Volume of violation of Bell-type inequalities as a measure of nonlocality

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Motivated by the recent proposal for the quantification of the Bell nonlocality [Phys. Rev. A **92**, 030101(R) (2015)], we provide an extensive analysis of this concept, namely, the volume of violation. This new measure of nonlocality has been proposed to deliver an evidence that the anomaly between maximally entangled states and states that maximally violate a Bell inequality is caused by the method, which has been applied to quantify nonlocality and the anomaly disappears when the volume of violation in used. We prove that such conclusion is not true for all bipartite quantum systems with dimension $d \otimes d$. In fact, if one assumes the same condition as in Phys. Rev. A **92**, 030101(R) (2015), it is limited to $d \leqslant 7$. Furthermore, we discuss several type of local measurements and their influence on the quantification of nonlocality by mean of volume of violation. In particular, we propose a set of local observers that significantly enhance the volume of violation, which may have important meaning in a real-world application. Finally, we present a comparison between the volume of violation and the maximal violation of a Bell inequality.

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I. INTRODUCTION

In the pioneering paper of Bell [1] it was proven that the predictions of quantum theory are incompatible with any realistic interpretation of deterministic world formalized in terms of the local hidden variable (LHV) theory. Bell considered a situation of two spin-$\frac{1}{2}$ particles in a singlet state and showed that the results of measurements performed independently on each of the particles are correlated in a way that cannot be explained by any local model. This constraint, expressed as so-called Bell inequality, represents one of the most profound developments in the foundations of physics. The experiments testing Bell inequalities are able to answer the question whether the quantum mechanics is needed to describe the world or whether the LHV theory is sufficient.

Over time, Bell inequalities have been introduced in many varieties [2]. In general, they can be characterized by the number of parties making measurements, n, the number of measurement settings, p, and the number of possible outcomes for each measurement, d. In particular, the most famous Bell-type inequality, the Clauser-Horne-Shimony-Holt (CHSH) inequality [3], has been proposed for $n = p = d = 2$. Later, more general scenarios have been offered. For instance, the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality developed for $n = p = 2$ and general d [4] or its alternative version defined in Ref. [5], a set of Bell-type inequalities for $p = 2$ and an arbitrary n and d proposed by Son *et al.* [6] (hereafter referred to as the SLK inequality), to name a few.

Based on all these results several important properties of nonlocality have been revealed. First of all, the early research showed that the CHSH-type inequalities are violated even for $d \to \infty$, but never exceeds the violation by two qubits, which is in agreement with the Cirel'son limit [7]. Some authors speculated that the correspondence principle suggests that nonlocality should diminish with growing d [8,9]. However, these studies of nonlocality were confined to some certain measurements performed on maximally entangled states (MESs) of two qudits [10,11]. Kaszlikowski *et al.* [12] proved that the increase of the outcomes number d may result in even stronger violation of local realism than it is for two qubits if one considers general observables. Although the measure of nonlocality used by Kaszlikowski *et al.* was based on the resistance against noise, the same findings are reproduced by the CGLMP inequality [13] given as

$$I_d(\rho) = \sum_{k=0}^{\lfloor d/2 \rfloor -1} \left(1 - \frac{2k}{d-1}\right)[\mathcal{P}(k) - \mathcal{P}(-k-1)] \leqslant 2, \tag{1}$$

where

$$\mathcal{P}(k) = P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k) \tag{2}$$

and

$$P(A_a = B_b + k) = \sum_{j=0}^{d-1} P(A_a = j, B_b = j + k \bmod d), \tag{3}$$

with $P(A_a, B_b) = \mathrm{Tr}(A_a \otimes B_b \rho)$ and A_a, B_b denoting two different d-outcome positive measurements for spatially separated observers A and B, respectively.

Further studies reveal that any bipartite pure entangled state of d-dimensional subsystems violates the Bell-type inequality [11,14,15], which is known as the Gisin's theorem. Moreover, the detailed investigations of Acín *et al.* [13] (and

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later continued in Refs. [5,16]) performed on such system disclosed another intriguing result, namely, the existence of nonmaximally entangled states that lead to greater violation of the CGLMP inequalities compared with MESs. The existence of such anomaly was a rather unexpected result since it was long believed that the MESs must also be the states of maximal violation of Bell inequalities, in the same way as the CHSH inequality is violated by two qubit states [14,17]. Nowadays it is known that "for all measures of nonlocality invented to date, it happens almost always that the most nonlocal state is not the maximally entangled one" [18,19], which is caused by the fact that entanglement and nonlocality are different resources [20]. We note that the anomaly has been also reported for other measures of nonlocality [21–23]. The above description provides a potential definition of the maximally nonlocal state (MNS) as a state that provides the maximal value of the Bell expressions I that can be attained for quantum measurements on an entangled state. However, in order to avoid further confusions we will call such states as the optimal states [13,22] (or asymmetric states [24,25]).

Interestingly, the anomaly cannot be confirmed by another set of Bell-type inequalities, namely the SLK inequalities, which are maximally violated by the MESs of two qudits [6,26,27]. Another two examples of Bell-type inequalities that are maximally violated by the MESs are given in Refs. [28,29]. The SLK inequalities are defined as

$$I_d^{\mathrm{SLK}} = \sum_{n=0}^{d-1} f(n)\mathcal{P}(n) \leqslant I_d^{\mathrm{SLK}}(LR), \tag{4}$$

where $f(n) = \frac{1}{\sqrt{2}}[\cot(\frac{\pi}{d}[n+\frac{1}{4}]) - 1]$, $I_d^{\mathrm{SLK}}(LR) = \frac{1}{\sqrt{2}}(3\cot\frac{\pi}{4d} - \cot\frac{3\pi}{4d}) - 2\sqrt{2}$, and $\mathcal{P}(n)$ is given by Eq. (2). In particular, if one uses a specific set of local observables proposed in Refs. [12] the maximal violation of SLK inequality is directly proportional to the amount of entanglement, $I_d^{\mathrm{SLK}} = 2\sqrt{2}(d-1)\mathcal{C}$, where $\mathcal{C}$ stands for two-qudit pure states concurrence [27]. Such set of local observables (hereafter called M_1) is defined by a group of local phase shifts, which are represented in the Schmidt decomposition basis as $|j\rangle \to e^{i\phi_j}|j\rangle$, followed by unbiased d-port beam splitters performing a unitary transformation described by the Fourier matrix $[U_A^{M_1}]_{kl} = d^{-1/2}e^{i(2\pi kl/d)}$ for observer A and the inverse Fourier matrix $[U_B^{M_1}]_{kl} = d^{-1/2}e^{-i(2\pi kl/d)}$ for observer B. Note that M_1 is also sufficient to confirm the existence of the anomaly reported for the CGLMP inequality [13].

