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Time deformations of master equations

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Convolutionless and convolution master equations are the two mostly used physical descriptions of open quantum systems dynamics. We subject these equations to time deformations: local dilations and contractions of time scale. We prove that the convolutionless equation remains legitimate under any time deformation (results in a completely positive dynamical map) if and only if the original dynamics is completely positive divisible. Similarly, for a specific class of convolution master equations we show that uniform time dilations preserve positivity of the deformed map if the original map is positive divisible. These results allow witnessing different types of non-Markovian behavior: the absence of complete positivity for a deformed convolutionless master equation clearly indicates that the original dynamics is at least weakly non-Markovian; the absence of positivity for a class of time-dilated convolution master equations is a witness of essentially non-Markovian original dynamics.

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I. INTRODUCTION

A physical quantum system is never isolated in practice, which leads to a concept of an open quantum system. The state of such a system is described by a density operator ϱ on some Hilbert space $\mathcal{H}$ (positive semidefinite operator with unit trace). Time evolution of the open system is governed by the total Hamiltonian H of "system + environment" and the initial state of the environment Ω_E. If the system and environment are initially factorized, i.e., their state is $\varrho \otimes \Omega_E$, then the system dynamics is defined by the standard reduction

$$\varrho(t) = \mathrm{Tr}_E\{e^{-iHt}\varrho \otimes \Omega_E e^{iHt}\}. \tag{1}$$

Formula (1) defines a dynamical map $\Phi(t)[X] = \mathrm{Tr}_E\{e^{-iHt}X \otimes \Omega_E e^{iHt}\}$, which has an important property of being completely positive (CP) and trace preserving. Complete positivity means that $\Phi(t) \otimes \mathrm{Id}_k$ maps any (possibly entangled) density operator of the system + k-dimensional ancilla into a legitimate density operator.

Physical environments usually have enormously many degrees of freedom, which makes the dynamics $\varrho(t)$ intractable via formula (1) unless suitable approximations are made [1–5]. Microscopic derivations of system evolution with the help of projection operator techniques result in either a convolutionless master equation [2]

$$\frac{d\varrho(t)}{dt} = L(t)[\varrho(t)] \tag{2}$$

with a time-local generator $L(t) : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ or a convolution master equation [2,6–8]

$$\frac{d\varrho(t)}{dt} = \int_0^t \mathcal{K}(t,t')[\varrho(t')]dt' \tag{3}$$

with a memory kernel $\mathcal{K}(t,t') : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$.

Only some sufficient conditions on the time-local generator $L(t)$ and memory kernel $\mathcal{K}(t,t')$ are known, which guarantee complete positivity and trace preservation of the corresponding dynamical map $\Phi(t)$ [9–20].

Suppose that master equations (2) and (3) define a legitimate quantum dynamics, i.e., a completely positive and trace-preserving dynamical map $\Phi(t)$. From the quantum information science perspective, the evolution process $\Phi(t)$ can have peculiar divisibility properties. If the dynamical map $\Phi(t)$ can be represented in the form of concatenation $\Phi(t_2) = V(t_2,t_1)\Phi(t_1)$ with CP intermediate map $V(t_2,t_1)$ for all $t_2 > t_1 \geqslant 0$, then the process $\Phi(t)$ is called CP divisible. Analogously, if $V(t_2,t_1)$ is positive (P) for all $t_2 > t_1 \geqslant 0$, then the process $\Phi(t)$ is called P divisible. P indivisible processes are also referred to as essentially non-Markovian, whereas CP indivisible but P divisible processes are sometimes called weakly non-Markovian [21]. CP divisibility and P divisibility are only two approaches to define Markovian quantum processes [22,23]; many other approaches include decreasing distinguishability of system states [24,25], monotonicity of quantum mutual information [26], decreasing capacity of quantum channels [27], independence of evolution with respect to events preceding the causal break when the system's state is actively reset [28], and others. The reviews of the current status in the discussion of quantum non-Markovianity are given in the papers [5,29–31].

The goal of this paper is to relate divisibility properties of $\Phi(t)$ and the behavior of master equations under time deformations. By time deformation of a master equation we understand the transformation

$$\varrho(t) \to \widetilde{\varrho}(\tau), \quad dt \to d\tau, \tag{4}$$

where

$$\tau(t) = \int_0^t \alpha(t')dt', \quad \frac{d\tau}{dt} = \alpha(t), \tag{5}$$

and $\alpha(t)$ is non-negative real function quantifying the local time stretching.

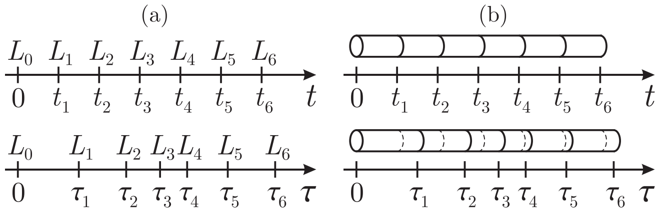

FIG. 1. (a) Time deformation of convolutionless master equation given by time-dependent generator $L(t)$. (b) Time deformation of CP divisible dynamics. Intermediate channels remain valid channels under dilations and contractions of time scale.

