Reference-frame-independent Einstein-Podolsky-Rosen steering

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(Received 14 October 2016; published 31 August 2018)

Protocols for testing or exploiting quantum correlations—such as entanglement, Bell nonlocality, and Einstein-Podolsky-Rosen steering—generally assume a common reference frame between two parties. Establishing such a frame is resource intensive and can be technically demanding for distant parties. While Bell nonlocality can be demonstrated with high probability for a large class of two-qubit entangled states when the parties have one or no shared reference direction, the degree of observed nonlocality is measurement-orientation dependent and can be arbitrarily small. In contrast, we theoretically prove that steering can be demonstrated with 100% probability for a larger class of states, in a rotationally invariant manner, and experimentally demonstrate rotationally invariant steering in a variety of cases. We also show, by comparing with the steering inequality of Cavalcanti *et al.* [J. Opt. Soc. Am. B **32**, A74 (2015)], that the steering inequality we derive is the optimal rotationally invariant one for the case of two settings per side and maximally mixed local qubit states.

DOI: 10.1103/PhysRevA.98.022333

I. INTRODUCTION

Shared quantum correlations are a topic of significant foundational interest and an important resource for quantum information and communication protocols. Quantum steering (also known as Einstein-Podolsky-Rosen (EPR) steering) corresponds to a class of correlations stronger than those required to merely witness entanglement, but which need not violate any Bell inequality [1]. Moving down this hierarchy of correlation strength, from Bell nonlocality to steering to entanglement, gives access to protocols which are more robust to noise [2,3]. The cost is that, while Bell inequality violations require neither party (Alice or Bob) to be trusted, steering requires one (here, Bob) to be trusted, and regular entanglement witnessing requires full trust in both parties [4]. Steering therefore represents an interesting and important case, providing for strong [3,10], even loophole-free [11], tests of nonlocality, but without the extreme noise suppression required to achieve Bell inequality violations.

Typically, these correlation tests, and the quantum information tasks that derive from them, assume a shared reference frame between the parties, Alice and Bob. This assumption has been given relatively little attention to date. However, it will be of significant practical concern in future field deployments, as in the recent entanglement distribution over long distances [12]. Establishing such a common reference frame is a nontrivial issue in experimental situations. For instance, in quantum communication, a time-varying temperature can change the orientation of the polarization reference frame in optical fiber. Likewise, the relative measurement settings between a satellite and earth could be time varying. In both cases, active compensation of these changes presents a considerable challenge [13]. Such compensation becomes unnecessary if encoding in optical orbital angular momentum [14] or complicated entangled states [15]. However, such states are very susceptible to loss and noise, and generating and manipulating such systems may be difficult. Therefore, it is of interest to reduce reference-frame dependence in quantum information tasks.

Can nonlocality be demonstrated simply without having established a common reference frame? This question was recently answered theoretically [13,16,17] and experimentally [18,19] for Bell nonlocality. Here, we demonstrate that a quantum steering protocol between two parties can be performed without establishing a reference frame. We can contrast our results with the case for Bell violations, which are measurement-orientation dependent and can be arbitrarily small; our technique surpasses these limitations.

To investigate quantum steering without a reference frame, we derived and experimentally tested a rotationally invariant steering (RIS) inequality, which is very robust and can certify steering with 100% probability for a large class of two-qubit entangled states. We compare the case where the parties share one measurement direction (e.g., derived from line of sight between them or the propagation axis of an optical fiber) to the case where they share none.

This paper is structured as follows. In Sec. II we derive our class of RIS inequalities. In Sec. III we compare it to previous work: the binary-outcome qubit steering inequality of Cavalcanti *et al.* [20] (CFFW); and the semi-device-independent steering inequalities and steering-related inequalities of Moroder *et al.* [21]. In Sec. IV we explain our experimental setup, and in Sec. V present the experimental results. We studied the two-setting case both with and without a shared reference direction, and compare our RIS inequality to the CFFW inequality. This enables us to demonstrate that our inequality is the optimal rotationally invariant one for the

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experimental situation studied. We also studied the three-setting case, demonstrating the advantage conferred by each party having more settings.

II. ROTATIONALLY INVARIANT STEERING (RIS) INEQUALITIES

A general quantum steering protocol between two parties, Alice and Bob, proceeds as follows. In each round, Bob receives a quantum system and announces two randomly chosen measurement settings: $j \in \{1,\ldots,m\}$ for Alice; and $k \in \{1,\ldots,n\}$ for himself (in many previous realizations $j=k$). Alice announces a corresponding measurement outcome A_j, which may be the result of a genuine measurement on her half of an entangled pair that she shares with Bob, or the result of a strategy that she (or some adversary controlling her equipment) is using to try to cheat, i.e., to convince Bob of shared quantum correlations which do not exist. Bob measures a pre-agreed observable $\hat{B}_k$ on Hilbert space H_B, with outcome B_k. Over many runs Bob is able to estimate the correlation matrix $M_{jk} := \langle A_j B_k\rangle$, and test whether it is compatible with a local hidden state (LHS) model for his system, i.e., a set of states $\{\hat{\varrho}_\lambda\}$ on H_B such that

$$M_{jk} = \langle A_j B_k\rangle = \int d\lambda\, p(\lambda)\, \langle A_j\rangle_\lambda\, \langle \hat{B}_k\rangle_{\hat{\varrho}_\lambda}. \qquad (1)$$

Here λ labels an underlying variable with probability density $p(\lambda)$, $\langle \hat{B}_k\rangle_{\hat{\varrho}_\lambda} := \mathrm{Tr}[\hat{\varrho}_\lambda \hat{B}_k]$, and $\langle A_j\rangle_\lambda$ is an arbitrary function of λ, bounded by the maximum and minimum of the set of the allowed values of A_j. If no such LHS model exists then Alice is said to be able to steer Bob's system via her measurements.

