

Several nonlocal sets of multipartite pure orthogonal product states

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It is known that there exist sets of pure orthogonal product states which cannot be perfectly distinguished by local operations and classical communication (LOCC). Such sets are nonlocal sets which exhibit nonlocality without entanglement. These nonlocal sets can be completable or uncompletable. In this work both completable and uncompletable small nonlocal sets of multipartite orthogonal product states are constructed. Apart from nonlocality, these sets have other interesting properties. In particular, the completable sets lead to the construction of a class of complete orthogonal product bases with the property that if such a basis is given then no state can be eliminated from that basis by performing orthogonality-preserving measurements. On the other hand, an uncompletable set of the present kind contains several Shifts unextendible product bases (UPBs) that belong to qubit subspaces. Identifying these subspace UPBs, it is possible to obtain a class of high-dimensional multipartite bound entangled states. Finally, it is shown that a two-qubit maximally entangled Bell state shared between any two parties is sufficient as a resource to distinguish the states of any completable set (of the above kind) perfectly by LOCC. This constitutes an example where the amount of entanglement, sufficient to accomplish the aforesaid task, depends neither on the dimension of the individual subsystems nor on the number of parties.

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In quantum information processing protocols, classical information is encoded within the state of a quantum system. Therefore, it is necessary to determine the state of a system in order to extract information. In this sense, after information encoding, it is required to discriminate among the possible states of a given system to decode that information. If the possible states of a given system are pairwise orthogonal to each other then the state of that system can in principle be determined perfectly by performing a suitable measurement on the *whole* system. Nevertheless, if the constituent parts (subsystems) of a composite quantum system are distributed among several spatially separated parties then it is not possible to perform measurements on the whole system. Such a constraint allows the parties to perform any sequence of quantum operations only on their individual subsystems and to make strategies they can communicate with each other classically. This class of operations on a distributed quantum system is commonly known as local operations and classical communication (LOCC).

It is an established fact that the set of operations on a distributed quantum system which can be implemented by LOCC is a strict subset of all physically realizable quantum operations on the whole system. Thus, after information encoding the task of determining the state of a distributed quantum system perfectly by LOCC is not always possible even if the possible states of the system are pairwise orthogonal to each other. However, to study various properties of any composite quantum system, distributed among several spatially separated

parties, LOCC plays a crucial role. Hence, to explore the properties of distributed quantum systems and to implement information processing tasks via distributed quantum systems, it is highly important to determine the states of such systems by LOCC. This particular task of determining the unknown state of a distributed quantum system by LOCC when a set of possible states is given is known as the LOCC state discrimination problem or local state discrimination problem (LSDP).

The LSDP, as it is understood now, was first considered by Peres and Wootters [1]. During the last couple of decades, the LSDP has gotten considerable attention [2–13]. For a given set of pairwise orthogonal quantum states, if it is not possible to distinguish all the states perfectly by LOCC then the states are locally indistinguishable and the set is called a locally indistinguishable set or a nonlocal set. On the contrary, for a given set of pairwise orthogonal quantum states, if it is possible to distinguish all the states perfectly by LOCC then the states are locally distinguishable and the set is called a locally distinguishable set. If a nonlocal set forms a basis in a particular Hilbert space corresponding to a given quantum system then the set is said to be a nonlocal basis. Nonlocal sets have also found practical applications in data hiding [14,15], quantum secret sharing [16], etc.

Due to the discovery of quantum nonlocality without entanglement by Bennett *et al.* [17], it is now understood that product states can also lead to local indistinguishability. In the paper just cited, the authors constructed two distinct nonlocal sets of orthogonal product states, forming complete bases in $\mathbb{C}^3 \otimes \mathbb{C}^3$ and in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Clearly, these bases constitute nonlocal separable operations, that is, separable operations which cannot be implemented by LOCC, though it is true that

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all quantum operations that can be implemented by LOCC are necessarily separable. In general, a nonlocal set of orthogonal product states that can be extended to a complete orthogonal product basis (COPB) always corresponds to a separable operation that cannot be realized by LOCC. In this context, it is important to mention that the mathematical structure of LOCC is still to be completely understood while separable operations are rich with mathematical structure. Again, if a quantum operation on a composite quantum system cannot be implemented by separable operations then that task must not be implemented by LOCC. This implies an important significance of exploring different types of separable operations.

There are other types of sets of pure orthogonal product states, i.e., unextendible product bases (UPBs), uncompletable product bases (UCPBs), strongly uncompletable product bases (SUCPBs). The details regarding these sets can be found in Refs. [18,19]. These sets cannot be extended to COPBs. In fact, the orthogonal states of a UPB or that of an SUCPB cannot be perfectly distinguished by LOCC, whereas the orthogonal states within a UCPB cannot be perfectly distinguished by local projective measurements and classical communication [19]. Again, an important property of a UPB is that it leads to the generation of a bound entangled state [18,19], a mixed entangled state from which no pure entangled state can be obtained by LOCC even if an arbitrary number of identical copies of the state are given. Examples of such states were first constructed in Ref. [20]. Since then there has been no easy technique to detect bound entangled states. Therefore, constructing different classes of bound entangled states is always a nontrivial task. In the following paragraph a few important results regarding different nonlocal sets of orthogonal product states are discussed.

