Entanglement sudden death and revival in quantum dark-soliton qubits

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We study the finite time entanglement dynamics between two dark-soliton qubits due to quantum fluctuations in quasi-one-dimensional Bose-Einstein condensates. Recently, dark solitons are proved to be an appealing platform for qubits due to their appreciably long lifetime. We explore the entanglement decay for an entangled state of two phonon coherences and the qubits to be in the diagonal basis of so-called Dicke states. We observe the collapse and revival of the entanglement, depending critically on the collective damping term but independent of the qubit-qubit interaction for both initial states. The collective behavior of the dark-soliton qubits demonstrate the dependence of entanglement evolution on the interatomic distance.

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I. INTRODUCTION

Two basic features to distinguish the classical world from the quantum world are the superposition and entanglement, cornerstone ingredients in the rapidly developing field of quantum information and computation [1]. In recent years, the interaction between qubits and the environment has attracted a great deal of attention motivated by the possibility of controlling and exploiting the entanglement dynamics [2–5]. The occurrence of spontaneous emission due to the coupling of qubits with the reservoir, leading to the irreversible loss of quantum information therein encoded, has been regarded as the main obstacle in practical usages of entanglement.

Much work has been done to understand the decoherence dynamics for a pair of qubits interacting with different reservoirs [6–8]. As it is currently known, two spatially separated qubits, initially prepared in a product of two pure states, get entangled as time evolves, leading to the creation of a so-called *transient entanglement* in the system [6,7]. Conversely, Yu and Eberly discovered a finite-time disentanglement of two initially entangled qubits in contact with pure dissipative environments [9,10]. This effect is currently known under the name of *entanglement sudden death*, and has been confirmed with experiments performed both with photonic [11] and atomic systems [12].

In this context, Bose-Einstein condensates (BECs) have attracted a great deal of interest during the last decades, since the macroscopic character of the wave function allows BECs to display pure-state entanglement at macroscopic scales [13–15]. A scheme to generate entangled states in a bimodal BEC has been announced in Refs. [16,17]; the investigation of the macroscopic superposition based on matter waves has been achieved with BEC Josephson junctions [18–20]. Moreover, light scattering with BECs have been used to enhance their nonlinear properties in superradiance experiments [21], and to show the possibility of matter wave amplification [22] and nonlinear wave mixing [23]. Entanglement dynamics for coupled BECs have been investigated in [24].

Another important manifestation of the macroscopic nature of BEC is the dark soliton (DS), a structure resulting from the detailed balance between the dispersive and nonlinear effects, appearing also in other physical systems [25–27]. The dynamics and stability of DSs in BECs have been a subject of intense research over the last decade [28,29]. The dynamical evolution of DS entanglement and how its stability is affected by quantum fluctuations has been studied in Ref. [30]. Collision-induced entanglement between fast moving matter-wave solitons using the Born approximation has been studied in BECs displaying attractive interactions [31]. Moreover, the study of collective aspects of soliton gases [32] bring DSs towards applications in many-body physics [33]. In a recent publication, we have shown that DSs trapping an impurity can behave as qubits in quasi-1D BECs [34], being excellent candidates to store information as a consequence of their appreciably long lifetimes (~ 0.01-1 s). DS qubits thus offer an appealing alternative to quantum optical of solid-state platforms, as information processing involves only phononic degrees of freedom: the quantum excitations on top of the BEC state.

Ghasemian *et al.* [35] described the collapse-revival phenomenon, after studying the dynamics of BEC atoms interacting with a single-mode laser field. The possibility of generating entangled Schrödinger cat states by using a BEC trapped in a double-well potential has been investigated [36,37]. In these works, it is reported that the revival of the initial state can be used as an unambiguous signature of the coherent macroscopic superposition, as opposed to an incoherent mixture. Also, two-impurity qubits surrounded by a BEC reveal the influence

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of quantum reservoirs on effects such as sudden death, revival, trapping, and generation of entanglement [38].

In this paper we investigate the dynamics of the entanglement produced between two dark-soliton qubits in a quasi-one-dimensional Bose-Einstein condensate. As described in Ref. [34], the qubits are produced by trapping impurities inside the potential created by the dark solitons. Moreover, the phonons (quantum fluctuations on top of the background density) play the role of a quantum reservoir. We show the occurrence of entanglement sudden death by computing the time evolution of the Wootters' concurrence and showing that it vanishes for a finite time. We further demonstrate that the concurrence dynamics critically depends on the distance between the DS qubits.

The paper is organized as follows: In Sec. II we model the mean-field dynamics of the BEC and the impurities by using the Gross-Pitaevskii (GP) and Schrödinger equations, respectively. We compute the coupling between the DS qubits and the phonons. In Sec. III we describe the Markovian master equation and extract the density matrix elements for the collective DS qubit states. The use of concurrence as a measure of the entanglement is discussed in Sec. IV. Finally, a summary and discussion of the investigation is presented in Sec. V.