The naive interpretation of (4) would be the replacement of $\Phi(t)$ by $\Phi(\tau(t))$ but this is not the case if the generator $L(t)$ or memory kernel $\mathcal{K}(t,t')$ is time dependent. In fact, a time deformation (4) may result in a nonlegitimate master equation. Surprisingly, nonlegitimacy of a deformed master equation is closely related with the divisibility property of the undeformed dynamics. In this paper, we reveal this relation.

The paper is organized as follows.

In Sec. II, we show that the time deformation of the convolutionless master equation (2) results in a legitimate dynamical map if and only if the original dynamics is CP divisible. In Sec. III, we relate legitimacy of time deformation of convolution master equation (3) with P divisibility of the original dynamics. In Sec. IV, brief conclusions are given.

II. DEFORMATION OF CONVOLUTIONLESS MASTER EQUATIONS

Master equation (2) formally defines a dynamical map $\Phi(t) = T_{\leftarrow}\exp\left(\int_0^t L(t')dt'\right)$, where $T_{\leftarrow}$ is the Dyson time-ordering operator. The intermediate map $V(t_2,t_1)$ in concatenation $\Phi(t_2) = V(t_2,t_1)\Phi(t_1)$ reads $V(t_2,t_1) = T_{\leftarrow}\exp\left(\int_{t_1}^{t_2} L(t')dt'\right)$.

Time deformation of Eq. (2) results in a modified (*inequivalent*) master equation

$$\frac{d\widetilde{\varrho}(\tau(t))}{d\tau(t)} = L(t)[\widetilde{\varrho}(\tau(t))], \tag{6}$$

where the density operator $\widetilde{\varrho}(\tau(t))$ describes evolution in the deformed time and the original generator $L(t)$ is applied at time moments $\tau(t)$; see Fig. 1(a).

In terms of the original time t Eq. (6) reads

$$\frac{d\varrho(t)}{dt} = \frac{d\tau}{dt}\frac{d\varrho}{d\tau} = \alpha(t)L(t)[\varrho(t)]. \tag{7}$$

We will refer to Eq. (7) as a time deformation of the original time-convolutionless master equation (2).

If L is time independent, i.e., $\Phi(t) = e^{Lt}$ is a semigroup, then (7) results in a deformed map $\widetilde{\Phi}(t) = \Phi(\tau(t))$. However, if $L(t)$ is time dependent, then $\widetilde{\Phi}(t) \neq \Phi(\tau(t))$. Moreover, $\widetilde{\Phi}(t)$ can become not CP even if the original map $\Phi(t)$ is legitimate (CP and trace preserving), which can be illustrated by the following example.

Example 1: Consider a qubit map $\Phi(t) : \mathcal{B}(\mathcal{H}_2) \mapsto \mathcal{B}(\mathcal{H}_2)$ given by the generator [32]

$$L(t)[\varrho] = \frac{1}{2}\sum_{i=1}^{3}\gamma_i(t)(\sigma_i\varrho\sigma_i - \varrho), \tag{8}$$

where $\sigma_1,\sigma_2,\sigma_3$ is the conventional set of Pauli operators, $\gamma_1(t) = \gamma_2(t) = 1$, and $\gamma_3(t) = -\tanh(t)$. The map Φ_t is CP and trace preserving for all $t \geqslant 0$, so it is a legitimate dynamical map that can be realized physically, e.g., in the deterministic collision model [33].

It was shown in Ref. [34] that the time-deformed master equation

$$\frac{d\varrho(t)}{dt} = \alpha L(t)[\varrho(t)] \tag{9}$$

(obtained via constant time stretching $\tau = \alpha t$) results in a CP map $\widetilde{\Phi}(t)$ if and only if $\alpha \geqslant 1$. Thus, if the original master equation is subjected to a uniform time dilation $(0 < \alpha < 1)$, then the map $\widetilde{\Phi}_\alpha(t)$ is not CP and does not correspond to any physical evolution (of initially factorized system and environment).

Note that $\widetilde{\Phi}(t) \neq \Phi(\alpha t)$ because the decoherence rates $\gamma_k(t)$ are time dependent. ■

Nonlegitimacy of the deformed map $\widetilde{\Phi}(t)$ in the example above can be attributed to the fact that the master equation (8) describes so-called eternal non-Markovian evolution, i.e., CP indivisible dynamical map $\Phi(t)$, where $V(t_2,t_1)$ is not CP for all $t_2 > t_1$ [32,35,36]. On the other hand, if the original dynamical map were CP divisible, then all the decoherence rates would be positive. Time stretching would not affect positiveness of decoherence rates and $\widetilde{\Phi}(t)$ would still be a valid dynamical map. This leads us to the following main result.

