mu_2)}. \quad (58)$$

The Casimir force per unit area between the plates is given by Eq. (I6.32). This follows from Eq. (I4.4) where the product of the interaction (37) and Green's function (47) multiplied

by $-q$ are integrated. With extension to plates with different dielectric constants this becomes

$$f_{\text{sur}} = -\frac{1}{\pi\beta} \sum_{n=0}^{\infty} \int_{\zeta_n}^{\infty} q^2 dq I Q S. \quad (59)$$

$$I = \int_0^{\infty} \int_0^{\infty} \frac{e^{-(q_{\varepsilon_2+q})(u_2+a)} e^{-(q_{\varepsilon_1+q})u_1}}{q_{\varepsilon_1} q} du_1 du_2 \\ = \frac{e^{-(q_{\varepsilon_2+q})a}}{q_{\varepsilon_1} q (q_{\varepsilon_1+q})(q_{\varepsilon_2+q})}. \quad (60)$$

(with $z = z_2 - z_1 = u_1 + u_2 + a$). The prime means that the $n = 0$ term is to be taken with half weight, and $\zeta_n = 2\pi n / (\beta \hbar c)$ as follows from Eq. (22). The Q is the $(9y/2)^2 A$ term of (I6.32) which with (I5.5) is $(\varepsilon - 1)^2/4$. For different media this becomes

$$Q = (\varepsilon_1 - 1)(\varepsilon_2 - 1)/4. \quad (61)$$

Further from Eq. (I5.5), $3y = (4\pi/3)\rho\beta\langle s^2 \rangle$ which is the classical case. Now the polarizability $\alpha = \beta\langle s^2 \rangle/3$ by which $3y = 4\pi\rho\alpha$. This also holds for the quantum case where nonzero frequencies contribute, $\alpha = \alpha(\omega)$. This follows from Sec. 5 of Ref. [17]. Finally the last factor is

$$S = S_{EE} = S^{\parallel} + S^{\perp}, \quad S^{\parallel} = D_E^{\parallel} \langle H_{\varepsilon 1} H_1^* \rangle, \\ S^{\perp} = D_E^{\perp} \langle H_2^2 \rangle, \quad (62)$$

where $H_1 = g_1 g_2$, $g_i = iq\hat{s}_{\perp i} \pm k_{\perp} \hat{s}_{\parallel i}$, i.e., $H_1 = H_{\varepsilon 1}$ for $\varepsilon = 1$. And from the Fourier transform relation (30), $H_1(-k_{\perp})$ is the quantity needed. For the complex conjugate we have $H_1^*(k_{\perp}) = H_1(-k_{\perp})$.

With extension of Eqs. (I6.27)–(I6.30) to different, but nonmagnetic, media we now get by using Eqs. (44) and (45) and then Eqs. (38) and (43) ($\langle \hat{s}_{\perp i}^2 \rangle = 1/3$, etc.)

$$\langle H_{\varepsilon 1} H_1^* \rangle = \frac{1}{9} (k_{\perp}^2 + q_{\varepsilon 1} q) (k_{\perp}^2 + q_{\varepsilon 2} q), \quad (63)$$

$$k_{\perp}^2 + q_{\varepsilon i} q = \frac{1}{\varepsilon_i - 1} [\varepsilon_i q^2 - q_{\varepsilon i} + (\varepsilon_i - 1) q_{\varepsilon i} q] \\ = \frac{q}{\varepsilon_i - 1} (\varepsilon_i - \kappa_i) (q + q_{\varepsilon i}), \quad (i = 1, 2), \quad (64)$$

$$\langle H_2^2 \rangle = \frac{1}{9} \zeta^4;$$

$$\zeta^2 = \frac{1}{\varepsilon_i - 1} (q_{\varepsilon i}^2 - q^2) \\ = \frac{q}{\varepsilon_i - 1} (\kappa_i - 1) (q_{\varepsilon i} + q). \quad (65)$$

Note these expressions for $k_{\perp}^2 + q_{\varepsilon i} q$ and ζ^2 are valid only for $\mu_1 = \mu_2 = 1$. Altogether for this case with I , Q , and S inserted into Eq. (59) the known Lifshitz result (I2.9) for the force is recovered [21]

$$f_{\text{sur}} = -\frac{1}{\pi\beta} \sum_{n=0}^{\infty} \int_{\zeta_n}^{\infty} q^2 dq \left[\frac{A_n e^{-2qa}}{1 - A_n e^{-2qa}} + \frac{B_n e^{-2qa}}{1 - B_n e^{-2qa}} \right], \quad (66)$$

with A_n and B_n given by Eqs. (53) and (58) respectively and with $q_{\varepsilon i}$ ($i = 1, 2$) given by Eq. (43).