We restrict our attention to the case where all outcomes are labeled by ± 1, and Bob's measurements correspond to a set of orthogonal spin directions on the Bloch sphere, i.e., $\hat{B}_k = \boldsymbol{b}_k \cdot \hat{\boldsymbol{\sigma}}$ with $\boldsymbol{b}_k \cdot \boldsymbol{b}_{k'} = \delta_{kk'}$. Here $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ is the vector of Pauli operators in some fixed basis. It is shown in Appendix A that any LHS model for this case must satisfy the steering inequality

$$\|M\|_{\mathrm{tr}} := \mathrm{tr}\sqrt{M^\top M} \leqslant \sqrt{m}. \qquad (2)$$

That is, there is an experimentally measurable *steering parameter* (here, the trace-norm of the correlation matrix M) which, for a LHS model, has an upper *bound* independent of the results (here, the square root of the number of Alice's settings). Thus, violation of this inequality is sufficient for Alice to be able to steer Bob.

Suppose now that Alice and Bob genuinely wish to achieve violation of steering inequality (2) for some shared two-qubit state $\hat{\rho}$, by each choosing a set of mutually orthogonal measurement directions. Thus, Alice measures a set of spin operators $\hat{A}_j := \boldsymbol{a}_j \cdot \boldsymbol{\sigma}$ with $\boldsymbol{a}_j \cdot \boldsymbol{a}_{j'} = \delta_{jj'}$, and $M_{jk} = \boldsymbol{a}_j^\top T \boldsymbol{b}_k$, where T is the 3×3 spin correlation matrix for state $\hat{\rho}$, i.e., $T_{pq} := \mathrm{Tr}[\hat{\rho}\hat{\sigma}_p \otimes \hat{\sigma}_q]$. The steering parameter in Eq. (2) is then predicted to be (see Appendix B)

$$\|M\|_{\mathrm{tr}} = \|P_A T P_B\|_{\mathrm{tr}}, \qquad (3)$$

where $P_A := \sum_j \boldsymbol{a}_j \boldsymbol{a}_j^\top$ and $P_B := \sum_k \boldsymbol{b}_k \boldsymbol{b}_k^\top$ are the respective 3×3 projection matrices onto the span of Alice's and Bob's measurement directions.

As a first example, if Alice and Bob each choose a *triad* of mutually orthogonal directions, i.e., $m=n=3$, then $P_A = P_B = I_3$ and Eqs. (2) and (3) simplify to $\mathrm{tr}\sqrt{T^\top T} \leqslant \sqrt{3}$, independently of the particular triads chosen. Thus, the degree of steerability, as quantified by a violation of Eq. (2), is invariant under local rotations of the triads, and so can be established even when Alice and Bob do not share any reference directions. In particular, for a Werner state—a probabilistic mixture of a maximally entangled singlet state with a symmetric noise state parametrized by the mixing probability, or Werner parameter, W—one has $T = -W I_3$, implying that a (constant) violation is guaranteed for any $W > 1/\sqrt{3}$. In comparison, a corresponding violation of the Bell inequalities in Refs. [17–19] is only guaranteed for $W=1$, and the degree of violation can be arbitrarily small.

As a second example, consider the case where Alice and Bob each choose a *pair* of mutually orthogonal directions, i.e., $m=n=2$, P_A and P_B are the projections onto the planes spanned by their measurement directions. Hence, the corresponding degree of steerability witnessed by the steering inequality is invariant under any local rotations that leave the measurement directions within these planes. In particular, if Alice and Bob only share a single reference direction $\boldsymbol{r}$, then they can determine a degree of steerability invariant under arbitrary rotations about this direction, by choosing their measurement directions to lie in the plane orthogonal to $\boldsymbol{r}$. For a Werner state, violation is guaranteed for any $W > 1/\sqrt{2}$ (the best possible bound for this case [20]). In comparison, a violation of the Bell inequalities in Refs. [17–19] is again only guaranteed for $W=1$ and may be arbitrarily small.

III. COMPARISON WITH OTHER STEERING AND STEERING-RELATED INEQUALITIES

It is interesting to compare our RIS with some related, but not equivalent, work in the area of steering inequalities.

A. Steering inequality that always detects steering for maximally mixed outcomes

For $m=n=2$, it is of interest to compare the RIS inequality (2) with a recent steering inequality derived by Cavalcanti, Foster, Fuwa, and Wiseman (CFFW) [20]. It is an inequality that is necessary and, for the case of maximally mixed marginals [22], sufficient for the correlation matrix M to admit a qubit LHS model for Bob:

$$|M^\top u_+| + |M^\top u_-| \leqslant \sqrt{2}, \qquad (4)$$

with $u_\pm := (1,\pm 1)^\top/\sqrt{2}$. The "necessary" part of the preceding sentence is of course what is meant by a steering inequality; it is the "sufficient" part that makes the CFFW inequality special. Note that we have normalized Eq. (4) differently from the inequality in Ref. [20] so that it has the same bound as Eq. (2) for $m=2$. If Alice and Bob share a two-qubit state $\hat{\rho}$, and Alice measures in two orthogonal directions $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$, the predicted steering parameter in Eq. (4) reduces to (see Appendix B)

$$|M^\top u_+| + |M^\top u_-| = |P_B T^\top \boldsymbol{a}_+| + |P_B T^\top \boldsymbol{a}_-|, \qquad (5)$$

with $\boldsymbol{a}_\pm = (\boldsymbol{a}^{(1)} \pm \boldsymbol{a}^{(2)})/\sqrt{2}$.

It is clear from Eq. (5) that, unlike the RIS inequality, the CFFW inequality is not invariant under rotations in the plane of Alice's measurement directions, since such rotations change $\boldsymbol{a}_{\pm}$. However, minimizing the left-hand side of Eq. (4) over all such rotations yields, by construction, a steering parameter that is rotationally invariant and which is bounded by $\sqrt{2}$. Moreover, since the CFFW inequality is necessary and sufficient for maximally mixed marginals, this minimization must yield the best possible rotationally invariant inequality for a two-qubit state with zero Bloch vectors for Alice and Bob. Carrying out the minimization of the CFFW steering parameter explicitly, one in fact recovers the rotationally invariant steering parameter in Eq. (3), as shown in Appendix C. In this sense, our RIS inequality (2) is the best possible for $m = n = 2$. We conjecture that it is similarly optimal for $m = n = 3$.