After the discovery of nonlocal COPBs by Bennett *et al.*, certain useful techniques were developed [21,22] to prove the nonlocality of those COPBs. There are many other papers [23–48] in which nonlocal sets of orthogonal product states (OPBs) were constructed and different properties of such sets were studied. But multipartite systems are less explored with respect to the bipartite systems. This is in a sense that, apart from multipartite UPBs, mainly various structures of other nonlocal sets were studied. In Ref. [24], different classes of m -partite nonlocal sets of OPBs were constructed, where the dimension of any subsystem is dependent on the number of parties. Later [26,28], local distinguishability and indistinguishability of the OPBs which belong to $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ were extensively studied. In Refs. [30,32], nonlocal COPBs were constructed, where the OPBs are associated with $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. Xu *et al.* showed that there exists a small nonlocal set of m -partite OPBs with only $2m$ members in $(\mathbb{C}^2)^{\otimes m}$ [37]. In Ref. [40], both bipartite and multipartite UPBs were presented. Furthermore, in Refs. [41–43,46], other forms of nonlocal sets of multipartite OPBs were shown. However, in this work not only distinct nonlocal sets of multipartite OPBs are introduced but also a few interesting properties of such sets are investigated.

Another direction of research is to explore entanglement as a resource to distinguish quantum states of nonlocal sets [49–58] by LOCC. In particular, the problem of distinguishing product states of given nonlocal sets by LOCC using entanglement as a resource was considered in Refs. [50,56,58]. In Ref. [50],

Cohen constructed entanglement-assisted *local protocols* (protocols implementable by LOCC) to distinguish the orthogonal product states of several unextendible product bases. Later [56], separate entanglement-assisted local protocols were constructed to distinguish the states of a class of nonlocal bipartite COPBs. Recently [58], the problem of product state discrimination by LOCC was considered, where multiple copies of a maximally entangled state in $\mathbb{C}^2 \otimes \mathbb{C}^2$ are used as resource. Nonetheless, the present result of entanglement-assisted product state discrimination by LOCC is different from the existing results, and the resource state in this scenario is used quite efficiently.

The remaining portion of this paper is arranged as follows. In Sec. II, necessary definitions and other preliminary concepts are presented. The forms of completable and uncompletable small nonlocal sets, associated with a three-qutrit, tripartite quantum system, are shown in Sec. III. In the same section, a few interesting properties of those sets are also discussed. Multipartite generalization of such sets is given in Sec. IV. Next, in Sec. V, an entanglement-assisted product state discrimination protocol is constructed to distinguish the states of any completable set of the present kind by LOCC. Finally, the conclusion is drawn in Sec. VI with some open problems for further studies.

II. PRELIMINARIES

Before presenting the definitions, it is important to mention that, like other works on multipartite quantum systems [24,26,28,30,32,37,40–43,46], in this work also only multipartite pure orthogonal fully separable states are considered and the states within a set are equally probable. The parties who are holding the subsystems are spatially separated and, thus, they are restricted to perform LOCC only. In a local discrimination protocol, there could be several rounds and, to complete such a protocol successfully, it is necessary to eliminate states of a given set by LOCC in several rounds. Here the product states are pairwise orthogonal to each other and thus it is quite relevant to consider the perfect discrimination of such product states. Note that in a multiround protocol it is necessary to preserve the orthogonality of the states after each round to distinguish the states perfectly.

In quantum theory, a measurement on a system of dimension d can be expressed by a set of positive operator-valued measure (POVM) elements $\{\pi_l\}$. Such elements satisfy the completeness relation, that is, $\sum_l \pi_l = \mathcal{I}_{d \times d}$, where $\mathcal{I}_{d \times d}$ is a $d \times d$ identity matrix.

Definition 1. While distinguishing an unknown state of a given set, if all POVM elements of a measurement are proportional to the identity matrix then such a measurement is said to be a trivial measurement since such a measurement is not efficient to extract information useful for state discrimination. On the other hand, if not all POVM elements of a measurement are proportional to the identity matrix then the measurement is said to be a nontrivial measurement.

Definition 2. Consider a measurement to distinguish a fixed set of pairwise orthogonal quantum states. After performing that measurement, if the postmeasurement states are also pairwise orthogonal to each other then such a measurement is said to be an orthogonality-preserving measurement.

Suppose a set of multipartite orthogonal quantum states is given. If no party is able to begin with a nontrivial orthogonality-preserving measurement then it is not possible to eliminate any state from the given set keeping the postmeasurement states orthogonal to each other. In fact, this guarantees nonlocality of the given set. This fact follows directly from the arguments given in Refs. [21,22].