V. INDUCED INTERACTION WITH MAGNETIC PROPERTIES INCLUDED

In this section we want to show that expression (66) for the Casimir force is valid also when magnetic interactions are present, i.e., μ_1 and/or μ_2 are different from unity. In Sec. IV the electric field from dipoles and their interactions for a pair of half-planes were found. This also included the induced magnetic fields and influence from magnetic permeabilities. Thus the coefficients D_E^{\parallel} and D_E^{\perp} will stay unchanged. Also the main structure of the surface force (59) and integral (60) will be the same. However, expression (62) will have additional terms that involve magnetic moments too. For these additional terms the Q given by Eq. (61) is modified by replacing ε_i ($i = 1, 2$) with μ_i in the ways possible. Further the symmetry of electric and magnetic fields in Maxwell's equations (13) is utilized. The fully new term to obtain is the induced interaction between electric and magnetic dipole moments.

With $\mu \neq 1$ expressions (62) and (63) for induced interaction between electric dipole moments will remain unchanged, but results (64) and (65) will not hold any longer. We find Eq. (64) can be modified in two useful ways ($q_{\varepsilon}^2 = k_{\perp}^2 + \varepsilon\mu\zeta^2$, $\kappa = q_{\varepsilon}/q$):

$$k_{\perp}^2 + q_{\varepsilon} q = \frac{1}{\varepsilon - 1} [\varepsilon k_{\perp}^2 - k_{\perp}^2 + (\varepsilon - 1) q_{\varepsilon} q] \\ = \frac{1}{\varepsilon - 1} [(\varepsilon q^2 - \varepsilon \zeta^2) - (q_{\varepsilon}^2 - \varepsilon \mu \zeta^2) + (\varepsilon - 1) q_{\varepsilon} q] \\ = \frac{q}{\varepsilon - 1} [(\varepsilon - \kappa)(q + q_{\varepsilon}) + (\mu - 1) \varepsilon \zeta^2], \quad (67)$$

and likewise from symmetry with respect to ε and μ

$$k_{\perp}^2 + q_{\varepsilon} q = \frac{q}{\mu - 1} [(\mu - \kappa)(q + q_{\varepsilon}) + (\varepsilon - 1) \mu \zeta^2]. \quad (68)$$

Due to symmetry the contributions from the induced interactions between magnetic dipole moments will be like the ones of Eq. (62) since with $q_{\varepsilon}^2 = k_{\perp}^2 + \varepsilon\mu\zeta^2$ or with Eqs. (67) and (68) the $k_{\perp}^2 + q_{\varepsilon} q$ stays unchanged by interchange of μ and ε

$$S_{HH} = D_H^{\parallel} \langle H_{\varepsilon 1} H_1^* \rangle + D_H^{\perp} \langle H_2^2 \rangle. \quad (69)$$

Further comparing with expressions (52) and (57) with interchange of μ and ε one finds

$$D_E^{\perp} = \mu_1 \mu_2 D_H^{\parallel}, \quad D_H^{\perp} = \varepsilon_1 \varepsilon_2 D_E^{\parallel}. \quad (70)$$

The remaining terms are those from induced interactions between electric and magnetic dipoles. Thus we need the magnetic fields due to the electric ones and vice versa that follows from symmetry.