As an example of practical interest, let Φ denote the angle between Alice's and Bob's measurement planes, and α denote the angle that the line of intersection of these planes makes with Alice's measurement direction $\boldsymbol{a}_1$. For a Werner state, with $T = -W I_3$, the RIS parameter of Eq. (3) simplifies to

$$\|M\|_{\mathrm{tr}} = W(1 + |\cos\Phi|), \tag{6}$$

independently of α, while the CFFW parameter of Eq. (4) is given by

$$\begin{aligned}|M^\top u_+| &+ |M^\top u_-| \\ &= W\big(\sqrt{1+\cos^2\Phi + \sin 2\alpha \sin^2\Phi} \\ &\quad + \sqrt{1+\cos^2\Phi - \sin 2\alpha \sin^2\Phi}\,\big)/\sqrt{2}.\end{aligned} \tag{7}$$

Minimizing the latter over all rotation angles α recovers Eq. (6). More generally (i.e., for any two-qubit state, rather than the particular case of a Werner state), for a fixed angle Φ between the two measurement planes, the CFFW steering parameter will vary with the rotation angle α in Alice's measurement plane, whereas the RIS steering parameter will remain constant, corresponding to the minimum value of the CFFW parameter. This prediction is experimentally tested below.

B. Semi-device-independent steering-related inequalities

In Ref. [21], Moroder *et al.* introduced a steering-related inequality [with two cases, given in Eqs. (13) and (14) of that paper] that makes no assumptions about the measurements Alice and Bob perform. This remarkable feat obviously makes their inequality reference-frame independent. However, their inequality does assume dimensions, d_A and d_B, for Alice's and Bob's Hilbert spaces, respectively. It is the assumption that Alice has a Hilbert space at all that means that their inequality is not strictly a steering inequality, because the phenomenon of EPR steering makes no assumptions about Alice's system [1]. However, in the limit $d_A \to \infty$ Alice's system is large enough to support any description in terms of local hidden variables and hence the inequality of Moroder *et al.* will become a true steering inequality. For a finite number n of settings by Bob (n_B in the notation of Ref. [21]), the relevant inequality in the limit $d_A \to \infty$ is their Eq. (14),

$$|\det(D)| \leqslant \left(\frac{\sqrt{d_B}}{n+1}\right)^{n+1}, \tag{8}$$

where D is the "data matrix" defined in Ref. [21].

We can compare the above steering inequality to our RIS inequality most easily by assuming zero marginals, which is the case relevant to our experiment and to the CFFW inequality, and for the case $m = n = 3$ for which they give the "data matrix" explicitly below their Eq. (14). For this situation it is easy to show that

$$|\det(D)| = |\det(M)|/(12\sqrt{3}). \tag{9}$$

Thus their steering inequality (8) reduces in this case to

$$|\det(M)| \leqslant (3\sqrt{3})/16 \approx 0.325. \tag{10}$$

In contrast, our RIS, Eq. (2), for this case is

$$\mathrm{tr}\sqrt{M^\top M} \leqslant \sqrt{3}. \tag{11}$$

Now using the inequality $(abc)^{1/3} \leqslant (a+b+c)/3$ for the three singular values of M, it follows that $|\det(M)| \leqslant [\mathrm{tr}\sqrt{M^\top M}]^3/27$. Thus our RIS implies, for the case being considered, that

$$|\det(M)| \leqslant \sqrt{3}/9 \approx 0.192. \tag{12}$$

Thus, our RIS is strictly stronger than the steering inequality of Ref. [21].

For completeness, we compare Eq. (12) to the relevant steering-related inequality of Moroder *et al.* that assumes that Alice's results come from measuring a qubit ($d_A = 2$). For the case considered above, with $m = n = 3$, their steering inequality reduces to $|\det(D)| \leqslant 1/108$ [their Eq. (15)]. Assuming zero marginals as above, this turns into exactly the same inequality, Eq. (12), which we derived from our RIS, Eq. (11). That is, our rotationally invariant steering inequality is even as strong as the steering-related inequality in [21] that assumes Alice's results are qubit born. But we remind the reader that Eq. (11) assumes that Bob's three measurement directions are orthogonal (in the Bloch-sphere sense) while the inequalities of Moroder *et al.* make no such assumptions.

IV. EXPERIMENTAL SETUP

As shown in Fig. 1, we implemented these steering protocols using polarization-entangled states generated from a spontaneous parametric down-conversion (SPDC) source. A 10-mm-long periodically poled potassium titanyl phosphate (ppKTP) crystal, mounted in a polarization Sagnac ring interferometer [23,24], was pumped bidirectionally by a 410-nm fiber-coupled continuous-wave laser with an output power (after fiber) of 2.5 mW.

To test the quality of the generated entangled state, quantum state tomography [25] was performed at several stages throughout the experiment—in each case, we achieved a fidelity of about 98% with the singlet state $(|HV\rangle - |VH\rangle)/\sqrt{2}$. We measured the correlations in our experiment by rotating the QWPs and HWPs in front of polarizing elements to set measurement directions and implement projective measurements for Alice's m and Bob's n settings, and counted coincident detections. We calculated each steering parameter from the observed correlations, and determined its error from those in the correlation matrix elements: $\Delta M_{jk} = \sqrt{(\Delta M_{jk}^{(\mathrm{sys})})^2 + (\Delta M_{jk}^{(\mathrm{stat})})^2}$. The error consists of a systematic error due to small imperfections in Bob's measurement settings, which could lead to an overestimation of the correlations