II. THEORETICAL MODEL AND QUANTUM FLUCTUATIONS

At the mean-field level, the system is governed by Gross-Pitaevskii (GP) equations,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\Psi}{\partial x^2} + g|\Psi|^2\Psi + \chi|\Phi|^2\Psi, \tag{1}$$

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m_2}\frac{\partial^2\Phi}{\partial x^2} + \chi|\Psi|^2\Phi, \tag{2}$$

where χ is the BEC-impurity coupling constant, g is the BEC particle-particle interaction strength, and m_1 and m_2 denote the BEC particle and impurity masses, respectively. Here the discussion is restricted to repulsive interactions ($g > 0$) where the dark solitons are assumed to be not disturbed by the presence of impurities. To achieve this, the impurity gas is considered to be much less massive than the BEC particles and sufficiently dilute, i.e., $|\Psi|^2 \gg |\Phi|^2$ (an experimental realization can be found in [34]). Therefore, the soliton behaves like a potential for the impurities (considered to be free particles), i.e.,

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m_2}\frac{\partial^2\Phi}{\partial x^2} + \chi|\psi_{\rm sol}|^2\Phi, \tag{3}$$

where a singular nonlinear solution corresponding to the ith ($i = 1, 2$) soliton profile is given by [39,40]

$$\psi_{\rm sol}^{(i)}(x) = \sqrt{n_0}\tanh\left[(x - x_i)/\xi\right], \tag{4}$$

where $x_i = \pm d/2$ determines the position of the solitons, $\xi = \hbar/\sqrt{m_1 n_0 g}$ is the healing length, and $n_0 \sim 10^8$–$10^9 m^{-1}$ is the typical linear density. For well separated solitons $d \gg \xi$, the internal level structure of the two qubits is assumed to be equal. Each qubit is characterized by its ground g_i and excited levels e_i ($i = 1, 2$), separated by a gap frequency

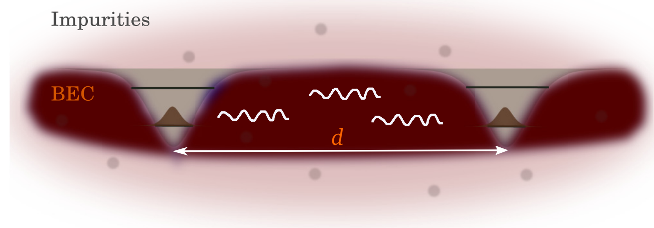

FIG. 1. Schematic representation of two dark-soliton qubits in cigar shaped quasi-1D Bose-Einstein condensates immersed in a dilute gas of impurities. The localized depressions in the density represent the dark solitons, while the wiggly lines represent the phonons, i.e., the elementary excitations composing the quantum reservoir.

$\omega_0 = \hbar(2\nu - 1)/(2m\xi^2)$ (see Fig. 1). Here

$$\nu = -1 + \sqrt{1 + 4\chi/g} \tag{5}$$

is a parameter controlling the number of bound states created by the DS, which operates as a qubit (two-level system) in the range $0.33 \leqslant \nu < 0.80$ [34]. Typical experimentally accessible conditions provide longitudinal and transverse trap frequencies of $\omega_z/2\pi \sim (15$–$730)$ Hz $\ll \omega_r/2\pi = (1$–$5)$ kHz, and the corresponding length $l_z = (0.6$–$3.9)$ μm [41]. More recent experiments lead to much less pronounced trap inhomogeneities by creating much larger traps, $l_z \sim 100$ μm [42].

Quantum fluctuations

The total BEC quantum field includes the DS wave functions and quantum fluctuations, $\Psi_i(x) = \psi_{\rm sol}^{(i)}(x) + \delta\psi_i(x)$, where

$$\delta\psi_i(x) = \sum_k \left(u_k^{(i)}(x)b_k + v_k^{*(i)}(x)b_k^\dagger\right). \tag{6}$$

Here b_k are the bosonic operators verifying the commutation relation $[b_k, b_q^\dagger] = \delta_{k,q}$. The amplitudes $u_k(x)$ and $v_k(x)$ satisfy the normalization condition $|u_k(x)|^2 - |v_k(x)|^2 = 1$ and are explicitly given by [43]

$$u_k^{(i)}(x) = e^{ik(x-x_i)}\sqrt{\frac{1}{4\pi\xi}\frac{\mu}{\epsilon_k}}\left\{\left((k\xi)^2 + \frac{2\epsilon_k}{\mu}\right)\right.$$
$$\left.\times\left[\frac{k\xi}{2} + i\tanh\left(\frac{x - x_i}{\xi}\right)\right] + \frac{k\xi}{\cosh^2\left(\frac{x-x_i}{\xi}\right)}\right\},$$

$$v_k^{(i)}(x) = e^{-ik(x-x_i)}\sqrt{\frac{1}{4\pi\xi}\frac{\mu}{\epsilon_k}}\left\{\left((k\xi)^2 - \frac{2\epsilon_k}{\mu}\right)\right.$$
$$\left.\times\left[\frac{k\xi}{2} + i\tanh\left(\frac{x - x_i}{\xi}\right)\right] + \frac{k\xi}{\cosh^2\left(\frac{x-x_i}{\xi}\right)}\right\}.$$