Definition 3. Consider an m -partite quantum system $\mathcal{H} = \otimes_{i=1}^m \mathcal{H}_i$. Also consider a set $S \in \mathcal{H}$ of pure orthogonal product states. The set S constitutes a COPB if it spans \mathcal{H} while the set S is said to be an incomplete orthogonal product basis (ICOPB) if it spans a subspace \mathcal{H}_S of \mathcal{H} .

Definition 4. An ICOPB in a fixed Hilbert space \mathcal{H} constitutes a UCPB if the complementary subspace \mathcal{H}_S^\perp contains a smaller number of pure orthogonal product states with respect to its dimension. On the other hand, an ICOPB is completable if the complementary subspace \mathcal{H}_S^\perp is spanned by a set of orthogonal product states.

Definition 5. Assume that an ICOPB is given. It is said to be a UPB if there is no product state in the complementary subspace \mathcal{H}_S^\perp .

Consider a set S of pure orthogonal product states $\{|\psi_k\rangle\}_{k=1}^n \in \mathcal{H} = \otimes_{i=1}^m \mathcal{H}_i$. Suppose this set constitutes an unextendible product basis and it spans \mathcal{H}_S of \mathcal{H} . Then, the normalized projector onto the complementary subspace \mathcal{H}_S^\perp is given by

$$\rho = \frac{1}{D-n} \left(\mathcal{I} - \sum_{k=1}^n |\psi_k\rangle\langle\psi_k| \right), \quad (1)$$

where D is the net dimension of \mathcal{H} and \mathcal{I} is the identity operator onto \mathcal{H} . The state ρ is an entangled state with positive partial transpose in each bipartition, that is, a multipartite bound entangled state [18]. Next, the definition of another type of product basis is presented.

Definition 6. Suppose a UCPB is given in a fixed Hilbert space \mathcal{H} . This UCPB is said to be an SUCPB if it cannot be completable in any locally extended Hilbert space \mathcal{H}_{ext} , where $\mathcal{H}_{\text{ext}} = \otimes_{i=1}^m (\mathcal{H}_i \oplus \mathcal{H}'_i)$.

Note that for a given Hilbert space a UPB is an obvious example of an SUCPB but an SUCPB may not be a UPB. However, here the definitions of UCPB, UPB, and SUCPB are given according to Refs. [18,19].

III. TRIPARTITE SYSTEM

Consider a tripartite quantum system associated with a Hilbert space $\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$. In the following, an explicit construction of a small set S of pure orthogonal product states $\{|\psi_k\rangle\} \in \mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$ is presented with $k = 1, \dots, 12$. To normalize the states of this section, consider $|a \pm b\rangle \equiv (1/\sqrt{2})(|a\rangle \pm |b\rangle)$ for $a, b = 0, 1, 2$. The states are given by

$$\begin{aligned} |\psi_1\rangle &= |0\rangle|1\rangle|0+1\rangle, & |\psi_2\rangle &= |0\rangle|1\rangle|0-1\rangle, \\ |\psi_3\rangle &= |0\rangle|2\rangle|0+2\rangle, & |\psi_4\rangle &= |0\rangle|2\rangle|0-2\rangle, \\ |\psi_5\rangle &= |1\rangle|0+1\rangle|0\rangle, & |\psi_6\rangle &= |1\rangle|0-1\rangle|0\rangle, \\ |\psi_7\rangle &= |2\rangle|0+2\rangle|0\rangle, & |\psi_8\rangle &= |2\rangle|0-2\rangle|0\rangle, \\ |\psi_9\rangle &= |0+1\rangle|0\rangle|1\rangle, & |\psi_{10}\rangle &= |0-1\rangle|0\rangle|1\rangle, \\ |\psi_{11}\rangle &= |0+2\rangle|0\rangle|2\rangle, & |\psi_{12}\rangle &= |0-2\rangle|0\rangle|2\rangle. \end{aligned} \quad (2)$$

Clearly, the set S does not form a COPB in $\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$ but it is possible to extend this set to a COPB by considering suitable product states (pairwise orthogonal). One such choice is given by

$$\begin{aligned} &|0\rangle|0\rangle|0\rangle, |0\rangle|1\rangle|2\rangle, |0\rangle|2\rangle|1\rangle, |1\rangle|0\rangle|2\rangle, |1\rangle|1\rangle|1\rangle, \\ &|1\rangle|1\rangle|2\rangle, |1\rangle|2\rangle|0\rangle, |1\rangle|2\rangle|1\rangle, |1\rangle|2\rangle|2\rangle, |2\rangle|0\rangle|1\rangle, \\ &|2\rangle|1\rangle|0\rangle, |2\rangle|1\rangle|1\rangle, |2\rangle|1\rangle|2\rangle, |2\rangle|2\rangle|1\rangle, |2\rangle|2\rangle|2\rangle. \end{aligned} \quad (3)$$

The states of the set S and that of the above equation together form a COPB in $\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$. However, this set leads to an interesting property which is given in the following theorem.

Theorem 1. Let \mathcal{B} be a complete orthogonal product basis in a three-qutrit, tripartite Hilbert space. Then no state from \mathcal{B} can be eliminated by performing orthogonality-preserving measurements if \mathcal{B} contains S .