Theorem 1: Master equation (2) with nonsingular generator $L(t)$ describes CP divisible dynamics if and only if the deformed map remains CP under any time deformation (7).

Proof. Necessity. Suppose the process $\Phi(t)$ is CP divisible and $L(t)$ is not singular; then the generator $L(t)$ has the time-dependent Gorini-Kossakowski-Sudarshan-Lindblad form [37,38]

$$L(t)[\varrho] = -i[H(t),\varrho] + \sum_k \gamma_k(t)\left(A_k(t)\varrho A_k^\dagger(t) - \frac{1}{2}\{A_k^\dagger(t)A_k(t),\varrho\}\right), \tag{10}$$

where all the rates $\gamma_k(t) \geqslant 0$. Multiplication of the Hamiltonian $H(t)$ by $\alpha(t)$ preserves its Hermiticity, and $\alpha(t)\gamma_k(t) \geqslant 0$, so $\alpha(t)L(t)$ is still a valid generator of the dynamical map (see, e.g., [9]).

Sufficiency. Let $\alpha(t) = \begin{cases} 0 & \text{if } 0 \leqslant t < t_1, \\ 1 & \text{if } t \geqslant t_1; \end{cases}$ then the deformed map $\widetilde{\Phi}(t) = T_{\leftarrow}\exp\left(\int_0^t \alpha(t')L(t')dt'\right) = \begin{cases} \mathrm{Id}, & \text{if } 0 \leqslant t < t_1, \\ V(t,t_1) & \text{if } t \geqslant t_1. \end{cases}$ Therefore, if the deformed map $\widetilde{\Phi}(t)$ remains CP under any deformation, then $V(t,t_1)$ is CP too for all $t > t_1$, i.e., the original map $\Phi(t)$ is CP divisible. ■

Therefore, CP divisible dynamics preserves the property of being CP divisible (and consequently CP) under any time

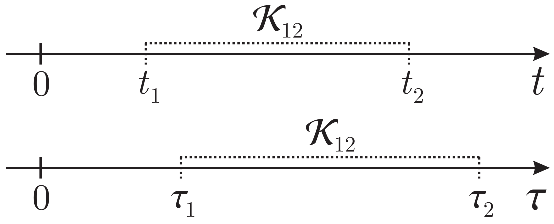

FIG. 2. Time deformation of convolution master equation governed by memory kernel $\mathcal{K}(t,t')$.

deformation; see Fig. 1(b). More importantly, if the original dynamical map is not CP divisible, then this fact can be revealed by a suitable time deformation under which the deformed map becomes nonlegitimate.

Remark 1: Nonsingularity of generator $L(t)$ is needed to guarantee invertibility of $\Phi(t)$. If $\Phi(t)$ is not invertible, then CP divisibility of $\Phi(t)$ does not require positivity of rates $\gamma_k(t)$; see Refs. [39,40]. However, the generator is not uniquely defined by the dynamical map $\Phi(t)$ in this case. In particular, if the process is CP divisible, then there exists a corresponding (possibly singular) time-local generator with non-negative rates. Theorem 1 holds true for such generators too.

III. DEFORMATION OF CONVOLUTION MASTER EQUATIONS

In this section, we consider time deformations of the convolution master equation (3) and make implications on P divisibility of the dynamical map $\Phi(t)$.

Continuing the same line of reasoning as before, let us assume that the same kernel $\mathcal{K}(t,t')$ is applied at deformed time moments $\tau(t)$ and $\tau(t')$; see Fig. 2. As a result, we obtain a time deformation of Eq. (3) of the form

$$\frac{d\widetilde{\varrho}(\tau(t))}{d\tau(t)} = \int_0^{\tau(t)} \mathcal{K}(t,t')[\widetilde{\varrho}(\tau(t'))]d\tau(t'), \tag{11}$$

which in terms of the original time t reads

$$\frac{d\varrho(t)}{dt} = \int_0^{t} \alpha(t)\alpha(t')\mathcal{K}(t,t')[\varrho(t')]dt'. \tag{12}$$

If $\mathcal{K}(t,t') = \delta(t-t')L(t')$, then (12) reduces to $\frac{d\varrho(t)}{dt} = \alpha^2(t)L(t)[\varrho(t)]$, i.e., to the time deformation of the convolutionless master equation considered before.

We assume that the open system dynamics does not depend on the particular choice of time moment $t=0$, when the system starts interacting with environment. Due to this time invariance $\mathcal{K}(t,t') = K(t-t')$ [6,7]. In local time deformations (5), the modified kernel $\alpha(t)\alpha(t')K(t-t')$ exhibits time invariance only if $\alpha(t)$ is time independent. For this reason, we consider only uniform time deformations $\tau(t) = \alpha t$, $\alpha = \text{const}$.