Recently, an alternative measure of nonlocality and hence, an alternative definition of MNS, has been proposed by Fonseca and Parisio [30]. The concept, called the volume of violation, is given as follows: Suppose one has (preferably tight) Bell-type inequality I. Then, the state ρ_1 is more nonlocal than ρ_2 by means of a given Bell-type inequality I if ρ_1 violates I, no matter by what extent, for a larger number of experimental configurations than ρ_2. To formalize this idea, let X be a space of all possible configurations of parameters x_i which determine the local observables. If for a particular state ρ one distinguishes the subspace $\Gamma_{\rho,I} \subset X$ that contains all configurations leading to violation of inequality I then, the volume of violation is given by

$$V(\rho, I) = \int_{\Gamma_{\rho,I}} d^n x, \tag{5}$$

where $d^n x = dx_1 \dots dx_n$. According to the above definition, all states with $\Gamma_\rho = \emptyset$ have volume $V = 0$. Consequently, all nonviolating states are equally local [31]. The volume of violation can also be expressed as $V(\rho, I) = p_V V(X)$, where $p_V = p(\rho, I)$ denotes the probability of violation of inequality I [32,33] and $V(X)$ stands for a total volume of space X [31]. In this context, the state ρ_1 is more nonlocal than ρ_2 by mean of volume of violation if the probability to obtain a violation of inequality I is larger for ρ_1. Here, we assume that both states ρ_1 and ρ_2 have the same total volume $V(X)$. Furthermore, since for a given Bell scenario $V(X)$ is just a constant value without playing any significant role, we put $V(X) = 1$. At this point we note that some properties of the volume of violation of qubit states have been recently studied in Ref. [34]. In this paper, V is called as a nonlocality fraction.

When the concept of the volume of violation has been applied to the CGLMP inequality, it turns out that the supposed anomaly disappears when the M_1 transformation is used [31] and there is no discrepancy between MES and MNS. Therefore, the hypothesis that the anomaly may be an artifact of the applied measure has been putted [30]. However, the calculations presented in Ref. [30] are limited to some special states for $d = \{3, 4\}$ under a certain kind of measurement namely, M_1. For that reason, it is important to verify whether the remarks described in Ref. [30] are valid for $d > 4$. In particular, whether V attains its maximum for MES and thus, one observes the disappearance of the anomaly for general qudit case if the same circumstances as in Refs. [30,31] are assumed.

Furthermore, the analysis performed in Ref. [30] can be extended to other kinds of local observers. Specifically, if $V(\rho, I)$ is considered to be a nonlocality measure for an arbitrary state ρ, one can ask whether the unitary transformation M_1 is adequate for proper detection and/or quantification of nonlocality. In other words, for the considerations of the maximal violation of a given Bell inequality I and a given state ρ, it is sufficient to indicate even one set of optimal measurements, regardless if other optimal measurements exist [4]. However, if one is interested in the number of experimental configurations providing a violation of inequality I then possible the most general unitary transformation should rather be applied. Then, it is an open question what kind of local observers (as simple as possible) is sufficient in order to fully map the nonlocality [24] and, hence, to provide an universal frame of quantification of V for all states. On the other hand, motivated by Refs. [32,33], one can reverse the above question. It is known that all nonlocal states are also entangled. However, the detection of nonlocal correlation and hence entanglement based on the standard violation of Bell inequality require an initial information about analyzed state. For that reason, one can consider $V(\rho, I)$, which is based on the random measurements, as an experimentally friendly entanglement witness [32]. In this context, a carefully chosen local measurement should be employed in order to obtain possible the greatest $V(\rho, I)$ and hence to detect nonlocal

TABLE I. The Schmidt coefficients λ_i for the optimal state $|\psi_O\rangle = \sum_{i=0}^{d-1} \lambda_i |ii\rangle$. Due to the relation $\lambda_{d-1-i} = \lambda_i$ it is sufficient to write only the first $\lceil \frac{d}{2} \rceil$ values of λ_i.

d	λ_1	λ_2	λ_3	λ_4	λ_5
2	0.70711	...	...	...	...
3	0.61689	0.48876	...	...	...
4	0.56857	0.42039	...	...	...
5	0.53684	0.38592	0.35459	...	...
6	0.51370	0.36443	0.32141	...	...
7	0.49574	0.34932	0.30113	0.28824	...
8	0.48118	0.33788	0.28725	0.26793	...
9	0.46904	0.32877	0.27701	0.25412	0.24742
10	0.45866	0.32125	0.26904	0.24404	0.23344

correlation with a sufficient probability [33]. Both approaches entail the need to investigate the influence of the adopted unitary transformation on the final results.

In this paper we present an extensive investigations of volume of violation. In particular, we show that for higher d the anomaly between MES and MNS does not disappear when the volume of violation in used. Furthermore, we discuss several sets of local measurements (well known in the literature) and show how strongly they affect the result. Finally, in order to provide qualitative meaning of $V(\rho, I)$ as a measure of nonlocality, we compare it with the violation of I_d to answer the question about the relation between these two quantities. At this point we note that there are also other measures of nonlocality such as Kullback-Leibler distance [22], a measure of so-called useful nonlocality [35,36], covariance Bell inequalities [37], and measures based on the trace distance [38].

We also wish to emphasize that in our calculations none of the possible relabeling of measurements setting and/or outcomes is taken into account. In other words, we consider only one Bell inequality, not the whole equivalent class obtained by some permutation of the labels, settings and outcomes [32,33,39].