Proof. To prove the above, first it is shown that the states of the set S allow each party to perform only trivial measurements on an entire three-dimensional subsystem if they want to preserve the orthogonality of the states.

Assume that the first party performs a measurement on his (or her) three-dimensional subsystem. Suppose this measurement is defined by a set of POVM elements $\{\pi_l\}$, $\sum_l \pi_l = \mathcal{I}_{3 \times 3}$. The matrix form of $\pi_l = M_l^\dagger M_l$ can be written in the $\{|0\rangle, |1\rangle, |2\rangle\}$ basis and it is given by

$$\pi_l = M_l^\dagger M_l = \begin{pmatrix} e_{00} & e_{01} & e_{02} \\ e_{10} & e_{11} & e_{12} \\ e_{20} & e_{21} & e_{22} \end{pmatrix}. \quad (4)$$

If this measurement is orthogonality preserving then after the measurement by the first party the postmeasurement states remains pairwise orthogonal to each other. So, the states $\{(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_k\rangle, k = 1, \dots, 12\}$ must be orthogonal to each other. Setting the inner product of the postmeasurement states $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_5\rangle$ and $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_7\rangle$ equals to zero, it is found that $\langle 1|M_l^\dagger M_l|2\rangle = e_{12} = \langle 2|M_l^\dagger M_l|1\rangle = e_{21} = 0$. Similarly, considering the inner product of the postmeasurement states $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_1\rangle$ and $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_5\rangle$, it turns out that $e_{01} = e_{10} = 0$. Again, the inner product of the postmeasurement states $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_3\rangle$, $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_7\rangle$ results in $e_{02} = e_{20} = 0$. In this way, it is proved that all *off-diagonal* entries of the above matrix are *zero*.

Now, considering the inner product of the postmeasurement states $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_9\rangle$ and $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_{10}\rangle$ it is found that $\langle 0+1|M_l^\dagger M_l|0-1\rangle = 0$ and this implies that $e_{00} = e_{11}$. Similarly, taking the inner product of $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_{11}\rangle$ and $(M_l \otimes \mathcal{I} \otimes \mathcal{I})|\psi_{12}\rangle$, it turns out that $e_{00} = e_{22}$. So, $e_{00} = e_{11} = e_{22}$, i.e., all *diagonal* entries of the above matrix are *equal*.

So, the POVM elements $\{\pi_l\}$ that define the measurement for the first party are all proportional to a 3×3 identity matrix. This implies that the first party can perform only trivial measurements if he (or she) wants to preserve the orthogonality of the states.

Notice that there is a symmetry present in the states of the set S . For instance, if the order of the subsystems is rearranged in a way that the third party holds the subsystem of the first party, the second party holds the subsystem of the third party, and the first party holds the subsystem of the second party, then the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are mapped to

$|\psi_5\rangle$ and $|\psi_6\rangle$, respectively. Applying the same rearrangement rules, the following transformations are obtained: $|\psi_5\rangle \rightarrow |\psi_9\rangle$, $|\psi_6\rangle \rightarrow |\psi_{10}\rangle$, $|\psi_9\rangle \rightarrow |\psi_1\rangle$, $|\psi_{10}\rangle \rightarrow |\psi_2\rangle$, $|\psi_3\rangle \rightarrow |\psi_7\rangle$, $|\psi_4\rangle \rightarrow |\psi_8\rangle$, $|\psi_7\rangle \rightarrow |\psi_{11}\rangle$, $|\psi_8\rangle \rightarrow |\psi_{12}\rangle$, $|\psi_{11}\rangle \rightarrow |\psi_3\rangle$, and $|\psi_{12}\rangle \rightarrow |\psi_4\rangle$. Because of this symmetry, if one party cannot start with a nontrivial orthogonality-preserving measurement then the other parties cannot either. So, the states of the set \mathcal{S} allow each party to start with only a trivial orthogonality-preserving measurement on their subsystems and this holds true for any basis \mathcal{B} if \mathcal{S} is fully contained in \mathcal{B} .

Again, to eliminate any state from \mathcal{B} by performing an orthogonality-preserving measurement, it is necessary that at least one party can start with a nontrivial measurement which also preserves the orthogonality of the states. In this sense, as no party can start with a nontrivial orthogonality-preserving measurement then it guarantees that no state from \mathcal{B} can be eliminated by performing orthogonality-preserving measurements. This completes the proof. ■

Obviously, the states of the set \mathcal{S} constitute a sufficient condition for the basis \mathcal{B} such that no state from this basis can be eliminated by performing orthogonality-preserving measurements. From the above discussion, it is also prominent that the states of \mathcal{S} cannot be perfectly distinguished by LOCC. This is because of the fact that for local distinguishability it is necessary to eliminate state(s) of a given set, which is not possible in the present scenario. In this way, Theorem 1 provides a sufficient condition but not a necessary one for local indistinguishability of the states within the basis \mathcal{B} in a three-qutrit tripartite Hilbert space. This particular notion is represented by the following corollary.