Denoting $(A*B)(t) = \int_0^t A(t-t')B(t')dt'$, master equation (3) takes the form $\frac{d}{dt}\Phi(t) = (K*\Phi)(t)$. Using the Laplace transform $\Phi_s = \int_0^\infty \Phi(t)e^{-st}dt$, the latter equation reduces to

$$\Phi_s = (s\,\text{Id} - K_s)^{-1}. \tag{13}$$

The uniformly deformed map $\widetilde{\Phi}(t)$ governed by Eq. (12) with $\alpha(t) = \alpha$ satisfies

$$\widetilde{\Phi}_s = (s\,\text{Id} - \alpha^2 K_s)^{-1}. \tag{14}$$

A straightforward algebra yields the following Laplace transform of the derivative $\frac{d}{dt}\widetilde{\Phi}(t)$:

$$\begin{aligned}\left(\frac{d\widetilde{\Phi}}{dt}\right)_s &= \frac{\alpha^2\left(\frac{d\Phi}{dt}\right)_s}{\text{Id} - (\alpha^2-1)\left(\frac{d\Phi}{dt}\right)_s}\\ &= \alpha^2\left(\frac{d\Phi}{dt}\right)_s \sum_{n=0}^{\infty}(\alpha^2-1)^n\left[\left(\frac{d\Phi}{dt}\right)_s\right]^n,\end{aligned} \tag{15}$$

where the second line represents a valid expansion if the norm $\|(\alpha^2-1)(\frac{d\Phi}{dt})_s\|_{1\to 1} < 1$. In the time domain one finds

$$\begin{aligned}\frac{d\widetilde{\Phi}}{dt} &= \alpha^2\frac{d\Phi}{dt} + \alpha^2(\alpha^2-1)\frac{d\Phi}{dt}*\frac{d\Phi}{dt} + \cdots\\ &\quad + \alpha^2(\alpha^2-1)^n\underbrace{\frac{d\Phi}{dt}*\cdots*\frac{d\Phi}{dt}}_{n+1\ \text{times}} + \cdots.\end{aligned} \tag{16}$$

Let us restrict ourselves to the commutative case, i.e., maps $\Phi(t)$ satisfying $\Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$ for all $t_1, t_2 \geqslant 0$. Commutative maps have time-independent eigenoperators, so the spectrum of $\frac{d\Phi}{dt}$ is merely the derivative of the spectrum of $\Phi(t)$. Denote eigenvalues of $\Phi(t)$ by $\lambda_k(t)$; then for P divisible $\Phi(t)$ one has $\frac{d|\lambda_k(t)|}{dt} \leqslant 0$ [41]. If $\Phi(t)$ is Hermitian, i.e., $\Phi(t)$ coincides with its dual map $\Phi^\dagger(t)$ in the Heisenberg picture, then $\lambda_k(t)$ are real. Therefore, for commutative Hermitian P divisible maps $\Phi(t)$ we have $\frac{d\lambda_k(t)}{dt} \leqslant 0$. On the other hand, if $\frac{d\lambda_k(t)}{dt} \leqslant 0$, then (16) implies $\frac{d\widetilde{\lambda}_k(t)}{dt} \leqslant 0$ provided $0 < \alpha < 1$. This way one arrives at the following result.

Proposition 1: Suppose the commutative Hermitian dynamical map $\Phi(t)$ is governed by a memory kernel $K(t)$. If the uniform time dilation $K(t) \to \alpha^2 K(t)$ with $0 < \alpha < 1$ and $(1-\alpha^2)\|(\frac{d\Phi}{dt})_s\|_{1\to 1} < 1$ results in a map $\widetilde{\Phi}(t)$ such that $\frac{d\widetilde{\Phi}}{dt}$ has at least one positive eigenvalue at some time t, then the original map $\Phi(t)$ is not P divisible.

The class of commutative Hermitian dynamical maps comprises conventional Pauli qubit maps $\Phi(t)[\varrho] = \frac{1}{2}(\text{tr}[\varrho]I + \sum_{k=1}^{3}\lambda_k(t)\text{tr}[\sigma_k\varrho]\sigma_k)$ as well as generalized Pauli channels [42,43]. For Pauli qubit maps one can find a simpler implication of Proposition 1.

Proposition 2: Suppose the Pauli map $\Phi(t)$ is governed by a memory kernel $K(t)$. If the uniform time dilation $K(t) \to \alpha^2 K(t)$ with $0 < \alpha < 1$ and $(1-\alpha^2)(1 - s\int_0^\infty \lambda_k(t)e^{-st}dt) < 1$ results in a map $\widetilde{\Phi}(t)$, which is not positive, then the original map $\Phi(t)$ is not P divisible.

Proof. Condition $(1-\alpha^2)[1 - s\int_0^\infty \lambda_k(t)e^{-st}dt] < 1$ guarantees the validity of expansion (16). Let $\widetilde{\Phi}(t)$ be nonpositive. Since the Pauli map $\widetilde{\Phi}(t)$ is positive if and only if $-1 \leqslant \widetilde{\lambda}_k(t) \leqslant 1$, then either $\widetilde{\lambda}_k(t) > 1$ or $\widetilde{\lambda}_k(t) < -1$ for some time t. Note that at the initial moment $\lambda_k(0) = 1$.