II. COMPARISON OF MES AND OPTIMAL STATES FOR M_1 MEASUREMENTS

Let us first analyze whether the volume of violation for MES is truly greater than the volume for the optimal states as it was suggested in Refs. [30,31]. For this purpose we consider bipartite quantum system composed of d-dimensional local parties. It is known that for such system any pure state $|\psi\rangle$ can be expressed in so-called Schmidt decomposition [40,41], $|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |jj\rangle$, where the Schmidt coefficients α_j satisfy $\alpha_j > 0$ and $\sum \alpha_j^2 = 1$. Here, we assume that the Schmidt rank, which determines the number of nonvanishing Schmidt coefficients, is equal to d. Based on this decomposition one can easily define MES of a bipartite qudit system as $|\psi_M\rangle = d^{-1/2} \sum_{j=0}^{d-1} |jj\rangle$. Similarly, the optimal states are given by $|\psi_O\rangle = \sum_{j=0}^{d-1} \lambda_j |jj\rangle$, where the corresponding Schmidt coefficients λ_j are given in Table I (cf. Ref. [5]).

For each of the analyzed states we have performed numerical Monte Carlo integration in a $4d$-dimensional space of angles ϕ_j as it is required by the M_1 transformation for $n = 10^{11}$ randomly chosen settings. Due to the computational demands related with the number of free parameters, our calculations are limited to $d = 10$. The numerical results for $V_M \equiv V(|\psi_M\rangle, I_d)$ and $V_O \equiv V(|\psi_O\rangle, I_d)$ are presented in Fig. 1 and Table II.

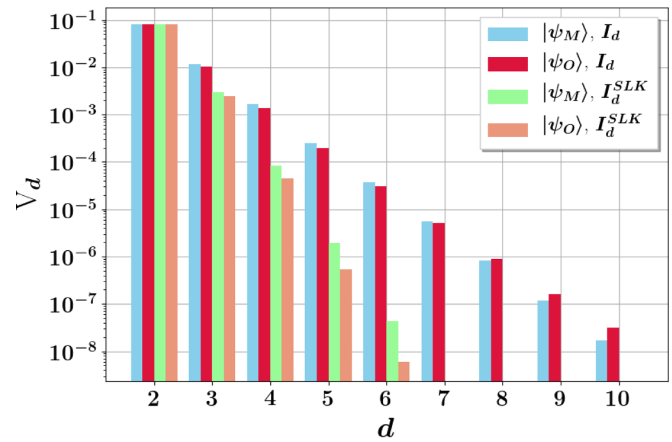

FIG. 1. The bars represent the volume of violation of the CGLMP inequality under the M_1 transformation for various number of outcomes d. The calculations are performed for the maximally entangled states $|\psi_M\rangle$ and the optimal states $|\psi_M\rangle$. Additionally, we present the results calculated for the SLK inequality and the same two groups of states.

As we see, for $d = 2$ the equivalence of V_M and V_O is observed [30,42,43]. The probability of violating I_d for randomly chosen angles ϕ_j is approximately equal to 8.01% and it is located between the probability of violating the Bell-CHSH inequality via random isotropic measurements and the probability of violating the Bell-CHSH inequality via random orthogonal measurements, $\frac{\pi-3}{2} \approx 7.08\%$ and 10.326% [32,33], respectively. A possible explanation of this feature could be the fact that random isotropic measurements represent a wider class than M_1. When $d > 2$ the exponential decrease of the volume of violations is observed for both V_M and V_O.

TABLE II. The estimation of volume of violation V for the maximally entangled $|\psi_M\rangle$ and the optimal $|\psi_O\rangle$ states. The results presented in the first two columns have been calculated for the CGLMP inequality I_d while the third column denotes the difference between $V(|\psi_M\rangle, I_d)$ and $V(|\psi_O\rangle, I_d)$. Similarly, the results for $I_d^{\rm SLK}$ are written in the last three columns. For each case, the number of random configurations used in calculations is equal to $n = 10^{11}$.

	$10^{-5}V$ for I_d		Difference	$10^{-5}V$ for $I_d^{\rm SLK}$		Difference
	$\|\psi_M\rangle$	$\|\psi_O\rangle$	%	$\|\psi_M\rangle$	$\|\psi_O\rangle$	%
2	8011.1	8011.1	0	8011.1	8011.1	0
3	1142.2	1015.9	12.431	301.16	245.60	22.622
4	168.98	136.61	23.696	8.2256	4.4130	86.395
5	24.992	19.602	27.500	0.1917	0.0542	253.69
6	3.7061	3.0237	22.571	0.0043	0.0006	616,66
7	0.5485	0.5011	9.447	–	–	–
8	0.0816	0.0875	−6.726	–	–	–
9	0.0119	0.0162	−26.634	–	–	–
10	0.0018	0.0030	−40.043	–	–	–

This means that despite the stronger violation of local realism for higher numbers of outcomes d proven in Refs. [12,13] the statistical probability for reaching such results for CGLMP inequality I_d drops rapidly with d. Of course, it is restricted to only one CGLMP inequality and we do not take into consideration the number of equivalent Bell expressions. The decrease of the volume of violations for both V_M and V_O is not identical for increasing d. In particular, when $3 \leqslant d \leqslant 7$ the volume for MES is greater than the volume for optimal states and our results for $d = \{3, 4\}$ are in good agreement with Ref. [30]. The highest deference between V_M and V_O occurs for $d = 5$ and it decreases for further growth of d (see Table II). In consequence, for $d = \{8, 9, 10\}$ one observes the opposite scenario and the volume V_O exceeds V_M. This means that for $d \geqslant 8$ the MES cannot be considered as the MNS either with respect to the maximal violation of CGLMP inequality or with respect to the volume of violation. These results are in contradiction with the hypothesis proposed in Ref. [30] and the anomaly does not disappear in general case. In order to verify our observation the numerical calculations for $d = 8$ have been repeated ten times. Based on this results we have found that the mean value $\overline{V}_M = 0.08140 \times 10^{-5}$ and $\overline{V}_O = 0.08738 \times 10^{-5}$ whereas the standard deviations are equal to $\Delta_M = 0.00042 \times 10^{-5}$ and $\Delta_O = 0.00092 \times 10^{-5}$, which confirms the quality of our observation.