Corollary 1. Let \mathcal{B}' be a complete orthogonal product basis in a three-qutrit tripartite Hilbert space. If no state from \mathcal{B}' can be eliminated by performing orthogonality-preserving measurements then the states of \mathcal{B}' cannot be perfectly distinguished by local operations and classical communication, though the converse is not always true.

In support of the above corollary, a COPB \mathcal{B}' in $\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$ is constructed from which certain states can be eliminated by performing an orthogonality-preserving measurement, though it is true that all the states of such a basis cannot be perfectly distinguished by LOCC.

Consider the states $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_5\rangle, |\psi_6\rangle, |\psi_9\rangle, |\psi_{10}\rangle\}$ of the set \mathcal{S} . It is known that these states cannot be perfectly distinguished by LOCC [37]. Thus, if a COPB \mathcal{B}' contains these states along with other product states then the basis must be a nonlocal basis. In order to construct such a basis \mathcal{B}' in $\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$, consider the following set of product states:

$$\begin{aligned} &|0\rangle|0\rangle|0\rangle, |0\rangle|0\rangle|2\rangle, |0\rangle|1\rangle|2\rangle, \\ &|0\rangle|2\rangle|0\rangle, |0\rangle|2\rangle|1\rangle, |0\rangle|2\rangle|2\rangle, \\ &|1\rangle|0\rangle|2\rangle, |1\rangle|1\rangle|1\rangle, |1\rangle|1\rangle|2\rangle, \\ &|1\rangle|2\rangle|0\rangle, |1\rangle|2\rangle|1\rangle, |1\rangle|2\rangle|2\rangle, \\ &|2\rangle|0\rangle|0\rangle, |2\rangle|0\rangle|1\rangle, |2\rangle|0\rangle|2\rangle, \\ &|2\rangle|1\rangle|0\rangle, |2\rangle|1\rangle|1\rangle, |2\rangle|1\rangle|2\rangle, \\ &|2\rangle|2\rangle|0\rangle, |2\rangle|2\rangle|1\rangle, |2\rangle|2\rangle|2\rangle. \end{aligned} \quad (5)$$

The product states given above and the product states $\{|\psi_k\rangle\}$ of \mathcal{S} with $k = 1, 2, 5, 6, 9, 10$ together form a COPB \mathcal{B}' in

$\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$. Interestingly, it is possible to define a nontrivial and orthogonality-preserving measurement by which the parties are able to eliminate certain states from \mathcal{B}' . The states given in the previous equation excluding two states $|0\rangle|0\rangle|0\rangle$ and $|1\rangle|1\rangle|1\rangle$ can be eliminated from \mathcal{B}' . This can be done by performing a two-outcome projective measurement, and corresponding measurement operators are given by

$$\pi_1 = |0\rangle\langle 0| + |1\rangle\langle 1|, \quad \pi_2 = |2\rangle\langle 2|. \quad (6)$$

This measurement can be performed by all three parties. In particular, if the measurement outcome is “2” due to the measurement by any of the parties then the state of the system can be perfectly determined by LOCC. Now, consider a three-qubit subspace \mathcal{V} spanned by the following states:

$$\begin{aligned} &|0\rangle|0\rangle|0\rangle, |0\rangle|0\rangle|1\rangle, |0\rangle|1\rangle|0\rangle, |0\rangle|1\rangle|1\rangle, \\ &|1\rangle|0\rangle|0\rangle, |1\rangle|0\rangle|1\rangle, |1\rangle|1\rangle|0\rangle, |1\rangle|1\rangle|1\rangle. \end{aligned} \quad (7)$$

Clearly, the states of the basis \mathcal{B}' that belong to the subspace \mathcal{V} cannot be perfectly distinguished by LOCC. This is solely because of the twisted states $\{|\psi_k\rangle\}$ of \mathcal{S} with $k = 1, 2, 5, 6, 9, 10$. So, the basis \mathcal{B} that contains \mathcal{S} displays an additional feature besides just showing local indistinguishability of the states.

Consider another small set \mathcal{S}' . This set contains only seven pure orthogonal product states $\{|\psi_k\rangle\} \in \mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$ with $k = 1, \dots, 7$. These states are given by

$$\begin{aligned} &|\psi_1\rangle = |1\rangle|0-1\rangle|0+1\rangle, |\psi_2\rangle = |0+1\rangle|1\rangle|0-1\rangle, \\ &|\psi_3\rangle = |0-1\rangle|0+1\rangle|1\rangle, |\psi_4\rangle = |2\rangle|0-2\rangle|0+2\rangle, \\ &|\psi_5\rangle = |0+2\rangle|2\rangle|0-2\rangle, |\psi_6\rangle = |0-2\rangle|0+2\rangle|2\rangle, \\ &|\psi_7\rangle = |0\rangle|0\rangle|0\rangle. \end{aligned} \quad (8)$$