The total Hamiltonian then reads

$$H = H_q + H_p + H_{\rm int}. \tag{7}$$

The term H_q describes the dark-solitons (qubits) Hamiltonian, which is given by

$$H_q = \sum_{i=1}^{2} \hbar\omega_0 \sigma_z^{(i)}, \tag{8}$$

with $\sigma_z^{(i)} = a_1^{(i)\dagger} a_1^{(i)} - a_0^{(i)\dagger} a_0^{(i)}$ being the effective spin operator of the respective qubit. The phonon (reservoir) Hamiltonian is given by

$$H_p = \sum_k \epsilon_k b_k^\dagger b_k, \tag{9}$$

with the Bogoliubov spectrum $\epsilon_k = \mu\xi\sqrt{k^2(\xi^2 k^2 + 2)}$ and the chemical potential $\mu = g n_0$. Finally, the interaction Hamiltonian $H_{\rm int}$ can be explicitly written as

$$H_{\rm int} = \sum_{i,j} \chi \int dx \Phi_j^\dagger \Psi_i^\dagger \Psi_i \Phi_j, \tag{10}$$

where $\Phi_j(x)$ describes the impurity wave function in the presence of DS potential and spannable in terms of bosonic operators a_l,

$$\Phi_j(x) = \sum_{n=0}^{1} \phi_n^{(j)}(x) a_n^{(j)}, \tag{11}$$

with the ground state $\phi_0(x) = \mathrm{sech}[(x-x_i)/\xi]/\sqrt{2\xi}$ and the excited state $\phi_1(x) = i\sqrt{3}\tanh[(x-x_i)/\xi]\phi_0(x)$. Therefore, Eq. (10) can be decomposed as

$$H_{\rm int} = H_{\rm int}^{(0)} + H_{\rm int}^{(1)} + H_{\rm int}^{(2)}, \tag{12}$$

containing zeroth-, first-, and second-order terms in the bosonic operators b_k and $b_k^\dagger$. The higher-order term $\sim O(b_k^2)$ is ignored, consistent with the Bogoliubov approximation performed in Eq. (6) owing to the small depletion of the condensate. This approximation is well justified in the case of two-level systems, as the inexistent higher excited states cannot be populated via two-phonon processes. The first part of Eq. (12) corresponds to

$$H_{\rm int}^{(0)} = n_0\chi \sum_{i=1}^{2} \sum_{n,n'=0}^{1} a_n^{\dagger(i)} a_{n'}^{(i)} f_{n,n'}^{(i)}, \tag{13}$$

with $f_{n,n'}^{(i)} = \int dx \phi_n^{\dagger(i)}(x)\phi_{n'}^{(i)}(x)\tanh^2[(x-x_i)/\xi]$, which can be omitted by renormalizing the qubit frequency $\tilde{\omega}_0 \approx \omega_0 + n_0\chi$. The first-order term is given by

$$H_{\rm int}^{(1)} = \sum_{k,i=1}^{2} \sum_{n,n'=0}^{1} a_n^{(i)\dagger} a_{n'}^{(i)} \big[b_k g_{n,n'}^{(i)}(k) + b_k^\dagger g_{n,n'}^{(i)*}(k) \big], \tag{14}$$

where

$$g_{n,n'}^{i,j}(k) = \sqrt{n_0}\chi \int dx \phi_n^{(j)\dagger}(x)\phi_{n'}^{(j)}(x) \tanh\left(\frac{x-x_i}{\xi}\right) u_k^{(i)}.$$

Equation (14) contains both interband ($n \neq n'$) and intraband ($n = n'$) terms. However, within the rotating wave approximation (RWA), the qubit transition can only be driven by near-resonant phonons, for which the intraband terms $|g_{00}^{(i)}(k)|$ and $|g_{11}^{(i)}(k)|$ are much smaller than the interband term $|g_{01}^{(i)}(k)| = |g_{10}^{(i)}(k)^*|$. As such, we obtain

$$H_{\rm int}^{(1)} = \sum_{k,i=1}^{2} (g^{(i)}(k)\sigma_+^{(i)} b_k + g^{(i)*}(k)\sigma_-^{(i)} b_k^\dagger) + \text{H.c.}$$

Here $g^{(i)}(k) = g_{n,n'}^{i,j}(k)$, $\sigma_+^{(i)} = a_1^{(i)\dagger} a_0^{(i)}$, and vice versa. The counter-rotating terms proportional to $b_k\sigma_-$ and $b_k^\dagger\sigma_+$ that do not conserve the total number of excitations are dropped by invoking the RWA. The accuracy of such an approximation can be verified in Ref. [34], where it is shown that the emission rate γ is much smaller than the qubit transition frequency ω_0 for DS qubits.