In order to explain why the anomaly appears for higher d, let us analyze the distribution of the volume of violation with respect to the degree of violation of CGLMP inequality. In other words, we study the modified volume V' that is limited to all random settings, which provide the violation of CGLPM inequality in the interval $\{I_d, I_d + \sigma\}$ with a given value $I_d \geqslant 2$. Here, we assume that all intervals are disjoint and fully cover the entire range of attainable I_d. We also note that the sum of V' after all such intervals gives the total volume of violation V. As we see in Fig. 2, for a particular case of $d = 4$ the modified volume $V'(|\psi_M\rangle) > V'(|\psi_O\rangle)$ for almost all I_d. Naturally, since the CGLPM inequality is more strongly violated by the optimal states than the MES, there exist such critical value I'_d (hereafter the crossing point) above, which $V'(|\psi_O\rangle)$ exceeds $V'(|\psi_M\rangle)$. In the discussed case such crossing point is located around $I'_d = 2.67$. Despite of that the enhance of $V'(|\psi_O\rangle)$ above the I'_d seems to be not sufficient to change the relation between total V_M and V_O when $d = 4$. However, for higher d the crossing point tends to the smallest values I_d. In the same time, the difference between $V'(|\psi_M\rangle)$ and $V'(|\psi_O\rangle)$ below the crossing point decreases and hence, V' corresponding to stronger violations I_d becomes more essential. Finally, when $d = 8$ the crossing point is located below $I'_d = 2$ and $V'(|\psi_O\rangle) > V'(|\psi_M\rangle)$ in the entire range of attainable I_d, which causes the appearance of the anomaly, i.e., $V_O > V_M$. This analysis provides also a strong argument that for $d > 8$ the anomaly should be even more visible, which is in line with Table II.

Finally, let us take the SLK inequality into consideration. Our calculations are limited to $d \leqslant 6$. It is caused by the fast decay of V (see Table II), which implies that we are not able to accumulate enough statistical data for higher d in a reasonable time. In the case of SLK inequality, the volume of violation for MES $V(|\psi_M\rangle, I_d^{\mathrm{SLK}})$ is greater than the volume for optimal states $V(|\psi_O\rangle, I_d^{\mathrm{SLK}})$, at least for the analyzed regime of d.

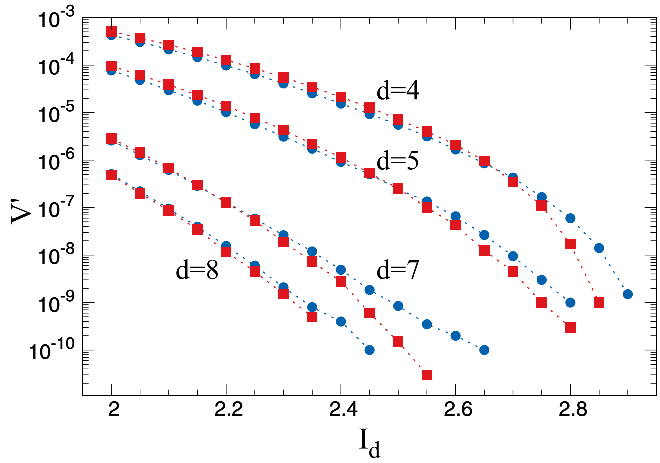

FIG. 2. The distribution of the modified volume of violation V' as a function of the violation of CGLPM inequality. The symbols represent the volume V' limited to all random settings providing the violation of CGLPM inequality in the interval $\{I_d, I_d + 0.1\}$. The square symbols correspond to the MES while circle symbols denote the optimal state. We present four pairs of $\{V'(|\psi_M\rangle), V'(|\psi_O\rangle)\}$ for $d = 4, 5, 7, 8$. For each case, the number of random configurations used in calculations is equal to $n = 10^{11}$.

However, it should be emphasized that the difference between V_M and V_O increases significantly faster with growing d than it is observed for the CGLMP inequality. Since the SLK inequality is maximally violated by MES, one can interpret this behavior as an indicator that MES is also the MNS also by mean of maximal volume of violation, even for $d > 6$. We note that further calculations are still needed to confirm this assumption. Nonetheless, different behavior of these two inequalities suggest that the disconnection between MES and MNS is rather hidden in the Bell-type inequality, not in the way of estimating the degree of nonlocality.

We note that the above results, although provide important counterexamples to the previous analysis [30], do not settle the question of the form of MNS estimated by means of V for increasing d. Despite for $d \geqslant 8$ we have not found a state with $V > V_O$, the large number of Schmidt coefficients and statistical fluctuation of numerical results do not allow us to answer this question conclusively.

III. INFLUENCE OF THE CHOSEN MEASUREMENTS

Now, let us investigate the influence of the chosen local observers on the volume of violation. In particular, we verify the usefulness of various kind of unitary transformations on the detection of V for other two-qudit states, not only MES and optimal state. Moreover, we examine how different choices of local observers can affect the mutual relationship between V_M and V_O discussed in the previous section. In order to address the first issues we analyze the volume of violation under several unitary transformations, which can be found in the literature, for the states given as

$$|\psi_\alpha\rangle = \alpha \sum_{j=0}^{d-2} |jj\rangle + \alpha_d |(d-1)(d-1)\rangle, \qquad (6)$$

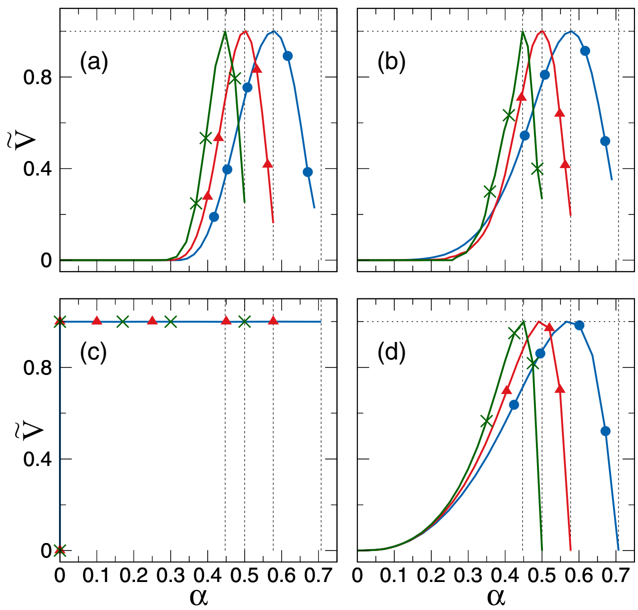

FIG. 3. The normalized volume of violation $\tilde{V}$ for various local observers: (a) M_1 transformation, (b) M_2 transformation, (c) $\mathcal{U}$ transformation, and (d) reshuffled $\mathcal{U}$ transformation. In order to give some meaning to the size of the volume, we have rescaled the volume of violation so that $V(|\psi_M\rangle, I_d) = 1$ for all transformation. In all panels the blue line with circuit symbols denotes the results for $d = 3$; red line with triangle symbols corresponds to $d = 4$; and green line with X symbols is related with $d = 5$.