Notice that the states $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_7\rangle\}$ ensure the fact that there is no other fully separable state which is orthogonal to these states and also belongs to the subspace \mathcal{V} (defined earlier). In this sense, the states $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_7\rangle\}$ together behave like a Shifts UPB [18,19] residing in the three-qubit subspace \mathcal{V} . If the states $\{|\psi_k\rangle\}$ with $k = 1, 2, 3, 7$ span \mathcal{V}_S of \mathcal{V} then \mathcal{V}_S^\perp must be an entangled subspace, where $\mathcal{V} = \mathcal{V}_S \oplus \mathcal{V}_S^\perp$ and the normalized projector ρ onto \mathcal{V}_S^\perp must be a tripartite bound entangled state. Now, consider another three-qubit subspace \mathcal{V}' which is spanned by

$$\begin{aligned} &|0\rangle|0\rangle|0\rangle, |0\rangle|0\rangle|2\rangle, |0\rangle|2\rangle|0\rangle, |0\rangle|2\rangle|2\rangle, \\ &|2\rangle|0\rangle|0\rangle, |2\rangle|0\rangle|2\rangle, |2\rangle|2\rangle|0\rangle, |2\rangle|2\rangle|2\rangle. \end{aligned} \quad (9)$$

In the same way, as described before, the states $\{|\psi_k\rangle\}$ with $k = 4, \dots, 7$ together behave like a Shifts UPB residing in the three-qubit subspace \mathcal{V}' . If the states $\{|\psi_k\rangle\}$ with $k = 4, \dots, 7$ span \mathcal{V}'_S of \mathcal{V}' then \mathcal{V}'_S^\perp must be another entangled subspace, where $\mathcal{V}' = \mathcal{V}'_S \oplus \mathcal{V}'_S^\perp$ and the normalized projector ρ' onto \mathcal{V}'_S^\perp is another tripartite bound entangled state. Note that the subspaces \mathcal{V} and \mathcal{V}' are not completely disjoint. In particular, \mathcal{V}_S and \mathcal{V}'_S have one-dimensional overlap with each other. However, with the help of ρ and ρ' , it is possible to define a new class of tripartite bound entangled states given by

$$\sigma(\lambda) = \lambda\rho + (1-\lambda)\rho', \quad (10)$$

where λ takes values between 0 and 1. The states $\sigma(\lambda)$ are supported in $\mathcal{V}_S^\perp \oplus \mathcal{V}'^\perp$ which is an entangled subspace. Therefore, the states $\sigma(\lambda)$ must be entangled states. Again, a convex combination of two bound entangled states cannot lead to distillable entanglement. Hence, the states $\sigma(\lambda)$ must be bound entangled states. In this context it is important to mention that both ρ and ρ' are separable in the bipartitions [19]. This results that the states $\sigma(\lambda)$ are also separable in the bipartitions.

Clearly, it is not possible to extend the set S' to a COPB in $\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$ as this set S' contains several UPBs residing in the subspaces \mathcal{V} and \mathcal{V}' . Thus, it is obvious that the states of S' cannot be perfectly distinguished by LOCC. This is because of the fact that the states within a UPB cannot be perfectly distinguished by LOCC [18]. Moreover, the set S' constitutes an SUCPB because those subspace UPBs lead to the uncompleteness in any locally extended Hilbert space.

In the next section, both the sets S and S' are generalized for an arbitrary number of parties with high-dimensional subsystems and then different properties of these small sets are explored.

IV. MULTIPARTITE SYSTEM

Consider a multipartite quantum system associated with a Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes m}$ ($d \geq 3, m \geq 3$), where m is the number of parties and each party holds a d -dimensional subsystem. In the following, a set \mathbb{S} of $2m(d - 1)$ pure orthogonal m -partite product states in $\mathcal{H} = (\mathbb{C}^d)^{\otimes m}$ is constructed. To normalize the states consider $|0 \pm i\rangle \equiv (1/\sqrt{2})(|0\rangle + |i\rangle)$ for $i = 1, \dots, (d - 1)$. The states are given by

$$\begin{aligned} |\psi_{1i}\rangle &= |0\rangle|0\rangle \cdots |0\rangle|i\rangle|0+i\rangle, \\ |\psi_{1i}^\perp\rangle &= |0\rangle|0\rangle \cdots |0\rangle|i\rangle|0-i\rangle, \\ |\psi_{2i}\rangle &= |0\rangle \cdots |0\rangle|i\rangle|0+i\rangle|0\rangle, \\ |\psi_{2i}^\perp\rangle &= |0\rangle \cdots |0\rangle|i\rangle|0-i\rangle|0\rangle, \\ &\vdots \\ |\psi_{mi}\rangle &= |0+i\rangle|0\rangle|0\rangle \cdots |0\rangle|i\rangle, \\ |\psi_{mi}^\perp\rangle &= |0-i\rangle|0\rangle|0\rangle \cdots |0\rangle|i\rangle. \end{aligned} \tag{11}$$