III. MASTER EQUATION

We derive the master equation (see Appendix) to describe the dynamics of the DS qubit density matrix ρ_q after taking trace over the phonon's degrees of freedom [5,44,45]

$$\frac{\partial\rho_q(t)}{\partial t} = -\frac{i}{\hbar}[H_q, \rho_q(t)] - \sum_{i\neq j}^{2} \eta_{ij}[\sigma_+^i\sigma_-^j, \rho_q(t)] + \sum_{ij=1}^{2} \Gamma_{ij}\left[\sigma_-^j \rho_q(t)\sigma_+^i - \frac{1}{2}\{\sigma_+^i\sigma_-^j, \rho_q(t)\}\right], \tag{15}$$

where

$$\Gamma_{ij} = 2L\int_0^\infty dk g_k^{(i)} g_k^{(j)*} \delta(\omega_k - \omega_0),$$
$$\eta_{ij} = \frac{L}{2\pi}\wp\int_0^\infty dk g_k^{(i)} g_k^{(j)*} \frac{1}{(\omega_k - \omega_0)}, \tag{16}$$

with $\wp$ standing for the principal value of the integral. For $i = j$, $\Gamma_{ii} \equiv \gamma$ denotes the spontaneous emission rate of the qubits due to the Bogoliubov excitations (phonons); for $i \neq j$, $\Gamma_{ij} \equiv \Gamma$ is the collective damping resulting from the mutual exchange of phonons. The coherent term $\eta_{ij} \equiv \eta$ represents the interaction between DS qubits. Both the coherent and incoherent terms are dependent on the distance d between DS qubits. Figure 2 depicts the dependence of the collective damping Γ and the coherent interaction η as a function of the DS qubit distance d. For large separations, i.e., $d \gg \xi$, both parameters are very small (i.e., $\Gamma = \eta \approx 0$).

The main concern of the present work is the study of the time evolution of the entanglement. To study this, the most adequate sates are the collective, or the so-called Dicke, states [46]

$$|g\rangle = |g_1, g_2\rangle,$$
$$|\pm\rangle = (|e_1, g_2\rangle \pm |g_1, e_2\rangle)/\sqrt{2},$$
$$|e\rangle = |e_1, e_2\rangle, \tag{17}$$

as schematically represented in Fig. 3. The density matrix of Eq. (17) becomes

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{e+} & \rho_{e-} & \rho_{eg} \\ \rho_{+e} & \rho_{++} & \rho_{+-} & \rho_{+g} \\ \rho_{-e} & \rho_{-+} & \rho_{--} & \rho_{-g} \\ \rho_{ge} & \rho_{g+} & \rho_{g-} & \rho_{gg} \end{pmatrix}, \tag{18}$$

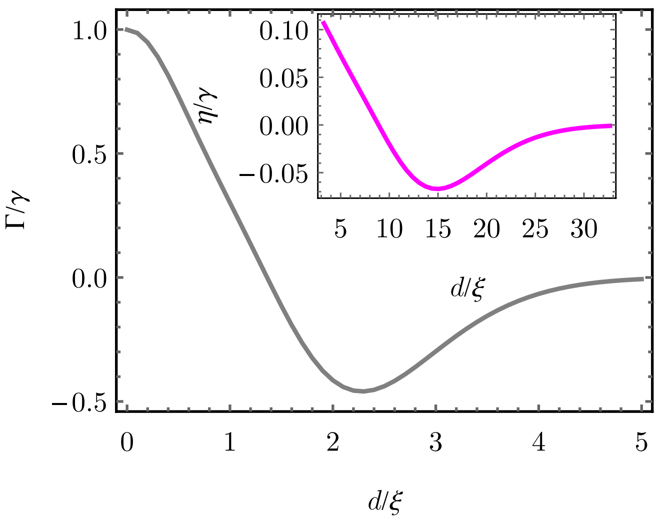

FIG. 2. Collective damping Γ and qubit-qubit interaction parameter η (inset) as a function of interatomic distance d. We have chosen $\nu = 0.75$, for which DS qubit is well defined.

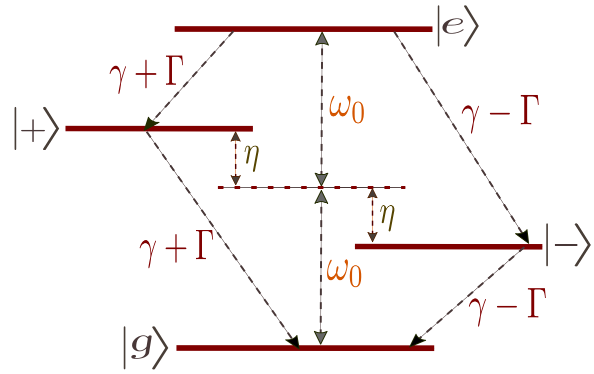

FIG. 3. Collective states of two dark-soliton qubits.

where $\rho_{ij} = \langle\psi_i|\rho|\psi_j\rangle$ with $i, j = +, -$, for example. The elements of the above density matrix can be determined by using Eq. (15),