Suppose $\widetilde{\lambda}_k(t) > 1$; then there exists a time moment $t_0 \in (0,t)$ such that $\frac{d\widetilde{\lambda}_k(t)}{dt}(t_0) > 0$. By Proposition 1, $\Phi(t)$ is not P divisible.

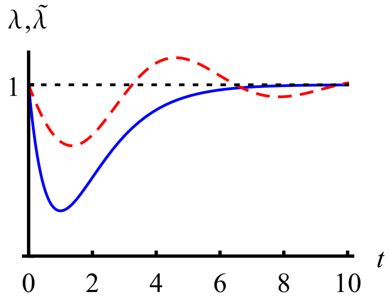

FIG. 3. Blue (solid) curve: eigenvalue $\lambda_1(t)$ of the original dynamical map governed by convolution master equation (18) with $\Gamma = 1$. Red (dashed) curve: eigenvalue $\tilde{\lambda}_1(t)$ of the time deformed map, Eq. (19), with $\Gamma = 1$ and the deformation coefficient $\alpha = \frac{1}{2}$.

Suppose $\tilde{\lambda}_k(t) < -1$; let us show that $\lambda_k(t) \ngeqslant 0$. Using expansion

$$\tilde{\Phi}(t) = \Phi(t) + (\alpha^2 - 1)\left(\frac{d\Phi}{dt} * \Phi\right)(t) + \cdots$$
$$+ (\alpha^2 - 1)^n \left(\underbrace{\frac{d\Phi}{dt} * \cdots * \frac{d\Phi}{dt}}_{n \text{ times}} * \Phi\right)(t) + \cdots, \quad (17)$$

one finds that if $\lambda_k(t) \geqslant 0$ and $\frac{d\lambda_k}{dt} \leqslant 0$, then a time deformation with $0 < \alpha < 1$ guarantees $\tilde{\lambda}_k(t) \geqslant 0$. As we consider the case $\tilde{\lambda}_k(t) < -1$, this contradiction proves that $\lambda_k(t) \ngeqslant 0$. As a result, the original Pauli map $\Phi(t)$ is not P divisible. ■

The physical meaning of Proposition 2 is that positivity is a topological property of Pauli P divisible process $\Phi(t)$, which is preserved under uniform time dilations.

Example 2: Consider a pure dephasing qubit map $\Phi(t)[\varrho] = \frac{1}{2}(\mathrm{tr}[\varrho]I + \sum_{k=1}^{3} \lambda_k(t)\mathrm{tr}[\sigma_k\varrho]\sigma_k)$ with $\lambda_1(t) = \lambda_2(t) = 1 - 2\Gamma t\, e^{-\Gamma t}$ and $\lambda_3(t) = 1$. This is a valid dynamical map if $\Gamma > 0$. Such a map is a solution of the convolution master equation

$$\frac{d\varrho(t)}{dt} = \int_0^t [\Gamma\delta(t - t') - \Gamma^2 \sin\Gamma(t - t')]$$
$$\times [\sigma_z\varrho(t')\sigma_z - \varrho(t')]dt'. \quad (18)$$

Condition $(1 - \alpha^2)(1 - s\int_0^\infty \lambda_k(t)e^{-st}dt) < 1$ is fulfilled automatically if $0 < \alpha < 1$. The uniform time dilation of the memory kernel $K(t - t') \to \alpha^2 K(t - t')$ results in the deformed Pauli map $\tilde{\Phi}(t)$ with

$$\tilde{\lambda}_1(t) = \tilde{\lambda}_2(t) = 1 - 2\alpha^2 e^{-\alpha^2\Gamma t}\frac{\sin(\sqrt{1 - \alpha^4}\,\Gamma t)}{\sqrt{1 - \alpha^4}} \quad (19)$$

and $\tilde{\lambda}_3(t) = 1$. When the trigonometric function $\sin(\cdot)$ takes negative values, $\tilde{\lambda}_1(t) = \tilde{\lambda}_2(t) > 1$, see Fig. 3 , so the deformed map $\tilde{\Phi}(t)$ is not positive. By Proposition 2, it clearly indicates that the original map $\Phi(t)$ is not P divisible.