It is possible to extend the set \mathbb{S} to a COPB in $\mathcal{H} = (\mathbb{C}^d)^{\otimes m}$. For this purpose, consider a different COPB \mathbf{B} in $\mathcal{H} = (\mathbb{C}^d)^{\otimes m}$. The form of the product states contained in \mathbf{B} is given by $|b_1\rangle|b_2\rangle \cdots |b_m\rangle$, where $b_i = 0, \dots, (d - 1)$. Next, consider a different set of orthogonal product states \mathbf{S} in $\mathcal{H} = (\mathbb{C}^d)^{\otimes m}$, given by

$$\begin{aligned} &|0\rangle|0\rangle \cdots |0\rangle|i\rangle|0\rangle, |0\rangle|0\rangle \cdots |0\rangle|i\rangle|i\rangle, \\ &|0\rangle \cdots |0\rangle|i\rangle|0\rangle|0\rangle, |0\rangle \cdots |0\rangle|i\rangle|i\rangle|0\rangle, \\ &\vdots \\ &|0\rangle|0\rangle|0\rangle \cdots |0\rangle|i\rangle, |i\rangle|0\rangle|0\rangle \cdots |0\rangle|i\rangle, \end{aligned} \tag{12}$$

where $i = 1, \dots, (d - 1)$. Notice that both the sets \mathbb{S} and \mathbf{S} span the same subspace \mathcal{H}' of $\mathcal{H} = (\mathbb{C}^d)^{\otimes m}$. Now, define another set of product states as $\mathbf{S}' = \mathbf{B} - \mathbf{S}$. The states of the set \mathbb{S} and that of \mathbf{S}' together form a COPB in $\mathcal{H} = (\mathbb{C}^d)^{\otimes m}$. Make a note that if one puts $m = d = 3$ then the set \mathbb{S} is exactly

the same as that of \mathcal{S} (defined in the previous section), only the states are labeled differently. However, the set \mathbb{S} leads to an interesting property which is captured by the next theorem. This theorem can be regarded as the generalized version of Theorem 1.

Theorem 2. Let \mathbb{B} be a complete orthogonal product basis in an m -qudit, m -partite Hilbert space. Then no state from \mathbb{B} can be eliminated by performing orthogonality-preserving measurements if \mathbb{B} contains \mathbb{S} .

Proof. To prove the above, it is sufficient to show that the states of the set \mathbb{S} allow each party to perform only trivial measurements on an entire d -dimensional subsystem if they want to preserve the orthogonality of the states. This argument follows from the proof of Theorem 1.

Now, assume that the $(m - 1)$ th party performs a measurement defined by a set of POVM elements $\{\pi_l\}; \pi_l = M_l^\dagger M_l$ and $\sum_l \pi_l = \mathcal{I}_{d \times d}$. Each π_l can be represented by a $d \times d$ matrix written in the $\{|0\rangle, |1\rangle, \dots, |d - 1\rangle\}$ basis as the following:

$$\pi_l = M_l^\dagger M_l = \begin{pmatrix} e_{00} & e_{01} & \cdots & e_{0,d-1} \\ e_{10} & e_{11} & \cdots & e_{1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{d-1,0} & e_{d-1,1} & \cdots & e_{d-1,d-1} \end{pmatrix}. \tag{13}$$

If this measurement preserves the orthogonality of the states then the postmeasurement states must be pairwise orthogonal to each other. Now, setting the inner product $\langle \psi_{1i} | \mathcal{I} \otimes \mathcal{I} \otimes \cdots \otimes M_l^\dagger M_l \otimes \mathcal{I} | \psi_{1i'} \rangle = 0$, it is found that $\langle i | M_l^\dagger M_l | i' \rangle = 0$, or $e_{ii'} = 0$ with $i, i' = 1, \dots, (d - 1)$ and $i \neq i'$. These $e_{ii'}$ are the off-diagonal entries of the above matrix. Next, consider the states $|\psi_{1i}\rangle$ and $|\psi_{mi}\rangle$. The inner product $\langle \psi_{mi} | \mathcal{I} \otimes \mathcal{I} \otimes \cdots \otimes M_l^\dagger M_l \otimes \mathcal{I} | \psi_{1i} \rangle$ must be zero and it turns out that $\langle 0 | M_l^\dagger M_l | i \rangle = e_{0i} = e_{i0} = 0$ for $i = 1, \dots, (d - 1)$. Thus, all the off-diagonal entries of the above matrix are zero.

Next, it is shown that the diagonal entries of the above matrix are all the same. For this purpose, consider the states $|\psi_{2i}\rangle$ and $|\psi_{2i}^\perp\rangle$. Again, the inner product $\langle \psi_{2i} | \mathcal{I} \otimes \mathcal{I} \otimes \cdots \otimes M_l^\dagger M_l \otimes \mathcal{I} | \psi_{2i}^\perp \rangle = 0$ and it is found that $\langle 0+i | M_l^\dagger M_l | 0-i \rangle = 0$, or $e_{00} = e_{ii}$. Hence, the diagonal entries of the above matrix are all the same.

In this way, it is proved that the POVM elements that define the measurement for the $(m - 1)$ th party are proportional to a $d \times d$ identity matrix. So, the $(m - 1)$ th party performs only trivial measurements.