With Eqs. (41), (46), and (47) the TM electric field for $z_2 > a$ can be written in the form $\mathbf{E} = (J_{\varepsilon}/s_2) D_E^{\parallel} g_{\varepsilon 1}(q_{\varepsilon 2}/k_{\perp}) \mathbf{u}_{\varepsilon 2 \pm}$. Thus with Eq. (50) the corresponding magnetic field will be

$$\zeta \mu_2 \mathbf{H} = \zeta \mathbf{B} = (J_{\varepsilon}/s_2) D_E^{\parallel} g_{\varepsilon 1}(\pm i k_{\perp} \varepsilon_2 \mu_2 \zeta^2 \hat{\mathbf{e}}_y) / k_{\perp}, \quad (71)$$

with $g_{\varepsilon 1}$ given by Eq. (45). This is to be multiplied with the magnetic dipole moment \mathbf{s}_2 (instead of electric dipole moment) whose unit vector has the transverse component $\hat{\mathbf{s}}_2^{\perp}$ (in the y direction) to obtain the equivalent of $H_{\varepsilon 1}$ in Eq. (47) in view

of the common prefactor $J_\varepsilon D_E^\parallel$

$$H_{\varepsilon EH}^\parallel = \frac{1}{\zeta \mu_2 J_\varepsilon D_E^\parallel} \mathbf{H} \cdot \mathbf{s}_2 = \pm \zeta g_{\varepsilon_1} \varepsilon_2 \delta_2^\perp. \quad (72)$$

This multiplied with the corresponding quantity in vacuum is averaged over the orientations of the dipole moments to give

$$\langle H_{\varepsilon EH}^\parallel H_{\varepsilon EH}^\parallel \rangle_1 = -\frac{1}{9} \zeta^2 (k_\perp^2 + q_{\varepsilon_1} q) \varepsilon_2. \quad (73)$$

Note in Eq. (72) the magnitude of the electric dipole moment s_2 is replaced by the magnetic one. This is taken into account by replacing $\varepsilon_2 - 1$ with $\mu_2 - 1$ in Eq. (61) for the factor Q to be used in Eq. (75) below. The $H_{\varepsilon EH}^\parallel = H_{\varepsilon EH}^\parallel$ for $\varepsilon_i = \mu_i = 1$ ($i = 1, 2$) with $\zeta \rightarrow -\zeta$. The $\zeta \rightarrow -\zeta$ (and $k_\perp \rightarrow -k_\perp$) again follows from the Fourier transform relation (30). Likewise one has the interaction between an electric dipole in half-plane 2 with a magnetic one in half-plane 1. Due to symmetry the corresponding expression must be

$$\langle H_{\varepsilon EH}^\parallel H_{\varepsilon EH}^\parallel \rangle_2 = -\frac{1}{9} \zeta^2 (k_\perp^2 + q_{\varepsilon_2} q) \varepsilon_1. \quad (74)$$

Finally for the TM mode one has the interaction between the magnetic moments in the two half-planes. But again due to symmetry of Maxwell's equations (13) this for the magnetic field (in the y direction) is equivalent to the TE field. Thus the corresponding magnetic interactions will be the same and follow from the general form of Eq. (65) as $\langle H_y^2 \rangle = \zeta^2/9$.

The resulting surface force is obtained by adding the various contributions multiplied with factors Q given by Eq. (61) with one or both ε replaced by μ . Adding together for the TM contribution the product QS^\parallel from Eqs. (61) and (62) is replaced by

$$\begin{aligned} S_Q^\parallel &= \frac{1}{4} [(\varepsilon_1 - 1)(\varepsilon_2 - 1) D_E^\parallel \langle H_{\varepsilon_1} H_1^* \rangle \\ &\quad + (\varepsilon_1 - 1)(\mu_2 - 1) D_E^\parallel \langle H_{\varepsilon EH}^\parallel H_{\varepsilon EH}^\parallel \rangle_1 \\ &\quad + (\varepsilon_2 - 1)(\mu_1 - 1) D_E^\parallel \langle H_{\varepsilon EH}^\parallel H_{\varepsilon EH}^\parallel \rangle_2 \\ &\quad + (\mu_1 - 1)(\mu_2 - 1) D_H^\perp \langle H_y^2 \rangle]. \end{aligned} \quad (75)$$