where $0 \leqslant \alpha \leqslant 1/\sqrt{d-1}$ and $\alpha_d = \sqrt{1-(d-1)\,\alpha^2}$. These states belong to the nontrivial family of highly symmetric states, namely the incomplete-permutation symmetry states, whose entanglement properties have been recently analyzed in Refs. [44,45]. The particular advantage of using such states is the fact that one can cross from the product states, $\alpha = 0$, to the d- and $(d-1)$-dimensional MES (for $\alpha = \frac{1}{\sqrt{d}}$ and $\alpha = \frac{1}{\sqrt{d-1}}$, respectively) by varying only one parameter. We note that in this section our calculations are restricted to $2 \leqslant d \leqslant 5$ and the number of random configurations $n = 10^{10}$.

In Fig. 3(a) we present the volume of violation $V_\alpha^{M_1} \equiv V(|\psi_\alpha\rangle, I_d)$ when the local measurements are defined by M_1 transformation. As we see, the volume $V_\alpha^{M_1}$ is equal to zero for $\alpha \leqslant \alpha_0$, where $\alpha_0 \approx \{0.38, 0.31, 0.27, 0.25\}$ for subsequent d, respectively. This means that pure entangled states with $\alpha \leqslant \alpha_0$ are local with respect to $V_\alpha^{M_1}$, which is in contrast with the Gisin's theorem. This outcome is caused by the fact that the transformation M_1 is not sufficient to reveal the nonlocality of all pure states [15,24]. In particular, the exact critical values α_0 below which the CGLMP inequality cannot be violated are equal to $\{0.3827, 0.3136, 0.2738, 0.2472\}$ for subsequent d, which naturally is in line with our results. As a consequence of existence of $\alpha_0 \neq 0$, the M_1 transformation in not sufficient to detect nonlocality for general qudit states.

In order to overcome this problem, one can use the unitary transformation, M_2, composed of two sets of phase shifters and unbiased multiport beam splitters [24]. In this case, $I_d > 2$ for all $\alpha > 0$ and, hence, one can expect $V_\alpha^{M_2} > 0$ for all $\alpha > 0$. However, our calculations reveal that for $d > 2$ the

TABLE III. The estimation of volume of violation V for I_d and two transformation: M_2 and unitary U(d) transformation. The maximally entangled and the optimal states are denoted as $|\psi_M\rangle$ and $|\psi_O\rangle$, respectively. The number of random configurations used in calculations is equal to $n = 10^{10}$.

	$10^{-5}V_d$ for M_2		Difference	$10^{-5}V_d$ for U(d)		Difference
d	$\lvert\psi_M\rangle$	$\lvert\psi_O\rangle$	%	$\lvert\psi_M\rangle$	$\lvert\psi_O\rangle$	%
3	7.2763	6.6522	9.382	4.7572	4.3376	9.621
4	0.6756	0.5146	31.290	0.8819	0.6743	30.79
5	0.0027	0.0019	42.105	–	–	–

volume $V_\alpha^{M_2}$ is qualitatively equal to $V_\alpha^{M_1}$ [Fig. 3(b)]. Specifically, when $d = 3$ and $\alpha \leqslant 0.2$ one has $V_\alpha^{M_2} < 7.8 \times 10^{-7}$ [i.e., around 1% of $V^{M_2}(|\psi_M\rangle, I_d)$ given in Table III] and it decreases rapidly to zero for decreasing α. For $d = 4$ ($d = 5$) such behavior is even more pronounced and the volume $V_\alpha^{M_2}$ for $\alpha \leqslant 0.18$ ($\alpha \leqslant 0.26$) is smaller than 10^{-9} [i.e., less than 0.1% of $V^{M_2}(|\psi_M\rangle, I_d)$]. For that reason, $V_\alpha^{M_2}$ of such states can be easily disturbed by statistical fluctuation of random configurations and hence one can assume $V_\alpha^{M_2} \approx 0$. On the other hand, a fast increase of the volume of violation is observed in the neighborhood of MES [see Fig. 3(b)]. In Table III detailed values of $V_M^{M_2}$ and $V_O^{M_2}$ are presented. As we see, although the differences between this two quantities for $d = \{4, 5\}$ are greater then the differences achieved for M_1 (Table II), we still have similar qualitative behavior as for M_1. In particular, we do not observe an exponential (or even linear) growth of such differences for successive d as it is presented for the SLK inequality. Therefore, one can expect that for greater d the volume $V_O^{M_2}$ may exceed $V_M^{M_2}$ as it is for M_1. However, due to the fast decay of V^{M_2} we cannot settle whether such inflection point for M_2 truly exists as it is for M_1.

Note that similar calculations can be done for other transformations of this kind [12,24], which iteratively approximate the outcomes for general unitary transformation presented in the next section. For that reason we do not report these results in this paper.