Note that for the equivalent original *convolutionless* equation, the uniform time deformation $\tau = \alpha t$ results in $\lambda_i'(t) = [\lambda_i(t)]^\alpha$, $i = 1, 2, 3$. In this case, the deformed map $\tilde{\Phi}'(t)$ remains CP and does not reveal P indivisibility of $\Phi(t)$. ■

Example 3: Let us consider a qubit evolution where the rescaling of the memory kernel is compatible with P divisibility of the dynamical map. Following [16], let $\Phi(t)$ be a Pauli qubit dynamical map governed by the memory kernel

$$K(t)[\varrho] = \frac{1}{2}\sum_{k=1}^{3} \varkappa_k(t)\sigma_k \mathrm{tr}[\sigma_k\varrho], \quad (20)$$

where the time-dependent eigenvalues $\varkappa_k(t)$ are defined (in the Laplace transform domain) via

$$(\varkappa_k)_s = \frac{-sf_s}{a_k - f_s}. \quad (21)$$

In the above definition the positive numbers $\{a_1, a_2, a_3\}$ satisfy triangle inequality $a_i^{-1} + a_j^{-1} \geqslant a_k^{-1}$ for all permutations of $\{i, j, k\}$, $f(t)$ is a real function satisfying $f(t) \geqslant 0$, and $f_0 = \int_0^\infty f(t)dt \leqslant 4(a_1^{-1} + a_2^{-1} + a_3^{-1})^{-1}$. The corresponding eigenvalues of $\Phi(t)$ are given by $\lambda_k(t) = 1 - a_k^{-1}\int_0^t f(t')dt'$.

The dynamical map $\Phi(t)$ is known to be P divisible if additionally $f(t)$ satisfies the requirement [16]

$$f_0 = \int_0^\infty f(t)dt \leqslant a_{\min}, \quad (22)$$

where $a_{\min} = \min\{a_1, a_2, a_3\}$. Suppose condition (22) is fulfilled; then $f_s \leqslant a_{\min}$ for all $s \geqslant 0$. The deformed eigenvalue

$$(\tilde{\lambda}_k)_s = \frac{1}{s\left(1 + \frac{\alpha^2 f_s}{a_k - f_s}\right)} = \frac{1}{s}\left(1 - \frac{f_s}{a_k}\right)\sum_{n=0}^{\infty}(1 - \alpha^2)^n\left(\frac{f_s}{a_k}\right)^n \quad (23)$$

in time domain is a convolution of two non-negative functions: the original eigenvalue $\lambda_k(t) \in (0, 1]$ and the inverse Laplace transform of $\sum_{n=0}^\infty (1 - \alpha^2)^n(\frac{f_s}{a_k})^n$. Hence $\tilde{\lambda}_k(t) \geqslant 0$. If $0 < \alpha < 1$, then the latter function is less than or equal to the inverse Laplace transform of $\sum_{n=0}^\infty (\frac{f_s}{a_k})^n = (1 - \frac{f_s}{a_k})^{-1}$. Therefore, $\tilde{\lambda}_k(t)$ is less than or equal to the inverse Laplace transform of function $(\lambda_k)_s(1 - \frac{f_s}{a_k})^{-1} = \frac{1}{s}$, i.e., $\tilde{\lambda}_k(t) \leqslant 1$. Thus the deformed map is positive for $0 < \alpha < 1$ because the original map is P divisible; see Eq. (22).

Interestingly, the map $\tilde{\Phi}(t)$ being positive and trace preserving is in general not completely positive and hence the kernel deformation $K(t) \to \alpha^2 K(t)$ does not lead to the legitimate dynamical map. In fact, consider the behavior of $\tilde{\lambda}_k(t)$ when $t \to \infty$. By the final value theorem

$$\lim_{t\to\infty}\tilde{\lambda}_k(t) = \lim_{s\to 0} s(\tilde{\lambda}_k)_s = \frac{1}{1 + \alpha^2\frac{f_0}{a_k - f_0}}. \quad (24)$$

Suppose $a_1 \leqslant a_2 \leqslant a_3$. The deformed map is CP if and only if the condition $\tilde{\lambda}_i + \tilde{\lambda}_j \leqslant 1 + \tilde{\lambda}_k$ is fulfilled for permutations of indices $\{i, j, k\}$. In the limit $t \to \infty$ this condition reduces to inequality

$$f_0\left(\frac{1}{a_2 - f_0} + \frac{1}{a_3 - f_0} - \frac{1}{a_1 - f_0}\right) + \frac{2\alpha^2 f_0^2}{(a_2 - f_0)(a_3 - f_0)}$$
$$+ \frac{\alpha^4 f_0^3}{(a_1 - f_0)(a_2 - f_0)(a_3 - f_0)} \geqslant 0. \quad (25)$$

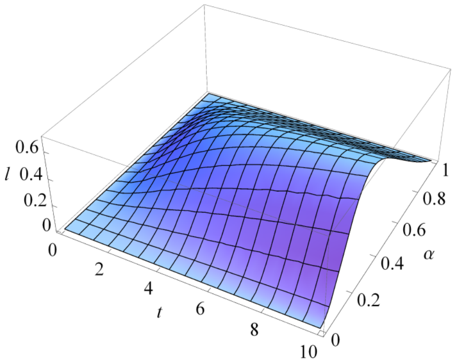

FIG. 4. Plot of $\ell(t) = \widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t) - \widetilde{\lambda}_3(t) - 1$ as a function of dimensionless time t and rescaling parameter α. One has $\ell(t) > 0$ for all $t > 0$ and $0 < \alpha < 1$, so the Fujiwara-Algoet condition of complete positivity is violated.