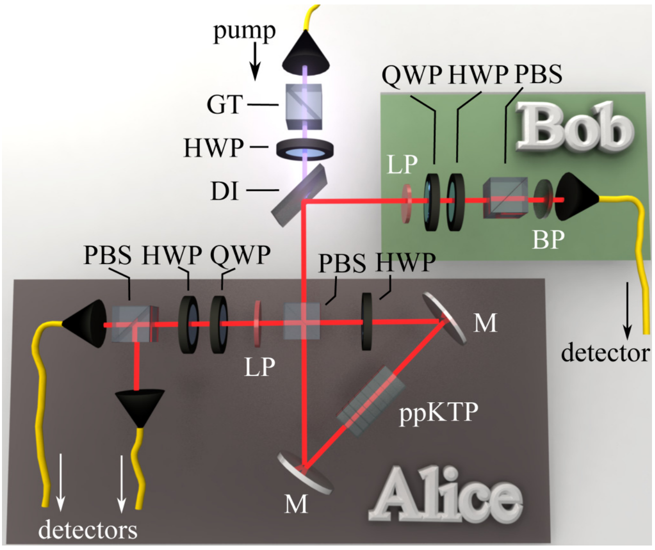

FIG. 1. Entangled photon pairs at 820 nm were produced via SPDC in a Sagnac interferometer consisting of a polarizing beam splitter (PBS), a dual-coated half-wave plate (HWP), two mirrors (M), and a periodically poled KTP crystal (ppKTP). The polarization of the pump is controlled by a Glan Taylor (GT) prism followed by a HWP. Different measurement settings are performed by rotating half- and quarter-wave plates (QWP) relative to the PBSs. Long pass (LP) filters, a dichroic mirror (DI), and an additional bandpass (BP) filter in Bob's line remove 410-nm pump photons copropagating with the 820-nm photons, before photons are coupled into single-mode fibers and detected by single-photon counting modules and counting electronics.

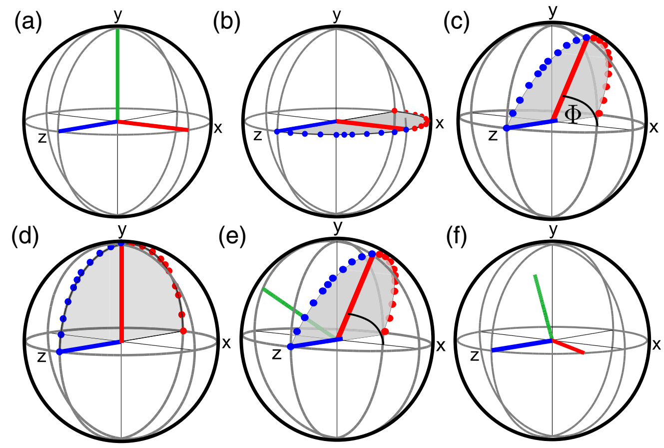

FIG. 2. Poincaré (Bloch) spheres contain vectors showing one of the eigenstates of the three relevant directions (blue, red, and green) in the experiments we performed. In each of the experiments, measurements are made along two or three of these directions. (a) Bob uses the same three measurement directions in each of the $n = 3$ experiments, while using only the red and blue directions for experiments with $n = 2$. (b) Alice's directions, in the case where Alice and Bob share a reference direction [shown as green in (a)]. We test the invariance of the $m = n = 2$ RIS inequality under rotations in the plane (grey), as the blue and red settings are rotated through 90° in steps (blue and red dots). (c) Alice's directions for $m = 2$ (blue and red dots) when her plane of measurement directions is tilted by $\Phi = 64°$ and the settings are rotated in that plane, while maintaining local orthogonality. (d) Same as (c), but with $\Phi = 90°$. (e) Alice's directions for testing the $m = n = 3$ RIS inequality (blue and red dots, and green axis), corresponding to measurement triads strongly misaligned with respect to Bob's directions in (a). (f) Nonorthogonal measurement directions for Alice.

[3], and the statistical error arising from Poissonian statistics in photon counting. Quantum steering usually requires that Bob chooses his settings independently from one measurement to the other. However, as we control Alice's realization and apparatus in this demonstration (i.e., we are not in an adversarial scenario), there is no need for a time ordering of the events and we collected data without shot-to-shot randomization [3]. However, this would have to be altered in a full deployment [11].

V. EXPERIMENTAL TESTS AND RESULTS

We investigated the rotational invariance of quantum steering in a series of experiments.

A. Shared reference plane

We first considered the case where Alice and Bob share a single reference direction and hence share a common reference plane orthogonal to this direction, and use $m = n = 2$ measurement settings—the minimal set size. The measurement directions lie in a plane orthogonal (on the Bloch sphere) to the shared direction, and the two settings on each side are locally orthogonal. This is a natural physical situation because a shared reference direction may be determined reliably, for example, by line of sight between the parties. Furthermore, it is natural to assume that Alice and Bob can reliably set local measurement directions. However, although Alice's and Bob's measurement directions will lie in the same plane, their relative orientation within this plane may be unknown. This situation also provides for a direct comparison between the RIS and CFFW inequalities.

In our experiment, the measurements lie in the σ_x-σ_z plane [Figs. 2(a) and 2(b)], corresponding to an angle of $\Phi = 0°$ between Alice's and Bob's measurement planes. While Bob's measurement directions were kept constant, Alice's were rotated through 90° in the plane, by angles $\alpha \in \{0°, 10°, 20°, 30°, 40°, 45°, 50°, 60°, 70°, 80°, 90°\}$. We observed a rotation-independent violation of both the RIS and CFFW inequalities for $\Phi = 0°$ [Figs. 2(b) and 3(a)], except for some deviation around $\alpha = 70°$. Across the remainder of the range, the measured steering parameters are close to the theoretically predicted value of 1.97 [Fig. 3(a), solid line], for both the RIS and CFFW correlation functions of a Werner state [26] having the same fidelity with the singlet as our entangled state. This value is close to the maximum value of 2 for an ideal singlet state. We attribute the experimental deviation near $\alpha = 70°$ to time-dependent fluctuations of the end state due to temperature shifts affecting the source. While this imperfection is undesirable, it serves to illustrate the point that the RIS inequality is tolerant to noise, due to the large gap at all relative angles α between the bound of $\sqrt{2}$ in Eq. (2) and the theoretical maximum value of 2.