Another example of unitary transformation that ensure the fulfillment of Gisin's theorem has been proposed by Chen *et al.* [15]. In this approach the unitary matrix of particle A is given by $\mathcal{U}(A) = \cos\zeta_a|0\rangle\langle 0| + \sin\zeta_a e^{-i\phi_a}|0\rangle\langle 1| + \sin\zeta_a e^{i\phi_a}|1\rangle\langle 0| - \cos\zeta_a|1\rangle\langle 1| + \sum_{n=2}^{d-1}|n\rangle\langle n|$ and for particle B the unitary transformation, $\mathcal{U}(B)$, has the same form as $\mathcal{U}(A)$. As it is presented in Fig. 3(c) the volume $V_\alpha^{\mathcal{U}} \equiv V(|\psi_\alpha\rangle, I_d)$ for the transformation $\mathcal{U}$ takes 0 for $\alpha = 0$ and constant value otherwise

$$V_\alpha^{\mathcal{U}} = \begin{cases} 0 & \text{for} \quad \alpha = 0 \\ 0.0353 & \text{for} \quad 0 < \alpha \leqslant 1/\sqrt{d-1} \end{cases}. \tag{7}$$

This outcome can be easily explained if one writes the corresponding I_d inequality explicitly: $I_d^{\mathcal{U}} = 2 + \frac{\alpha^2 d}{d-1}[2 - f(\zeta_a, \phi_a, \zeta_b, \phi_b)]$, where $f(\dots)$ is an universal (for all α and d) function of angles $\zeta_a, \dots, \phi_b$. As we see, $I_d > 2$ if and only if $2 - f(\zeta_a, \phi_a, \zeta_b, \phi_b) > 0$, which is independent of α and d. Therefore, there is one common subspace of configurations Γ_{α, I_d} yielding the violation of CGLMP inequality $I_d^{\mathcal{U}}$ for all

α and hence, constant value of $V_\alpha^{\mathcal{U}}$. In the same time, the violation of CGLMP inequality writes $\max_{\zeta_a,\ldots,\phi_b}(I_d) = 2 + 2.828\frac{\alpha^2 d}{d-1}$. In summary, all pure entangled states $|\psi_\alpha\rangle$ defined in an arbitrary d are equivalently nonlocal with respect to $V_\alpha^{\mathcal{U}}$, in spite of the fact that the violation of I_d is α and d dependent [15].

This observation has an important meaning if one is interested in detection of entanglement. As we see, the probability $p_V^{\mathcal{U}}$ of findings nonlocal correlations for the states $|\psi_\alpha\rangle$ (with including the MES) is around 3.53%, which is a significantly greater value than for other kinds of transformations where the probability drops exponentially with d (c.f. Tables II and III). For instance, when $d = 5$ the nonlocal correlations of MES are observed with probability around 2×10^6 greater than for the M_2 transformation, $p_V^{M_2} = 0.18 \times 10^{-5}$.

On the other hand, if one reshuffles the unitary matrices $\mathcal{U}(A)$ and $\mathcal{U}(B)$ in such a way that $|0\rangle \leftrightarrow |d-1\rangle$ and $|1\rangle \leftrightarrow |d-2\rangle$ then Eq. (7) is no longer valid. As it is presented in Fig. 3(d), the volume increases with α, reaching the maximum $V_\alpha^{\mathcal{U}} = 0.0353$ for MES and drops down to 0 for $\alpha = 1/\sqrt{d-1}$. This means that the increase of the probability of detecting nonlocal correlations and hence, entanglement, is truly achievable if MESs are considered. For other cases of $|\psi_\alpha\rangle$ states some initial reference frames are needed in order to properly choose the transformation $\mathcal{U}(A)$ and $\mathcal{U}(B)$. We note that there is no evidence that the $\mathcal{U}$ transformation is the most suitable for increasing the probability of detecting nonlocal correlations and it supplies interesting motivation for further researches.

All these examples clearly show a strong dependence of V on the adopted unitary transformation, not only quantitatively but also qualitatively. Therefore, the nonlocal properties achieved for different local observers cannot be mutually compared. In order to quantify the volume universally it is necessary to consider the same local observers, in particular, the general d-dimensional unitary transformation U, which belongs to the U(d) group.

IV. COMPARISON OF VOLUME OF VIOLATION AND CGLMP INEQUALITY FOR TWO-QUTRIT STATES

In this section we analyze the volume of violation, $V^{U(3)}$, for two-qutrit states given by

$$|\psi_3\rangle = \sin\beta(\cos\gamma|00\rangle + \sin\gamma|11\rangle) + \cos\beta|22\rangle, \quad (8)$$

when the general unitary transformation U(3) is applied (for parametrization, see, e.g., Refs. [46,47]). The construction of random U(3) matrices was performed by mean of the method proposed in Ref. [48]. In this section the numerical calculations are performed on $n = 10^{10}$ settings.

Let us start from the comparison with the known results. To do this, we begin with a special case of $\gamma = 45^\circ$. Recently, the probability of violation for this particular state has been examined in Ref. [25], where the method based on the linear programing have been used. As a result, the total probability of violation presented in Ref. [25] is a combination of all probabilities related with the violation of any Bell inequality. For that reason, only the qualitative agreement with our results is expected. Especially that we deal only with one form of CGLMP inequality not the whole class of equivalent

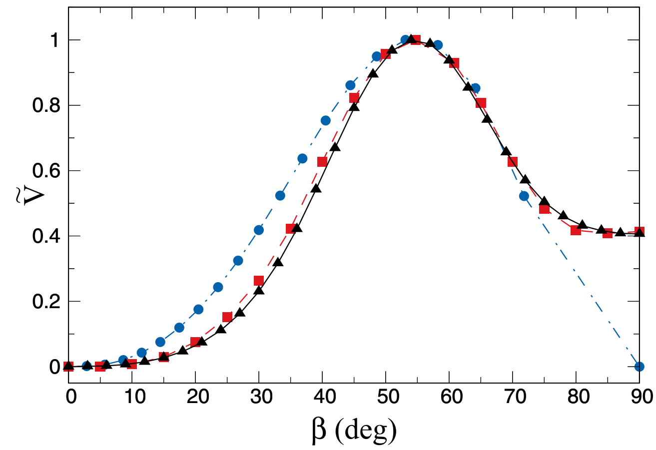

FIG. 4. The normalized volume of violation $\tilde{V}$ for two-qutrit states $|\psi_3\rangle = \frac{\sin\beta}{\sqrt{2}}(|00\rangle + |11\rangle) + \cos\beta|22\rangle$. Black curve with triangle symbols correspond to $\tilde{V}^{U(3)}$ while the red-dashed curve with rectangles denotes the normalized results obtained in Ref. [25]. Blue dot-dashed line with circles is the same as in Fig. 3(d), i.e., it is related with $\tilde{V}_\alpha^{\mathcal{U}}$ for the reshuffled unitary matrices $\mathcal{U}(A)$ and $\mathcal{U}(B)$ where $\alpha = \frac{\sin\beta}{\sqrt{2}}$ and $\alpha_d = \cos\beta$.