The obtained inequality is fulfilled for all $0 < \alpha < 1$ if and only if $(a_2 - f_0)^{-1} + (a_3 - f_0)^{-1} \geqslant (a_1 - f_0)^{-1}$, which is surprisingly equivalent to CP divisibility of the original map $\Phi(t)$; cf. Ref. [16]. Thus the dynamical map governed by the memory kernel (20) is CP divisible only if the deformed map is CP for all $0 < \alpha < 1$. ■

Example 4: Consider CP indivisible Pauli dynamical map $\Phi(t)$ as in Example 1 but now in terms of the convolution equation $\frac{d\Phi}{dt} = K * \Phi$. The explicit form of the kernel $K(t)$ is given in Ref. [35]. The uniform time deformation $K(t) \to \alpha^2 K(t)$ leads to the deformed eigenvalues $\widetilde{\lambda}_1(t) = \widetilde{\lambda}_2(t) = (1+\alpha^2)^{-1}[1 + \alpha^2 e^{-(1+\alpha^2)t}]$ and $\widetilde{\lambda}_3(t) = e^{-2\alpha^2 t}$. The deformed map $\widetilde{\Phi}(t)$ is never CP for $t > 0$ and $0 < \alpha < 1$ since the corresponding set of eigenvalues violates the Fujiwara-Algoet conditions for complete positivity [44] (cf. Fig. 4). ■

Considered examples allow us to make a *conjecture* that a general Pauli dynamical map $\Phi(t)$, defined by a convolution master equation, is CP divisible if and only if the deformed map $\widetilde{\Phi}(t)$ is CP for all $0 < \alpha < 1$. It is tempting to pose a similar conjecture for general dynamical maps governed by memory kernel master equations, namely, the map is CP divisible iff the corresponding rescaled kernel $\alpha^2 K(t)$ is physically legitimate for $0 < \alpha < 1$. This, however, requires further analysis.

IV. CONCLUSIONS

We have analyzed different forms of non-Markovianity in terms of the time deformations of governing master equations. If some deformation of the time-local equation results in a map, which is not CP, then the original map is not CP divisible (it is at least weakly non-Markovian). Analogously, if a deformation of the proper time-convolution equation results is a map, which is not P, then the original map is not P divisible (it is essentially non-Markovian).

As the analysis of convolution master equations is particularly complicated, we have managed to obtain only a necessary condition for P divisible Hermitian commutative dynamical maps (Proposition 1). We have illustrated implications of this condition for Pauli dynamical qubit maps (Proposition 2 and Example 2). We have also considered Examples 3 and 4 of Pauli dynamical maps defined via a convolution master equation, for which CP divisibility is equivalent to CP property of the deformed map for all uniform time dilations.

In addition to witnessing non-Markovianity, the achieved results clarify legitimate forms of dissipators and memory kernels, which naturally emerge due to relativistic and gravitational time dilation [45], as well as acceleration of quantum systems [46].

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[1] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Springer, Berlin, 1987).

[2] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[3] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics* (Springer, Berlin, 2004).

[4] H. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer, Berlin, 2009).

[5] L. Li, M. J. W. Hall, and H. M. Wiseman, Concepts of quantum non-Markovianity: A hierarchy, arXiv:1712.08879v2 [quant-ph].

[6] S. Nakajima, On quantum theory of transport phenomena: Steady diffusion, Prog. Theor. Phys. **20**, 948 (1958).

[7] R. Zwanzig, Ensemble method in the theory of irreversibility, J. Chem. Phys. **33**, 1338 (1960).

[8] F. Haake, *Statistical Treatment of Open Systems by Generalized Master Equations*, Springer Tracts in Modern Physics Vol. 66 (Springer, Berlin, 1973).

[9] M. J. W. Hall, Complete positivity for time-dependent qubit master equations, J. Phys. A: Math. Theor. **41**, 205302 (2008).

[10] S. M. Barnett and S. Stenholm, Hazards of reservoir memory, Phys. Rev. A **64**, 033808 (2001).

[11] A. A. Budini, Stochastic representation of a class of non-Markovian completely positive evolutions, Phys. Rev. A **69**, 042107 (2004).

[12] A. Shabani and D. A. Lidar, Completely positive post-Markovian master equation via a measurement approach, Phys. Rev. A **71**, 020101(R) (2005).

[13] S. Maniscalco, Limits in the characteristic-function description of non-Lindblad-type open quantum systems, Phys. Rev. A **72**, 024103 (2005).

[14] S. Maniscalco and F. Petruccione, Non-Markovian dynamics of a qubit, Phys. Rev. A **73**, 012111 (2006); **73**, 029902(E) (2006); **75**, 059905(E) (2007).

[15] H.-P. Breuer and B. Vacchini, Quantum Semi-Markov Processes, Phys. Rev. Lett. **101**, 140402 (2008).