Inserting from Eqs. (63), (70), (73), and (74) one finds

$$\begin{aligned} S_Q^\parallel &= \frac{D_E^\parallel}{36} [(\varepsilon_1 - 1)(\varepsilon_2 - 1)(k_\perp^2 + q_{\varepsilon_1} q)(k_\perp^2 + q_{\varepsilon_2} q) \\ &\quad - (\varepsilon_1 - 1)(\mu_2 - 1) \varepsilon_2 \zeta^2 (k_\perp^2 + q_{\varepsilon_1} q) \\ &\quad - (\varepsilon_2 - 1)(\mu_1 - 1) \varepsilon_1 \zeta^2 (k_\perp^2 + q_{\varepsilon_2} q) \\ &\quad + (\mu_1 - 1)(\mu_2 - 1) \varepsilon_1 \varepsilon_2 \zeta^4] \\ &= \frac{D_E^\parallel}{36} [(\varepsilon_1 - 1)(k_\perp^2 + q_{\varepsilon_1} q) - (\mu_1 - 1) \varepsilon_1 \zeta^2] \\ &\quad \times [(\varepsilon_2 - 1)(k_\perp^2 + q_{\varepsilon_2} q) - (\mu_2 - 1) \varepsilon_2 \zeta^2]. \end{aligned} \quad (76)$$

Finally with use of Eq. (67) the result is

$$S_Q^\parallel = \frac{D_E^\parallel}{36} q^2 (\varepsilon_1 - \kappa_1)(q + q_{\varepsilon_1})(\varepsilon_2 - \kappa_2)(q + q_{\varepsilon_2}). \quad (77)$$

This result is precisely the result for $S_Q^\parallel = QS^\parallel$ with Eqs. (63) and (64) inserted. However, the former result was only valid for $\mu = 1$ while result (77) is valid for general μ .

To obtain the resulting force the contribution from the TE mode with magnetic interaction included (i.e., $\mu \neq 1$) is also needed. Again one may find the magnetic interactions via the corresponding magnetic field in Eq. (55). However, we simplify by utilizing the symmetry between the electric and magnetic fields in Maxwell's equations. Thus for the magnetic field the TE mode will be similar to the TM mode with exchange of ε and μ . The various contributions will be similar to those of Eq. (75) including the S^\perp of Eq. (62). Adding together, Eq. (76) will be replaced by

$$\begin{aligned} S_Q^\perp &= \frac{D_E^\perp}{36 \mu_1 \mu_2} [(\mu_1 - 1)(k_\perp^2 + q_{\varepsilon_1} q) - (\varepsilon_1 - 1) \mu_1 \zeta^2] \\ &\quad \times [(\mu_2 - 1)(k_\perp^2 + q_{\varepsilon_2} q) - (\varepsilon_2 - 1) \mu_2 \zeta^2]. \end{aligned} \quad (78)$$

With Eqs. (68) and (70) this becomes

$$S_Q^\perp = \frac{D_E^\perp}{36 \mu_1 \mu_2} q^2 (\mu_1 - \kappa_1)(q + q_{\varepsilon_1})(\mu_2 - \kappa_2)(q + q_{\varepsilon_2}). \quad (79)$$

With $\mu_1 = 1$ and $\mu_2 = 1$ this result is the same as the result $S_Q^\perp = QS^\perp$ with Eq. (65) inserted into Eq. (62). Result (79) generalizes expression (66) for the Casimir force to arbitrary μ_1 and μ_2 .

VI. SUMMARY

We have studied the induced Casimir force between media that can have both dielectric and magnetic properties. Usually this force is attractive, but with both properties present, it turns out that it can be repulsive. It is not obvious how this can be understood on physical grounds. In Sec. II we established a simple oscillator model that shows precisely this induced repulsive behavior. The model is a simple extension of the model studied earlier [15,16], where we showed the analog of the attractive forces produced by the TM and TE modes of dielectric media. Then by use of the statistical mechanical method introduced in Ref. [17] we evaluated the induced force between a pair of particles possessing both dielectric and magnetic properties. Further in Sec. IV the statistical mechanical theory used in Ref. [18] was extended to a pair of half-planes with different dielectric constants. Finally, in Sec. V we extended the theory to a pair of half-planes that have the mentioned properties and are separated by a distance a . The result of the latter, as might be expected, is a generalization of the well-known Lifshitz formula as given by Eq. (66). Again the Casimir force is repulsive if one of the media is mainly dielectric while the other is mainly magnetic. With equal media in the two half-planes, the Casimir force is always attractive.

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