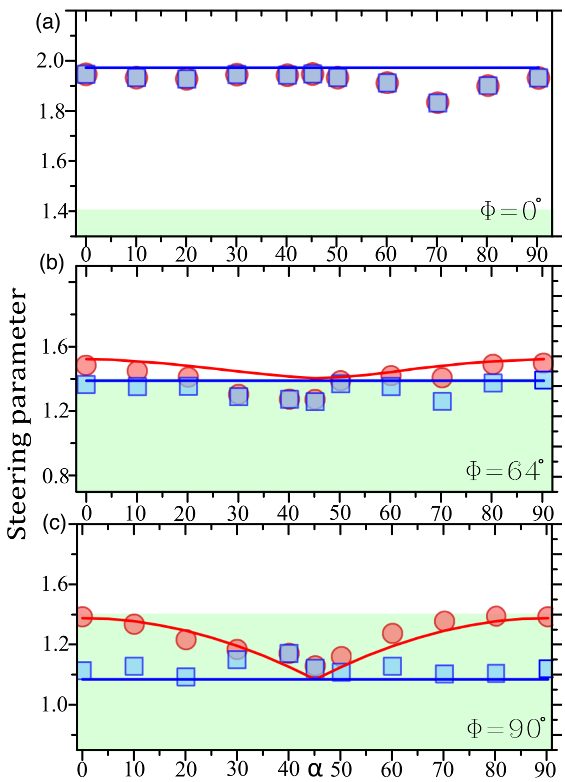

FIG. 3. Steering parameter vs rotation angle α (degrees) in Alice's measurement plane, for experiments with $m = n = 2$ measurement directions (cases in Secs. V A and V B of the main text). The plane tilt angle takes on the values (a) $\Phi = 0^\circ$, (b) $\Phi = 64^\circ$, and (c) $\Phi = 90^\circ$ [see Figs. 2(b)–2(d) respectively]. For all angles, we calculated the theoretically expected curves for the RIS inequality (2) (blue) and the CFFW inequality (4) (red) for the Werner state. The Werner parameter W of the closest Werner state was calculated from the average tomographic data. The RIS data are represented by blue squares, and the CFFW data by red circles. The error bars are too small to be seen. Data points in the upper white region imply steering of Bob by Alice.

B. Tilted reference planes

We repeated the first case, but allowed an offset in the previously shared reference direction (i.e., $\Phi \neq 0$), simulating the case when there is imperfection in sharing this direction. Due to robustness of the inequalities, we had to tilt Alice's measurement plane significantly, by 64°, to shift to a regime where the inequalities were not necessarily violated [Fig. 2(c)]. The RIS data stayed approximately rotation invariant [Fig. 3(b)], and comparable to the theoretically predicted value of 1.40 in Eq. (6) (less than the steering bound of $\sqrt{2}$) for the closest ideal Werner state. By contrast, the CFFW data showed an oscillatory behavior [Fig. 3(b), red line], as predicted by Eq. (7), with violation for $\alpha < 20^\circ$ and $\alpha > 70^\circ$. Again, the noise in the data is due to asymmetries in the state arising from state preparation imperfections caused by thermal fluctuations in the apparatus.

We also investigated the case where there was extreme misalignment in the supposedly shared reference direction. For this, Alice used measurement directions in the σ_z-σ_y plane, i.e., for $\Phi = 90^\circ$ [Fig. 2(d)]. Neither steering inequality was violated at any angle α. While the RIS data were approximately insensitive to rotations of Alice's measurement directions, the CFFW data showed an oscillatory behavior [Fig. 3(c)], as per the theoretical predictions in Eqs. (6) and (7), respectively, for the tomographically reconstructed state. For each of Figs. 3(a)–3(c) the RIS values are never greater than the CFFW values, as predicted.

C. More than two measurement directions

As the RIS inequality (2) is not restricted to $m = n = 2$, we extended the number of measurement directions to $m = n = 3$ directions for each party. First we studied the case where Alice's and Bob's orthogonal measurement triads were perfectly aligned, along the σ_x, σ_y, and σ_z directions [Fig. 2(a)]. For this case, we generated a state with a fidelity of 98.4% with a singlet state. The measured RIS steering parameter 2.93 ± 0.01 significantly exceeds the bound of $\sqrt{3}$ in inequality (2). The small deviation from the maximum possible value of 2.95 for a Werner state with $W = 0.984$ can be explained by imperfections of polarization optics and classical interference in the Sagnac interferometer.

We also calculated the average correlation between Alice's and Bob's results for $m \neq n$ directions. We analyzed a subset of the three-setting-per-side data to investigate if Alice is able to steer Bob's state for two cases: $m = 2$ (σ_x and σ_z) with $n = 3$; and $m = 3$ (σ_x, σ_y, and σ_z) with $n = 2$. In both cases, the corresponding RIS inequality bounds, $\sqrt{2}$ and $\sqrt{3}$ respectively, were violated, with respective steering parameters 1.96 ± 0.01 and 1.97 ± 0.01.

Finally, we investigated the rotational invariance of the RIS inequality for $m = n = 3$ directions on each side. In particular, we chose orthogonal triads for Alice [Fig. 2(e)] which were strongly misaligned relative to Bob's fixed measurement triad [Fig. 2(a)]. Alice's triads comprised two orthogonal measurement directions lying in a plane oriented at an angle of $\Phi = 64^\circ$ to Bob's σ_x–σ_z measurement plane [blue and red dots in Fig. 2(e)], and a third direction orthogonal to the first two directions (the green line in Fig. 2(e)). The experimentally obtained steering parameters for these triads had an average value of 2.89—well above the bound of $\sqrt{3}$ for the corresponding RIS inequality in Eq. (2)—and a standard deviation of 0.03. The small deviation confirms that the steering parameter is indeed insensitive to local rotations of Alice's and Bob's measurement triads. We further note that this choice of measurement directions would not lead to a steering demonstration using an ordinary linear steering inequality of the type given in Ref. [2], whereas the RIS is robust to such major misalignments.