[16] F. A. Wudarski, P. Należyty, G. Sarbicki, and D. Chruściński, Admissible memory kernels for random unitary qubit evolution, Phys. Rev. A **91**, 042105 (2015).
[17] D. Chruściński and A. Kossakowski, Sufficient conditions for a memory-kernel master equation, Phys. Rev. A **94**, 020103(R) (2016).
[18] B. Vacchini, Generalized Master Equations Leading to Completely Positive Dynamics, Phys. Rev. Lett. **117**, 230401 (2016).
[19] D. Chruściński and A. Kossakowski, Generalized semi-Markov quantum evolution, Phys. Rev. A **95**, 042131 (2017).
[20] J. Marshall, L. C. Venuti, and P. Zanardi, Noise suppression via generalized-Markovian processes, Phys. Rev. A **96**, 052113 (2017).
[21] D. Chruściński and S. Maniscalco, Degree of Non-Markovianity of Quantum Evolution, Phys. Rev. Lett. **112**, 120404 (2014).
[22] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Assessing Non-Markovian Quantum Dynamics, Phys. Rev. Lett. **101**, 150402 (2008).
[23] Á. Rivas, S. F. Huelga, and M. B. Plenio, Entanglement and Non-Markovianity of Quantum Evolutions, Phys. Rev. Lett. **105**, 050403 (2010).
[24] H.-P. Breuer, E.-M. Laine, and J. Piilo, Measure for the Degree of Non-Markovian Behavior of Quantum Processes in Open Systems, Phys. Rev. Lett. **103**, 210401 (2009).
[25] E.-M. Laine, J. Piilo, and H.-P. Breuer, Measure for the non-Markovianity of quantum processes, Phys. Rev. A **81**, 062115 (2010).
[26] S. Luo, S. Fu, and H. Song, Quantifying non-Markovianity via correlations, Phys. Rev. A **86**, 044101 (2012).
[27] B. Bylicka, D. Chruściński, and S. Maniscalco, Non-Markovianity and reservoir memory of quantum channels: A quantum information theory perspective, Sci. Rep. **4**, 5720 (2014).
[28] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Operational Markov Condition for Quantum Processes, Phys. Rev. Lett. **120**, 040405 (2018).
[29] Á. Rivas, S. F. Huelga, and M. B. Plenio, Quantum non-Markovianity: Characterization, quantification and detection, Rep. Prog. Phys. **77**, 094001 (2014).
[30] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Colloquium: Non-Markovian dynamics in open quantum systems, Rev. Mod. Phys. **88**, 021002 (2016).
[31] I. de Vega and D. Alonso, Dynamics of non-Markovian open quantum systems, Rev. Mod. Phys. **89**, 015001 (2017).
[32] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, Canonical form of master equations and characterization of non-Markovianity, Phys. Rev. A **89**, 042120 (2014).
[33] S. N. Filippov, J. Piilo, S. Maniscalco, and M. Ziman, Divisibility of quantum dynamical maps and collision models, Phys. Rev. A **96**, 032111 (2017).
[34] F. Benatti, D. Chruściński, and S. Filippov, Tensor power of dynamical maps and positive versus completely positive divisibility, Phys. Rev. A **95**, 012112 (2017).
[35] N. Megier, D. Chruściński, J. Piilo, and W. T. Strunz, Eternal non-Markovianity: From random unitary to Markov chain realisations, Sci. Rep. **7**, 6379 (2017).
[36] D. Chruściński and F. A. Wudarski, Non-Markovianity degree for random unitary evolution, Phys. Rev. A **91**, 012104 (2015).
[37] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of N-level systems, J. Math. Phys. **17**, 821 (1976).
[38] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).
[39] E. Andersson, J. D. Cresser, and M. J. W. Hall, Finding the Kraus decomposition from a master equation and vice versa, J. Mod. Opt. **54**, 1695 (2007).
[40] D. Chruściński, Á. Rivas, and E. Størmer, On Divisibility and Information Flow Notions of Quantum Markovianity for Non-Invertible Dynamical Maps, [Phys. Rev. Lett. (to be published)].
[41] D. Chruściński, C. Macchiavello, and S. Maniscalco, Detecting Non-Markovianity of Quantum Evolution via Spectra of Dynamical Maps, Phys. Rev. Lett. **118**, 080404 (2017).
[42] M. Nathanson and M. B. Ruskai, Pauli diagonal channels constant on axes, J. Phys. A **40**, 8171 (2007).
[43] D. Chruściński and K. Siudzińska, Generalized Pauli channels and a class of non-Markovian quantum evolution, Phys. Rev. A **94**, 022118 (2016).
[44] A. Fujiwara and P. Algoet, One-to-one parametrization of quantum channels, Phys. Rev. A **59**, 3290 (1999).
[45] I. Pikovski, M. Zych, F. Costa, and Č. Brukner, Time dilation in quantum systems and decoherence, New J. Phys. **19**, 025011 (2017).
[46] B. Richter, H. Terças, Y. Omar, and I. de Vega, Collective dynamics of accelerated atoms, Phys. Rev. A **96**, 053612 (2017).