$$\begin{aligned}
\rho_{ee}(t) &= e^{-2\gamma t}\rho_{ee}(0), \quad \rho_{++}(t) = e^{-(\gamma+\Gamma)t}\rho_{++}(0) \\
&\quad + \frac{(\gamma+\Gamma)}{(\gamma-\Gamma)}(e^{-(\gamma+\Gamma)t} - e^{-2\gamma t})\rho_{ee}(0), \\
\rho_{--}(t) &= e^{-(\gamma-\Gamma)t}\rho_{--}(0) + \frac{(\gamma-\Gamma)}{(\gamma+\Gamma)}(e^{-(\gamma-\Gamma)t} - e^{-2\gamma t})\rho_{ee}(0), \\
\rho_{eg}(t) &= e^{-\gamma t}\rho_{eg}(0), \\
\rho_{+-}(t) &= e^{-(\gamma-2i\eta)t}\rho_{+-}(0), \\
\rho_{e+}(t) &= e^{-\frac{1}{2}(3\gamma+\Gamma-2i\eta)t}\rho_{e+}(0), \\
\rho_{e-}(t) &= e^{\frac{1}{2}(3\gamma+\Gamma-2i\eta)t}\rho_{e-}(0), \\
\rho_{g+}(t) &= e^{-\frac{1}{2}(\gamma+\Gamma-2i\eta)t}\rho_{g+}(0) + \frac{(\gamma+\Gamma)}{(\gamma+2i\eta)}2e^{-\frac{1}{2}(2\gamma+\Gamma)t} \\
&\quad \times \sinh\left(\frac{t}{2}(\gamma+2i\eta)\right)\rho_{+e}(0), \\
\rho_{g-}(t) &= e^{-\frac{1}{2}(\gamma-\Gamma+2i\eta)t}\rho_{g-}(0) - \frac{(\gamma-\Gamma)}{(\gamma-2i\eta)}2e^{-\frac{1}{2}(2\gamma-\Gamma)t} \\
&\quad \times \sinh\left(\frac{t}{2}(\gamma-2i\eta)\right)\rho_{-e}(0),
\end{aligned} \tag{19}$$

with $\rho_{jk} = \rho^*_{kj}$ and $\rho_{gg} = 1 - \rho_{ee} - \rho_{++} - \rho_{--}$. Equation (19) depicts that all transition rates to and from the state ρ_{++} are equal to $\gamma + \Gamma$ while from state ρ_{--} are $\gamma - \Gamma$. Therefore, the state ρ_{++} decays with an enhanced (superradiant) rate and ρ_{--} with a reduced (subradiant) rate (see Fig. 3).

IV. MEASUREMENT OF ENTANGLEMENT

The amount of entanglement can be determined by using the Wootter's concurrence [47]

$$C(t) = \max(0, \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} - \sqrt{\varepsilon_3} - \sqrt{\varepsilon_4}),$$

where the ε_i's are the eigenvalues in decreasing order of magnitude of the matrix

$$\zeta = \rho\sigma_y \otimes \sigma_y\rho^*\sigma_y \otimes \sigma_y.$$

Here ρ^* represents the complex conjugate of ρ and σ_y is the Pauli matrix. Depending on the initial state, concurrence can reach a value equal to zero asymptotically or at some finite time. It is interesting to observe that locally equivalent initial states with the same concurrence can disentangle at different times, depending on the parameters Γ and η. In what follows, we investigate this aspect.

A. Entangled state

Let us assume that, initially, both or neither of the DS qubits are excited, i.e., the qubits are chosen to be prepared initially in an entangled state

$$|\Psi\rangle = \sqrt{1-\alpha}|g\rangle + \sqrt{\alpha}|e\rangle, \tag{20}$$

with $0 \leqslant \alpha \leqslant 1$. Therefore, the density matrix Eq. (18) becomes

$$\rho = \begin{pmatrix} \rho_{ee} & 0 & 0 & \rho_{eg} \\ 0 & \rho_{++} & 0 & 0 \\ 0 & 0 & \rho_{--} & 0 \\ \rho_{ge} & 0 & 0 & \rho_{gg} \end{pmatrix}.$$

The eigenvalues of the respective matrix ζ are thus given by

$$\begin{aligned}
\sqrt{\varepsilon_{1,2}} &= \sqrt{\rho_{ee}(t)\rho_{gg}(t)} \pm |\rho_{ge}(t)|, \\
\sqrt{\varepsilon_{3,4}} &= \tfrac{1}{2}[\rho_{++}(t) + \rho_{--}(t)] \pm \tfrac{1}{2}[\rho_{++}(t) - \rho_{--}(t)].
\end{aligned}$$

It is easy to verify that, depending on the largest eigenvalue (either $\sqrt{\varepsilon_1}$ or $\sqrt{\varepsilon_3}$), the concurrence $C(t)$ can be defined in

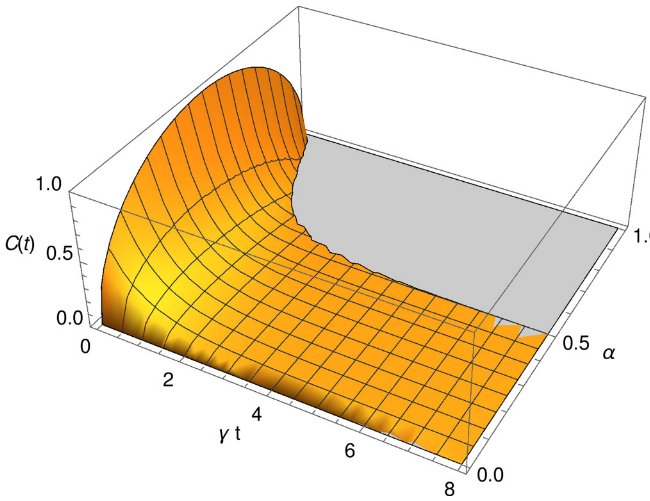

FIG. 4. Time evolution of concurrence $C(t)$ for for an initial entangled state $|\Psi\rangle$ in a noninteracting DS qubit system.