D. Nonorthogonal measurement directions

Finally, we observed whether Alice could demonstrate steering by measuring in nonorthogonal directions, while Bob's measured directions remained orthogonal [Fig. 2(f)]. We note that the steering inequalities [Eqs. (2) and (4)] are valid for any choice of Alice's measurement directions, whereas the predictions for the steering parameters in Eqs. (3)–(7) assume they are orthogonal. In this experiment, the state we generated had a fidelity of 97.2% with the singlet state.

First, we considered the case of two measurement directions for each party. Bob measured along σ_z and σ_x [Fig. 2(a), blue and red], with Alice's directions along σ_z and at a 60° angle therefrom in the same plane [Fig. 2(f), blue and red], corresponding to

$$\boldsymbol{a}_1 = (0, 0, 1), \qquad \boldsymbol{a}_2 = (\sqrt{3}/2, 0, 1/2). \tag{13}$$

With these measurement settings, both the RIS and CFFW inequalities, Eqs. (2) and (4), were violated, with steering parameters 1.85 ± 0.01 and 1.96 ± 0.01 compared to the bound of $\sqrt{2}$.

We concluded the experiment by measuring in three directions for each party. While Bob measured along σ_x, σ_y, and σ_z [Fig. 2(a)], Alice's directions formed a regular tetrahedron with the origin [Fig. 2(f)], corresponding to $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ as in Eq. (13) and $\boldsymbol{a}_3 = [1/(2\sqrt{3}), \sqrt{2/3}, 1/2]$. We violated the bound of $\sqrt{3}$ in Eq. (2) with a steering parameter of 2.74 ± 0.01, which is not far below the maximum possible value of 3 obtainable via mutually orthogonal directions and a maximally entangled state.

VI. CONCLUSIONS

We theoretically determined a rotationally invariant steering inequality. Sufficiently entangled states produce constant violations of the inequality under local rotations. For two measurement settings per side, we showed that the violation is constant under local rotations about a shared axis and that our RIS inequality is the optimal such inequality for this situation. Experimentally, we showed that for two settings per side and one shared reference direction only, the violation of both inequalities is independent of frame alignment between Alice and Bob, up to state preparation imperfections. Degradation of the shared direction eventually means that steering inequalities can no longer be violated. For three settings per side, the rotationally invariant inequality is violated even for maximal misalignment of the reference frames, unlike an ordinary steering inequality [2] and even in the presence of state preparation imperfections. In principle, using the appropriate (two- or three-setting) rotationally invariant inequality for one or zero shared measurement directions always provides a large buffer between the theoretically expected steering value and the bound, unlike the case for frame rotations in Bell tests [18,19]. As demonstrated by our data, this provides robustness to imperfections such as asymmetries in a real-world shared entangled state. Therefore our work shows how the steering task can be more tolerant to reference-frame misalignment and asymmetry than Bell tests, adding to the previous list (decoherence-tolerance [2] and loss-tolerance [3]) of noise sources where steering enjoys an advantage. Our demonstration of rotationally invariant steering holds potential application in ground-to-space satellite quantum communication [12,27] and in quantum key distribution [28,29].

ACKNOWLEDGMENTS

We acknowledge Sergei Slussarenko for helpful discussions and thank Cyril Branciard for useful comments, including on the role of maximally mixed marginals in the CFFW condition. This work was supported by Australian Research Council project DP140100648. S.W. acknowledges financial support by the Australian government.

APPENDIX A: PROOF OF RIS INEQUALITY

To prove the rotationally invariant steering inequality in Eq. (2), suppose that the measurement outcomes for Alice and Bob are restricted to ± 1, and that the correlation matrix M has a qubit LHS model for Bob's system as per Eq. (1) of the main text. Thus,

$$M_{jk} = \langle A_j B_k \rangle = \int d\lambda\, p(\lambda)\, \langle A_j \rangle_\lambda\, \langle \hat{B}_k \rangle_{\hat{\varrho}_\lambda}, \tag{A1}$$

with $\langle \hat{B}_k \rangle_{\hat{\varrho}_\lambda} = \mathrm{Tr}[\hat{\varrho}_\lambda \hat{B}_k]$ where

$$\hat{\varrho}_\lambda = \tfrac{1}{2}[1 + \boldsymbol{s}(\lambda) \cdot \hat{\boldsymbol{\sigma}}], \quad \hat{B}_k = \boldsymbol{b}_k \cdot \hat{\boldsymbol{\sigma}}, \quad \boldsymbol{b}_k \cdot \boldsymbol{b}_{k'} = \delta_{kk'}.$$

Hence, letting $\mathcal{A}(\lambda)$ denote the m vector with components $\mathcal{A}_j(\lambda) = \langle A_j \rangle_\lambda$, and $\mathcal{B}(\lambda)$ denote the n vector with components $\mathcal{B}_k(\lambda) = \boldsymbol{s}(\lambda) \cdot \boldsymbol{b}_k$, one can rewrite the correlation matrix in Eq. (A1) as

$$M = \int d\lambda\, p(\lambda)\, \mathcal{A}(\lambda)\, \mathcal{B}(\lambda)^\top. \tag{A2}$$

Taking the trace norm then yields the steering inequality

$$\begin{aligned} \|M\|_{\mathrm{tr}} &= \left\| \int d\lambda\, p(\lambda)\, \mathcal{A}(\lambda)\, \mathcal{B}(\lambda)^\top \right\|_{\mathrm{tr}} \\ &\leqslant \int d\lambda\, p(\lambda)\, \|\mathcal{A}(\lambda)\, \mathcal{B}(\lambda)^\top\|_{\mathrm{tr}} \\ &= \int d\lambda\, p(\lambda)\, |\mathcal{A}(\lambda)|\, |\mathcal{B}(\lambda)| \\ &\leqslant \int d\lambda\, p(\lambda)\, \sqrt{m} = \sqrt{m}, \end{aligned} \tag{A3}$$

as per Eq. (2). Here, the first inequality follows from the triangle inequality, the next line from the easily verified property $\|vw^\top\|_{\mathrm{tr}} = |v|\,|w|$, and the final inequality via $|\mathcal{A}(\lambda)|^2 = \sum_j \langle A_j \rangle_\lambda^2 \leqslant m$ and $|\mathcal{B}(\lambda)|^2 = \sum_k \boldsymbol{s}(\lambda)^\top \boldsymbol{b}_k \boldsymbol{b}_k^\top \boldsymbol{s}(\lambda) = \boldsymbol{s}(\lambda)^\top P_B \boldsymbol{s}(\lambda) = |P_B \boldsymbol{s}(\lambda)|^2 \leqslant |\boldsymbol{s}(\lambda)| \leqslant 1$.