inequalities [33,39]. To reach this goal, we have simply rescale the outcomes of Rosier *et al.* [25] as it is shown in Fig. 4. As we see, both curves present similar behavior. In particular, the ratio of the extremal points $V^{U(3)}(\beta = 54.74^\circ)/V^{U(3)}(\beta = 90^\circ) = \frac{7.2763\times10^{-5}}{2.9542\times10^{-5}} = 2.463$, while the corresponding proportion given in Ref. [25] yield 2.419. This small disagreement may be caused by the presence of local minimum around $\beta \to 84^\circ$ noticed in the referenced paper. The authors explained this surprising feature by the fact that CHSH and CGLMP inequalities have different functions representing the violation probability. In our case, such local minimum of $V^{U(3)}$ has not been found, which confirms this statement and may affect the ratio between the extremal points.

Similarly, one can compare $V^{U(3)}$ with the results presented in the previous section. Specifically, if one puts $\alpha = \sin\beta\cos 45^\circ$ and $\alpha_d = \cos\beta$ then the state $|\psi_3\rangle(\gamma = 45^\circ)$ becomes equivalent to $|\psi_\alpha\rangle$. As we see in Figs. 3 and 4, none of the previous transformation can approximate $V^{U(3)}$ in the satisfactory way, especially, for $\beta \geqslant 70^\circ$ (equivalently $\alpha \geqslant 0.66$). For this regime of angle β the volume $V^{U(3)}$ tends slowly to the constant value of 2.9542×10^{-5}, while in the previous cases almost linear decay of V is observed.

Now, we can relax the angle γ and determine the volume $V^{U(3)}$ for a general two-qutrit state (8). In this case only one maximum around $\{\gamma, \beta\} = \{45^\circ, 54.74^\circ\}$, i.e., the MES, has been found (Fig. 5). This result confirms that MES can be truly considered as MNS with respect to V for $d = 3$ and the volume for MES is around 9.6% greater than for the optimal states $V^{U(3)}(\psi_O)$ (as presented in Table III), which is close to the M_2 transformation. There is also no local minimum in the entire range of angles $\{\gamma, \beta\}$.

In order to compare these results with the violation of Bell inequality I_d, we first introduce the normalized parameter

$$E = \frac{I_d - 2}{\max I_d - 2}, \quad (9)$$

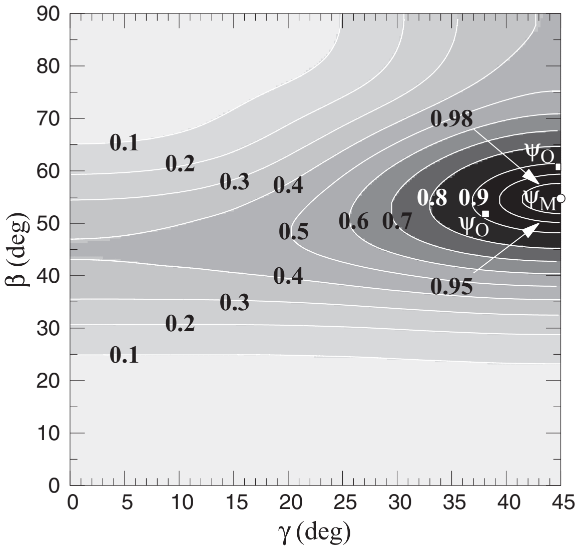

FIG. 5. The map of the volume of violation $\tilde{V}^{U(3)}$ for the state $|\psi_3\rangle$ and the general unitary transformation U(3). The solid lines correspond to the contour lines of $\tilde{V}^{U(3)}$. White rectangles represent two optimal states ($\{\gamma, \beta\} = \{38.1^\circ, 51.8^\circ\}$ and $\{45^\circ, 61.76^\circ\}$) equivalent under qutrit permutations while the white circle at $\{\gamma, \beta\} = \{45^\circ, 54.74^\circ\}$ denotes the MES.

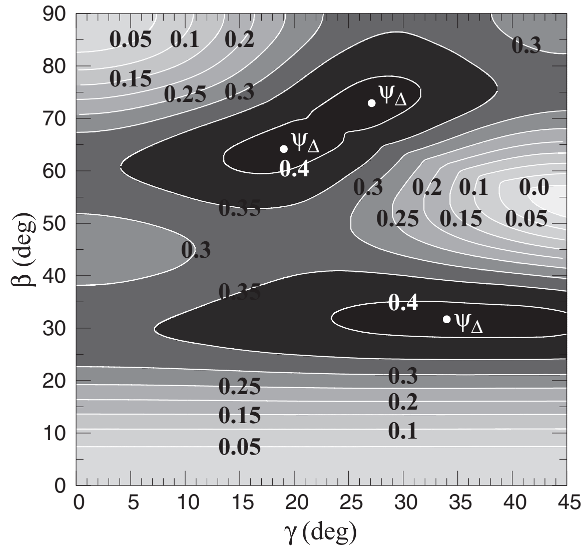

FIG. 7. The map of deference $\Delta = E - \tilde{V}^{U(3)}$ for the state $|\psi_3\rangle$ [Eq. (8)]. The solid lines denote the contour lines of Δ and white circles located at $\{\gamma, \beta\} = \{19.05^\circ, 64.1^\circ\}$, $\{27.11^\circ, 72.91^\circ\}$, and $\{34.0^\circ, 31.7^\circ\}$ correspond to the maximal points of Δ.

where max I_d is the maximal value of the Bell expression and the constant value indicating local realism [c.f. (1)] is subtracted. As a result, $E = 1$ denotes the optimal state and $E = 0$ corresponds to all local states. In Fig. 6 we see that for $E \leqslant 0.6$ the normalized volume $\tilde{V} = \langle 0, 0.25\rangle$ and is