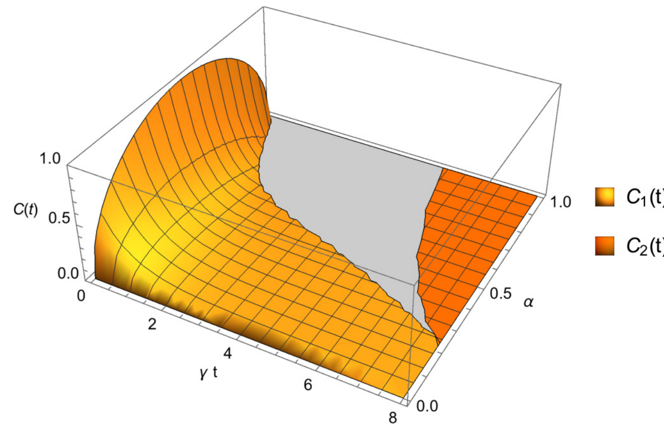

FIG. 5. Time evolution of concurrence $C(t)$ for an initial entangled state $|\Psi\rangle$ at distance $d = 6\xi/5$.

two alternative ways, i.e.,

$$C_1(t) = 2|\rho_{ge}(t)| - [\rho_{++}(t) + \rho_{--}(t)], \tag{21}$$

$$C_2(t) = |\rho_{++}(t) - \rho_{--}(t)| - 2\sqrt{\rho_{ee}(t)\rho_{gg}(t)}, \tag{22}$$

where $C_1(t)$ measures the entanglement produced by the state of Eq. (20) with the necessary condition $\rho_{eg}(t) \neq 0$, while $C_2(t)$ provides the entanglement formed by the states $|\pm\rangle$. Notice that the positiveness of $C_2(t)$ is guaranteed if the latter states are not equally populated. At $t = 0$, the system is entangled by the amount $C_1(0) = 2\sqrt{\alpha(1-\alpha)}$. By inspecting Eqs. (19), (21), and (22), it is possible to observe that DS qubits radiating independently ($\Gamma = 0$) cannot be entangled by following the criterion $C_2(t)$ because $\rho_{++}(t) = \rho_{--}(t)$. Thus, the time for the entanglement death due to spontaneous emission can be found via the condition $C_1(t) = 0$, which provides

$$t_{\text{death}} = \frac{1}{\gamma}\ln\left(\frac{\alpha}{\alpha - \sqrt{\alpha(1-\alpha)}}\right). \tag{23}$$

It is also pertinent to mention here that the condition for a finite-time disentanglement for independent DS qubits is $\alpha > 1/2$ (see Fig. 4).

The situation changes when we allow the qubits to interact. In this case, the entanglement death is followed by its revival at a larger time t_{revival}. Figure 5 depicts the time evolution of the concurrence for the collective interactive system. It is shown that the entanglement dies as a consequence of the spontaneous emission, but revives after a time $t_{\text{revival}} \simeq 8/\gamma$, for $\alpha \simeq 1/4$ and qubits placed at a distance $d = 6\xi/5$. After a careful inspection of Eq. (19), it is observed that the concurrence $C_1(t) < 0$ at long times. Therefore, finite time (t_{revival}) entanglement is determined by following $C_2(t)$ which yields

$$t_{\text{revival}} = \frac{2}{3\Gamma}\ln\left(\frac{4\gamma}{\sqrt{\alpha}(\gamma - \Gamma)}\right). \tag{24}$$

Moreover, it can be analyzed from Fig. 6 that entanglement vanishes around the time at which $\rho_{--}(t)$ is maximum, i.e., the state $\rho_{--}(t)$ is maximally populated, and that it does not undergo any revival. The latter is due to the impartiality between the term $\rho_{eg}(t)$ and $\rho_{--}(t)$. In other words, $\rho_{eg}(t)$ and $\rho_{--}(t)$ go to almost zero at long times while the population of $\rho_{++}(t)$ accumulates on the timescale $t = 1/(\gamma + \Gamma)$ which is sufficiently large at $\Gamma \approx -0.5\gamma$. At large distances, when $d \simeq 5\xi$, the collective damping is very small $\Gamma \approx 0$ and DS qubits act like an independent qubits (as in Fig. 4).

B. Mixed state

We now consider a two-qubit system to be initially prepared in a diagonal basis of the collective states. Therefore, the initial density matrix has the form

$$\rho(0) = \frac{1}{3}\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix}. \tag{25}$$

The initial concurrence is determined by $C_2(0) = 2(1 - \sqrt{\alpha(1-\alpha)})/3$, and the sudden-death time for independent DS qubits can be described by using criterion

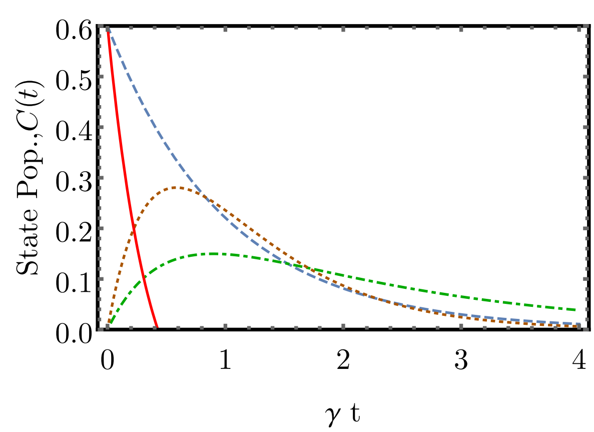

FIG. 6. State population [dashed curve for $\rho_{ge}(t)$, dotted-dashed for $\rho_{++}(t)$ and dotted for $\rho_{--}(t)$] and concurrence $C_1(t)$ (solid curve) at $d \simeq 5\xi/2$.