Similarly, it can be shown that all other parties can perform only trivial measurements if they want to preserve the orthogonality of the states. This is because of the symmetry present in the states of the set \mathbb{S} . Here the proof completes. ■

For any COPB $\mathbb{B} \in \mathcal{H} = (\mathbb{C}^d)^{\otimes m}$ that contains all the states of the set \mathbb{S} , it is straightforward from Theorem 2 that the states of \mathbb{B} cannot be perfectly distinguished by LOCC. If the construction is restricted up to a bipartite system, then there exists a set of $4(d - 1)$ product states that cannot be distinguished by LOCC. Precise construction of such a set is given in Ref. [32]. Again, Walgate *et al.* showed that there does not exist a product basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$ which is a nonlocal basis [22]. When the number of parties $m \geq 3$, and every party holds a qubit, then there are $2m$ product states in $(\mathbb{C}^2)^{\otimes m}$ that cannot

After the first party, these measurements are also performed by next $(m - 3)$ parties, that is, excluding the $(m - 1)$ th and m th parties. This completes the distinguishability of all the states except the states of the first two rows.

(iv) So, for the states of the first two rows, the $(m - 1)$ th party does a $(d - 1)$ -outcome measurement where the corresponding measurement operators are $M_i^{(m-1)} = |i0\rangle\langle i0| + |i1\rangle\langle i1|$. For each i , there are two orthogonal states remaining which can be distinguished further.

(v) Next, go back to step (i) again. If the measurement outcome is “2” due to the measurement by the m th party then there is another set of states like the set given in Eq. (17). This set can be distinguished in the same way as described above. In this way, the protocol completes. ■

Recall that the resource state in the above protocol is shared between the $(m - 1)$ th and the m th party. But because of the symmetry present within the states of the set \mathbb{S} , it is really not important which pair of parties holds the resource state. From the above protocol it is also proved that the amount of entanglement required to accomplish the task of distinguishing the OPSs of \mathbb{S} by LOCC does not depend on the dimension of the subsystems. Again, it does not depend on the number of parties either. So, in the above protocol, entanglement is employed more efficiently than a teleportation-based protocol. In this context, it is important to mention that a teleportation-based protocol to distinguish the OPSs of \mathbb{S} requires $(m - 1)\log_2 d$ ebits. However, Theorem 3 exhibits the first ever example where the OPSs of a completable set is distinguished so efficiently via an entanglement-assisted local protocol. From the dimensional point of view, the present resource is an optimal resource. But it is not known whether a two-qubit nonmaximally entangled state can be employed for the perfect discrimination of the states of \mathbb{S} by LOCC or not. This particular fact has also been pointed out before in the context of distinguishing the states that belong to a UPB [50]. Theorem 3 also depicts that, as long as the task of distinguishing the states of the set \mathbb{S} is concerned, there exists at least one separable operation which can accomplish this task perfectly and can be implemented by LOCC with the help of a two-qubit maximally entangled Bell state as resource.

From the above it is clear that for a given nonlocal COPB if the nonlocality of such a COPB is solely because of the states of the set \mathbb{S} then that COPB can be perfectly distinguished by LOCC with the help of a two-qubit maximally entangled Bell state shared between any two spatially separated parties. Note that the present discrimination protocol does not work for all uncompletable sets of this paper. Thus, it is yet to be known

which kind of entangled states are sufficient to distinguish the states of the uncompletable sets.

VI. CONCLUSION AND OPEN PROBLEMS

In this paper, different classes of nonlocal sets of pure orthogonal product states have been constructed for arbitrarily high-dimensional multipartite quantum systems. These constructions are important for a better understanding about the phenomenon *quantum nonlocality without entanglement*. In particular, the completable nonlocal sets give insight regarding the separable operations that are not implementable by LOCC. As useful by-products of present nonlocal sets, a class of COPBs has been introduced from which no state can be eliminated by performing orthogonality-preserving measurements and also a class of multipartite bound entangled states has been introduced. After the present constructions, one immediate open problem is to generalize these sets for any high-dimensional multipartite quantum systems where the parties do not hold the same dimensional subsystems. A local protocol has also been constructed to distinguish the product states of the completable sets using a two-qubit maximally entangled Bell state as a resource. Nevertheless, it is quite difficult to find out an optimal resource to distinguish the states of a given set of product states by LOCC. Here are two interesting open problems: The first is to find (if it is possible to construct) a nonlocal orthogonal product basis for which a teleportation-based protocol is an optimal protocol. Second, the amount of entanglement required to realize a nonlocal separable operation by LOCC is considered here while the task is fixed; that is, the state discrimination task. So, if it is possible to define a universal entanglement cost (task independent) of separable operation then it will be interesting. In a given Hilbert space, the sets constructed here are small sets; that is, the number of states contained in a set is much less than the net dimension of the Hilbert space. Hence, an essential search in this direction is to find out the number of orthogonal product states that is necessary to certify the fact that no state can be eliminated from a set in a given Hilbert space by performing orthogonality-preserving measurements. Explicit constructions of such sets are also desired for any multipartite quantum system.

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