APPENDIX B: RIS AND CFFW STEERING PARAMETERS FOR TWO-QUBIT STATES

Now, if Alice and Bob each make a set of mutually orthogonal measurements on a two-qubit state with spin correlation matrix T, with $T_{jk} = \mathrm{Tr}[\hat{\rho}\, \sigma_j \otimes \sigma_k]$, then

$$M = A^\top T B, \tag{B1}$$

where A and B denote the $3 \times m$ and $3 \times n$ matrices with columns corresponding to their respective spin directions, i.e., $A := (\boldsymbol{a}_1 \boldsymbol{a}_2 \ldots \boldsymbol{a}_m)$ and $B := (\boldsymbol{b}_1 \ldots \boldsymbol{b}_n)$. The steering parameter for the RIS inequality in Eq. (2), i.e., the trace norm of M, can then be evaluated as

$$\begin{aligned}\|M\|_{\rm tr} &= \|M^\top\|_{\rm tr} = {\rm Tr}\sqrt{A^\top T B B^\top T^\top A}\\ &= {\rm Tr}\sqrt{A^\top T P_B P_B T^\top A}\\ &= \|P_B T^\top A\|_{\rm tr} = \|(P_B T^\top A)^\top\|_{\rm tr}\\ &= {\rm Tr}\sqrt{P_B T^\top A A^\top T P_B}\\ &= {\rm Tr}\sqrt{P_B T^\top P_A P_A T P_B} = \|P_A T P_B\|_{\rm tr},\end{aligned} \tag{B2}$$

as per Eq. (3), where we have used $BB^\top = \sum_j \boldsymbol{b}_j \boldsymbol{b}_j^\top = P_B = P_B^2$, and the corresponding relations for $AA^\top$.

To evaluate the steering parameter for the CFFW inequality

$$|M^\top u_+| + |M^\top u_-| \leqslant \sqrt{2} \tag{B3}$$

in Eq. (4), note for any two-vector u that $|M^\top u|^2 = u^\top M M^\top u = u^\top A^\top T B B^\top T^\top A u = u^\top A^\top T P_B P_B T^\top A u = |P_B T^\top A u|^2$. Substitution into the above inequality, and recalling that $u_\pm = (1, \pm 1)^\top/\sqrt{2}$ and $\boldsymbol{a}_\pm = (\boldsymbol{a}_1 \pm \boldsymbol{a}_2)/\sqrt{2}$, immediately yields

$$|M^\top u_+| + |M^\top u_-| = |P_B T^\top \boldsymbol{a}_+| + |P_B T^\top \boldsymbol{a}_-| \tag{B4}$$

for the CFFW steering parameter, as per Eq. (5).

APPENDIX C: OPTIMALITY OF THE RIS INEQUALITY

Finally, we show that for $m = n = 2$, the RIS inequality is the best possible rotationally invariant steering inequality for states with maximally mixed marginals, as claimed in Sec. III. In particular, we show that the RIS steering parameter in Eq. (3) is given by minimizing the CFFW steering parameter in Eq. (5), over all orthogonal measurement pairs in Alice's and Bob's respective measurement planes. Noting that Eq. (5) [equivalent to Eq. (B4) above] only depends on Bob's measurement directions via P_B, it is sufficient to show that

$$\min_{R_A} |P_B T^\top R_A \boldsymbol{a}_+| + |P_B T^\top R_A \boldsymbol{a}_-| = \|P_A T P_B\|_{\rm tr}, \tag{C1}$$

where R_A ranges over all rotations that leave Alice's measurement plane invariant and a_+ and a_- are fixed. Now, for any vector $\boldsymbol{a}$ in this measurement plane, one has $P_A \boldsymbol{a} = \boldsymbol{a}$, and hence, using $P_B^2 = P_B$,

$$|P_B T^\top \boldsymbol{a}| = |P_B T^\top P_A \boldsymbol{a}| = \sqrt{\boldsymbol{a}^\top P_A T P_B T^\top P_A \boldsymbol{a}}. \tag{C2}$$

Defining $K := P_A T P_B T^\top P_A$, one therefore has

$$\begin{aligned}&|P_B T^\top R_A \boldsymbol{a}_+| + |P_B T^\top R_A \boldsymbol{a}_-|\\ &\quad = \sqrt{(R_A \boldsymbol{a}_+)^\top K (R \boldsymbol{a}_+)} + \sqrt{(R_A \boldsymbol{a}_-)^\top K (R \boldsymbol{a}_-)}.\end{aligned}$$