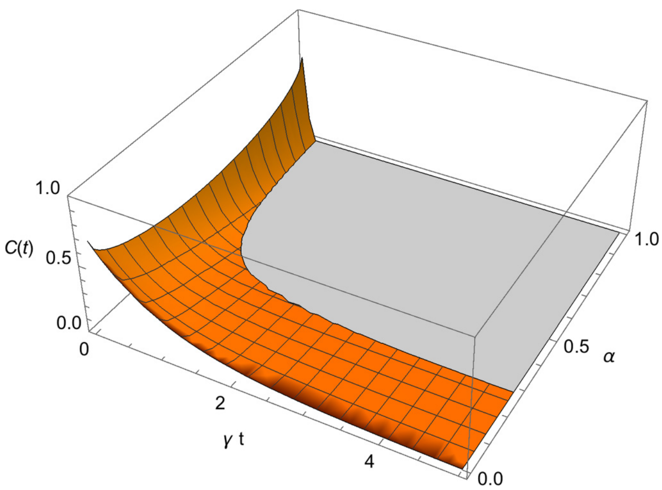

FIG. 7. Time evolution of concurrence $C(t)$ for a mixed initial state in a noninteracting DS qubit system.

$C_2(t) = 0$, which provides

$$t_{\text{death}} = \frac{1}{\gamma}\ln\left(\frac{\alpha}{\sqrt{(3\alpha^2+5\alpha)}-(1+\alpha)}\right). \tag{26}$$

It is obvious from Eq. (26) that the entanglement sudden death (ESD) is possible only for $\alpha \gtrsim 1/3$ (see Fig. 7). The time evolution of the concurrence for interacting qubits is depicted in Fig. 8. It is observed that the entanglement first decays and then revives for $\alpha \gtrsim 1/2$ at $d \approx 4\xi$. ESD happens at $t_{\text{death}} \sim 0.75/\gamma$, while entanglement revival is obtained at $t_{\text{revival}} \sim 1.4/\gamma$.

C. Experimental estimates

We consider a quasi-1D BEC of ^{85}Rb with a chemical potential (μ) of a few kHz. This yields a qubit gap frequency $\omega_0/2\pi \sim 0.5$ kHz, the spontaneous decay rate $\gamma/2\pi \sim 29$ Hz and a collective decay $\Gamma/2\pi \sim 6$ Hz, at $d \sim 6\xi/5 \sim 1\ \mu$m. These rates validate *a posteriori* the RWA and Markovian approximations. Therefore, the sudden death time for the maximally entangled state is $t_{\text{death}} \sim 19$ ms and the revival time is $t_{\text{revival}} \sim 35$ ms, where the period $\Delta t \sim 16$ ms is the dark period, i.e., the time interval during which $C(t) = 0$. For large separations, ESD occurs at $t_{\text{death}} \sim 2$ ms due to the balanced population of ρ_{eg} and ρ_{--}, whereas, for the mixed initial state, the dark period occurs for ~ 3.6 ms. Prolonging entanglement is essential for practical realization of quantum information and computation protocols based on entanglement. Therefore, the dark period of entanglement can be delayed or averted by carrying out local unitary operation on qubits [48,49].

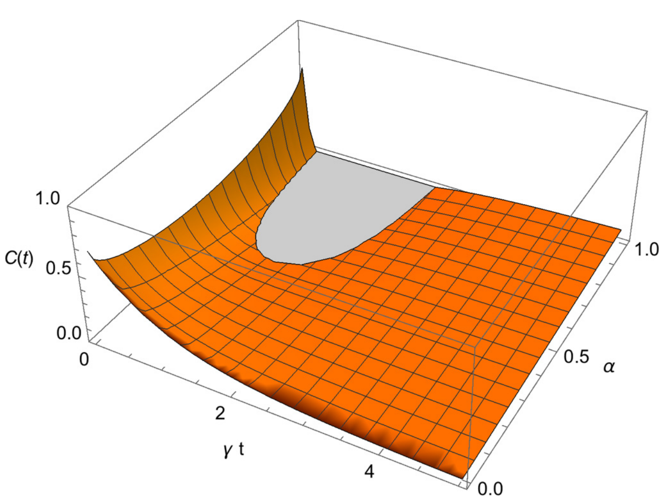

FIG. 8. Time evolution of concurrence $C(t)$ for a mixed initial state at distance $d \approx 4\xi$ between DS qubits.