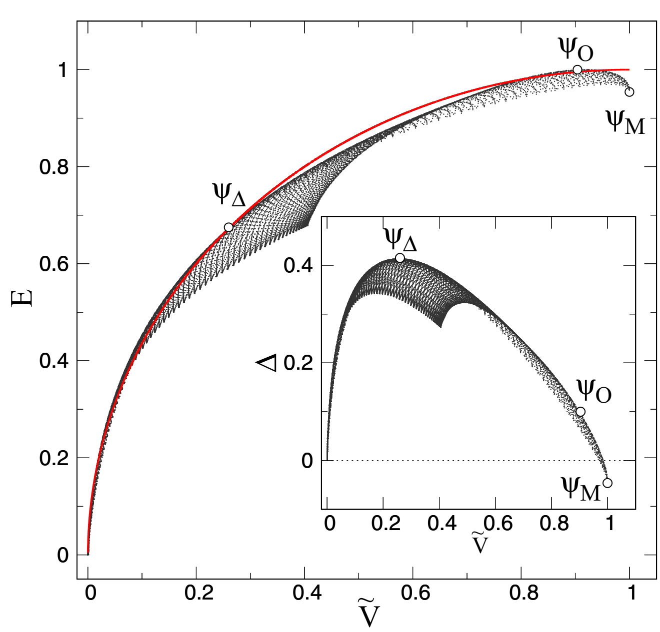

FIG. 6. Relation between the normalized volume of violation $\tilde{V}^{U(3)}$ and the normalized violation of CGLMP inequality E. Red line corresponds to the function $f_E(\tilde{V}) = \sqrt{1 - (\tilde{V} - 1)^2}$. The inset shows the difference $\Delta = E - \tilde{V}^{U(3)}$ vs $\tilde{V}^{U(3)}$. The labels ψ_M, ψ_O and ψ_Δ corresponds to the MES, optimal state and the state with highest Δ, respectively.

a slowly increasing function compared to E. Consequently, in order to estimate E for the small values of $\tilde{V}$ high-resolution experiments are needed to determine $\tilde{V}$ in proper precision. The opposite tendency takes place for $E \geqslant 0.85$. The highest difference $\Delta = E - \tilde{V}^{U(3)} = 0.41$ is observed for $\tilde{V}^{U(3)} \approx 0.26$, which corresponds to the state close to $|\psi_3\rangle = 0.436|00\rangle + 0.294|11\rangle + 0.851|22\rangle$ (up to the qutrits permutations). Finally, when $\tilde{V}$ is around 1 the difference $\Delta < 0$, which is related with the canceling of the anomaly by the volume of violation (see Fig. 7). In particular, $\Delta(\psi_M) = -0.046$ while $\Delta(\psi_O) = 0.088$. Due to the fact that there is no analytical solution for $\tilde{V}^{U(3)}$ we are not able to determine the extreme states of Fig. 7 but the upper bound can be approximated by the function $f_E(\tilde{V}) = \sqrt{1 - (\tilde{V} - 1)^2}$. Although this fitting is not tide, it supplies a simple picture of the relation between E and $\tilde{V}$.

We note that similar comparison can be done for $d > 3$. However, due to the large number of variables, $4(d^2 - 1)$, parametrizing four unitary transformations, which belong to U(d) group and the exponential decay of $V^{U(d)}$ caused be increasing d and/or number of variables the integration becomes extremely time demanding. For that reason, we limit our studies to MES and optimal states for $d = 4$. In this case, we have found that $V_M^{U(4)}$ is about 31% larger than $V_O^{U(4)}$ (Table III). Once again this result is similar to the outcome of the M_2 transformation. Consequently, one may expect the existence of the inflection point also in this case.

V. CONCLUSION

In this paper we have discussed the recent concept of nonlocality measure, namely the volume of violation. This new quantity has been proposed to overcome the existence of the anomaly between MESs and the states that maximally violate CGLMP inequality. In order to verify this hypothesis extensive numerical calculations have been done. We have shown that the predictions proposed in Ref. [30] are valid only for $d < 8$. When $d = \{8, 9, 10\}$ the opposite tendency is observed. The detailed analysis of the relation V_M/V_O reveal that our observation is not a coincidence and one should rather expect that the anomaly between MES and MNS for the CGLMP inequality cannot be canceled by the volume of violation in general.

When preparing the final version of the paper we became engaged in a recent manuscript [49] where the authors study a similar aspect of nonlocality without assuming any *a priori* Bell inequality. Although the calculations presented in Ref. [49] are limited to $d \leqslant 7$, there is no evidence that the anomaly could appear for $d = 8$ in the same way as described here. All these observations suggest that, at least for $d = 8$, the anomaly could be a special feature of a given facet of correlation polytope (corresponding to the CGLPM inequality) rather than a phenomenon of a whole polytope. Nonetheless, our observation highlights even stronger the question about the nature of the above-mentioned anomaly and offers a promising insight into a measure of nonlocality.

Furthermore, we have analyzed the influence of the chosen set of local observers on the volume of violation. In particular, we have shown that depending on the applied transformation the final results may differ by few orders of magnitude with respect to M_1 transformation. In order to ensure the universal frames to detect and quantify nonlocality by means of V for all pure states, one should use the general unitary transformation U(d). Nonetheless, the relation V_M/V_O for successive d seems to be analogical as for M_1 transformation and, hence, one may expect that for higher values of d the optimal states are also MNS with respect to the volume of violation.

On the other hand, we have presented an exemplary set of the local observers $\mathcal{U}$ for an arbitrary d, which allow for significant enhancement of the volume of violation, which has a strong practical consequence if one uses V for entanglement detection [32]. However, for this issue further calculations are needed to estimate the robustness of our results against decoherence.

Finally, we have discussed the relation between the volume of violation with the standard violation of CGLMP inequality, which is commonly used as a measure of nonlocality. We have found that two-qutrit pure states these two quantities can be approximated by $1 - E^2 \sim (V - 1)^2$, where E denotes the normalized expectation value of CGLMP inequality. In these calculations the general unitary transformation U(3) has been used. We have also shown that our results for some special two-qutrit states are compatible with those presented in Ref. [25].

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