Since $R_A \boldsymbol{a}_\pm$ and K only have support on Alice's measurement plane, minimizing this expression over R_A reduces to a 2×2 matrix problem. Further, since K is by definition a non-negative symmetric matrix, and $R_A \boldsymbol{a}_+$ and $R_A \boldsymbol{a}_-$ range over all pairs of orthogonal unit vectors in the measurement plane, we can choose coordinates such that

$$K \equiv \begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix}, \quad R_A \boldsymbol{a}_+ \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad R_A \boldsymbol{a}_- \equiv \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

on this plane, for some $k \geqslant k' \geqslant 0$ and $\theta \in [0, 2\pi]$. Thus,

$$\begin{aligned}&|P_B T^\top R_A \boldsymbol{a}_+| + |P_B T^\top R_A \boldsymbol{a}_-|\\ &\quad = \sqrt{k \cos^2\theta + k' \sin^2\theta} + \sqrt{k \sin^2\theta + k' \cos^2\theta}\\ &\quad = \sqrt{X + Y\cos 2\theta} + \sqrt{X - Y \cos 2\theta},\end{aligned} \tag{C3}$$

with $X := (k + k')/2$ and $Y := (k - k')/2$. It is straightforward to check that the function $f(x) := \sqrt{1+x} + \sqrt{1-x}$ is symmetric with a single maximum at $x = 0$. Hence, the minimum of the above expression is obtained at $\cos 2\theta = \pm 1$, yielding

$$\begin{aligned}&\min_{R_A} |P_B T^\top R_A \boldsymbol{a}_+| + |P_B T^\top R_A \boldsymbol{a}_-|\\ &\quad = \sqrt{X+Y} + \sqrt{X-Y} = \sqrt{k} + \sqrt{k'}\\ &\quad = {\rm Tr}[\sqrt{K}] = \|P_B T^\top P_A\|_{\rm tr},\end{aligned} \tag{C4}$$

using the definition of K. Finally, since the trace norm of a matrix is invariant under transposition, Eq. (C1) follows as desired.

Substantial generalizations of these results, with Alice and Bob not limited to orthogonal sets of measurements, and allowing for detector inefficiencies, will be discussed elsewhere.

[1] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).

[2] D. Saunders, S. Jones, H. Wiseman, and G. Pryde, Nat. Phys. **6**, 845 (2010).

[3] A. J. Bennet, D. A. Evans, D. J. Saunders, C. Branciard, E. G. Cavalcanti, H. M. Wiseman, and G. J. Pryde, Phys. Rev. X **2**, 031003 (2012).

[4] The exception is quantum-refereed games [5–9] where the need for trust can be mitigated by using additional quantum resources.

[5] F. Buscemi, Phys. Rev. Lett. **108**, 200401 (2012).

[6] E. G. Cavalcanti, M. J. W. Hall, and H. M. Wiseman, Phys. Rev. A **87**, 032306 (2013).

[7] C. Branciard, D. Rosset, Y.-C. Liang, and N. Gisin, Phys. Rev. Lett. **110**, 060405 (2013).

[8] S. Kocsis, M. J. W. Hall, A. J. Bennet, D. J. Saunders, and G. J. Pryde, Nat. Commun. **6**, 5886 (2015).

[9] E. Verbanis, A. Martin, D. Rosset, C. C. W. Lim, R. T. Thew, and H. Zbinden, Phys. Rev. Lett. **116**, 190501 (2016).

[10] D. Smith, G. Gillett, M. P. de Almeida, C. Branciard, A. Fedrizzi, T. J. Weinhold, A. Lita, B. Calkins, T. Gerrits, H. M. Wiseman *et al.*, Nat. Commun. **3**, 625 (2012).

[11] B. Wittmann, S. Ramelow, F. Steinlechner, N. Langford, N. Brunner, H. Wiseman, R. Ursin, and A. Zeilinger, New J. Phys. **14**, 053030 (2012).

[12] J. Yin, Y. Cao, Y. Li, S. Liao, L. Zhang, J. Ren, W. Cai, W. Liu, B. Li, H. Dai, G. Li, Q. Lu, Y. Gong, Y. Xu, M. Li, J. Jia, G. Ren, D. He, Y. Zhou, X. Zhang, N. Wang, X. Chang, Z. Zhu, N. Liu, Y. Chen, C. Lu, R. Shu, C. Peng, J. Wang, and J. Pan, Science **356**, 1140 (2017).
[13] A. Laing, V. Scarani, J. G. Rarity, and J. L. O'Brien, Phys. Rev. A **82**, 012304 (2010).
[14] V. D'Ambrosio, E. Nagali, S. P. Walborn, L. Aolita, S. Slussarenko, L. Marrucci, and F. Sciarrino, Nat. Commun. **3**, 961 (2012).
[15] A. Cabello, Phys. Rev. A **68**, 042104 (2003).
[16] Y.-C. Liang, N. Harrigan, S. D. Bartlett, and T. Rudolph, Phys. Rev. Lett. **104**, 050401 (2010).
[17] J. J. Wallman, Y.-C. Liang, and S. D. Bartlett, Phys. Rev. A **83**, 022110 (2011).
[18] P. Shadbolt, T. Vértesi, Y.-C. Liang, C. Branciard, N. Brunner, and J. L. O'Brien, Sci. Rep. **2**, 470 (2012).
[19] M. S. Palsson, J. J. Wallman, A. J. Bennet, and G. J. Pryde, Phys. Rev. A **86**, 032322 (2012).
[20] E. G. Cavalcanti, C. J. Foster, M. Fuwa, and H. M. Wiseman, J. Opt. Soc. Am. B **32**, A74 (2015).
[21] T. Moroder, O. Gittsovich, M. Huber, R. Uola, and O. Gühne, Phys. Rev. Lett. **116**, 090403 (2016).
[22] P. Girdhar and E. G. Cavalcanti, Phys. Rev. A **94**, 032317 (2016).
[23] T. Kim, M. Fiorentino, and F. N. C. Wong, Phys. Rev. A **73**, 012316 (2006).
[24] A. Fedrizzi, T. Herbst, A. Poppe, T. Jennewein, and A. Zeilinger, Opt. Express **15**, 15377 (2007).
[25] A. G. White, A. Gilchrist, G. J. Pryde, J. L. O'Brien, M. J. Bremner, and N. K. Langford, J. Opt. Soc. Am. B **24**, 172 (2007).
[26] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).
[27] G. Vallone, D. Bacco, D. Dequal, S. Gaiarin, V. Luceri, G. Bianco, and P. Villoresi, Phys. Rev. Lett. **115**, 040502 (2015).
[28] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, Phys. Rev. A **85**, 010301 (2012).
[29] J. A. Slater, C. Branciard, N. Brunner, and W. Tittel, New J. Phys. **16**, 043002 (2014).