V. SUMMARY AND DISCUSSION

To summarize, we investigate the finite-time disentanglement, or the entanglement sudden death, between two dark-soliton qubits produced in a quasi-one-dimensional BEC. We derive the master equation and extracted the time evolution of the relevant density matrix elements. The Wooter's concurrence is used as a measure of entanglement and we show the collapse and revival behavior, depending on the collective damping and on the initial state. For an initial entangled state, the concurrence cannot be revived at large distances in the range of 2–5 μm due to the impartial behavior of the populated states, while it revives for a mixed state. Therefore, it can be concluded that the collective behavior of the dark-soliton qubits reveals the dependence of entanglement evolution on the interatomic distance and it becomes quite different from that of independent dark-soliton qubits.

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APPENDIX: DERIVATION OF THE MASTER EQUATION

We begin by writing the total Hamiltonian by spanning the Hilbert space as

$$H(t) = H_q \otimes I + I \otimes H_p + H_{\text{int}}(t). \tag{A1}$$

The key ingredient for the application of the Born-Markov approximation is the assumption that $H_q \otimes H_p$ is small compared to the remaining terms, so that a perturbative treatment of the interaction is possible. To make this more explicit, we take $H_0 = H_q + H_p$ and move into the interaction picture

$$H_I(t) = e^{iH_0t} H_I e^{-iH_0t}. \tag{A2}$$

We make the Born approximation and assume that the density operator factorizes at all times as

$$\rho(t) \cong \rho_q(t) \otimes \rho_p, \tag{A3}$$

where the reservoir density operator is assumed to be time independent, i.e., $\rho_p = \rho_p(0)$. Therefore, within the interaction picture, the density operator evolves according to

$$\frac{d\rho(t)}{dt} = -\imath[H_I(t), \rho(t)], \tag{A4}$$

and the respective formal integral solution is given by

$$\rho(t) = \rho(0) - \imath \int_0^t [H_I(s), \rho(s)]ds. \tag{A5}$$

By plugging Eq. (A5) into Eq. (A4), and by taking the partial trace over reservoir degrees of freedom (phonons), we obtain

$$\frac{d\rho_q(t)}{dt} = -\int_0^t ds\, \mathrm{Tr}_P[H_I(t), [H_I(s), \rho_q(s)\rho_p]]. \tag{A6}$$

This equation is called the Redfield equation where the term $\mathrm{Tr}_P[H_I(t), \rho(0)]$ is disregarded. Furthermore, we make use of the Markov approximation to put Eq. (A6) into a more amenable form. This assures that the behavior of $\rho_q(t)$ is local in time. This master equation still depends on the choice of the initial state. However, making the substitution $s \to t - \tau$ and letting the upper integration limit to go to infinity, we obtain

$$\frac{d\rho_q(t)}{dt} = -\int_0^\infty d\tau\, \mathrm{Tr}_P[H_I(t), [H_I(t-\tau), \rho_q(t)\rho_p]]. \tag{A7}$$

The latter is the Born-Markov master equation. To gain a little more insight into the structure of Eq. (A7), it is useful to be more specific about the form of the interaction picture Hamiltonian

$$H_I(t) = S^\dagger B + SB^\dagger, \tag{A8}$$

where $S^\dagger(t) = \sum_{i=1}^2 \sigma_+^{(i)} = e^{\imath\omega_0 t}S^\dagger$, $B(t) = \sum_k g(k)e^{-\imath\omega_k t}b$, and vice versa. Here we use the identity

$$e^{\alpha A}Se^{-\alpha A} = S + \alpha[A, S] + \frac{\alpha^2}{2!}[A, [A, S]] + \cdots . \tag{A9}$$

Moreover, we invoke the cyclic property of trace to write the reservoir correlation function as $\mathrm{Tr}_P(b_k b_q^\dagger \rho_P) = \delta_{k,q}$. As such, the Born-Markov master equation (A7) can be finally rewritten as

$$\frac{d\rho_q(t)}{dt} = -\gamma[\rho_q(t)S^\dagger S - S\rho_q(t)S^\dagger] - \gamma[\rho_q(t)SS^\dagger - S\rho_q(t)S^\dagger] + \mathrm{H.c.}, \tag{A10}$$

where

$$\gamma = \sum_k g(k)g(k)^* \int_0^t d\tau e^{-\imath(\omega_k-\omega_0)(t-\tau)}.$$

The sum over the phonon k modes can be computed by taking the continuum limit

$$\sum_k \to \int_0^\infty D(k)dk, \tag{A11}$$

where $D(k) = L/2\pi$ is the density of states, L is the size of the system, and

$$\int_0^t d\tau e^{-i(\omega_k-\omega_0)(t-\tau)} = [\pi\delta(\omega_k - \omega_0) - i\wp/(\omega_k - \omega_0)]. \tag{A12}$$

Transforming Eq. (A.10) back in the Schrödinger picture, we finally obtain

$$\frac{d\rho_q(t)}{dt} = -\frac{i}{\hbar}[H_q, \rho_q(t)] - \sum_{i\neq j}^2 \eta_{ij}[\sigma_+^i\sigma_-^j, \rho_q(t)] + \sum_{ij=1}^2 \Gamma_{ij}\left[\sigma_-^j\rho_q(t)\sigma_+^i - \frac{1}{2}\{\sigma_+^i\sigma_-^j, \rho_q(t)\}\right]. \tag{A13}$$

Equation (A13) is the final form of the master equation used in this work.

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