Self-averaging of random quantum dynamics

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The stochastic dynamics of a quantum system driven by N statistically independent random sudden quenches in a fixed time interval is studied. We reveal that with increasing N the system approaches a deterministic limit, indicating self-averaging with respect to its temporal unitary evolution. This phenomenon is quantified by the variance of the unitary matrix governing the time evolution of a finite-dimensional quantum system which, according to an asymptotic analysis, decreases at least as $1/N$. For a special class of protocols (when the averaged Hamiltonian commutes at different times), we prove that for finite N the distance (according to the Frobenius norm) between the averaged evolution unitary operator generated by the Hamiltonian H and the unitary evolution operator generated by the averaged Hamiltonian $\langle H\rangle$ scales as $1/N$. Numerical simulations enlarge this result to a broader class of noncommuting protocols.

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I. INTRODUCTION

Self-averaging is a well-established concept in the statistical physics of disordered and random systems. Loosely speaking, a certain property X of a system is self-averaging if most realizations of the randomness have the same value of X in some limiting regime. More precisely, a system is self-averaging with respect to X if the relative variance of X tends to 0 in this limiting regime. If, e.g., we consider a system of combinatorial objects of size N, then the relative variance

$$\frac{\langle X_N^2\rangle - \langle X_N\rangle^2}{\langle X_N\rangle^2} \to 0 \tag{1}$$

as $N \to \infty$.

For a large class of randomly driven quantum systems such as *quenched disordered systems* [1,2], the question of self-averaging of their properties is essentially nontrivial [3]. There have been studies on self-averaging of a free energy for spin systems with short-range [4] or long-range interactions [5], self-averaging of diffusion in heterogeneous media [6], self-averaging of Lyapunov exponents in fluids [7], and self-averaging of the reduced density matrices [8], to mention only a few.

In this paper, we consider a broad class of randomly driven quantum systems (see the recent paper [9] on random evolution) for which the time evolution is universally self-averaging. In particular, we study quantum dynamics in the presence of subsequent random and independent steplike perturbations of finite-dimensional quantum systems. Such a driving corresponds to the *quantum quench* dynamics of closed quantum systems—a rapidly developing and intensively investigated research area [10] which recently has found experimental realizations [11]. Thermalization [12], quantum phase transitions [13,14], integrability [15], simple out-of-equilibrium quantum systems [16], and work fluctuations [17]—this is a far from complete list of examples of quantum quench scenarios that have been studied. We investigate the driving of a quantum system formed as a series of statistically independent random quenches—*multiple random quench* (MRQ)—and its continuous limit of an infinite number of quenches occurring in a finite time interval—*continuous random quench* (CRQ). Self-averaging of the unitary time evolution for the MRQ protocol occurs with an increasing number N of quenches in the fixed time interval. This phenomenon is quantified by vanishing variance of the unitary time-evolution matrix representation, which decreases at least as $1/N$. This behavior is formally proved for an arbitrary distribution supported on bounded intervals of the randomly controlled Hamiltonians. According to the self-averaging property, the considered unitary evolution almost surely converges to its mean value. We estimate this mean value for a special class of protocols when an instantaneous average of the Hamiltonian (with respect to the matrix ensemble) commutes at different time instants. We call this property "the commutation in the statistical sense." For this case we prove that the self-averaged unitary evolution converges to the evolution governed by a mean value of a random Hamiltonian and convergence is in the sense of the Frobenius (Hilbert-Schmidt) norm. In other words, in the basis where the average of the Hamiltonian is diagonal, off-diagonal elements with vanishing mean value contribute less and less to the time evolution as the number of quenches increases. Moreover, we have also performed numerical simulations in order to analyze the noncommuting case for a qubit. For some particular drivings we show that also in this case, in the CRQ limit, the evolution is generated by the mean value of the Hamiltonian, even though it does not commute in a statistical sense at different points in time (i.e., when instantaneous averages cannot be simultaneously diagonalized). For this noncommuting case and two other examples of the MRQ protocols for a qubit space, results of numerical simulation apparently exhibit the exact power law $1/N$, which is the lower asymptotic estimation predicted analytically.

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The layout of the paper is as follows. In Sec. II, we provide the necessary information on the theory of random matrices required for further reasoning. Next, in Sec. III, we formulate a unitary time evolution of quantum systems with random quenches and introduce the notion of the effective Hamiltonian of the system. In the same section, we define commutation of operators in the statistical sense. In Sec. IV, we discuss the statistics of the effective Hamiltonian of the MRQ control (with two main propositions concerning its properties), and as a consequence we formulate a self-averaging condition for the unitary time evolution. In Sec. V, we provide a numerical simulation for more general MRQ protocols. Finally, in Sec. VI, we summarize our results and we present some ideas for future work. We postpone proofs of the propositions formulated in Sec. IV to Appendixes A–F.

II. RANDOM MATRIX THEORY

In order to describe and define MRQ we utilize random matrix theory [18], a rapidly developing branch of mathematics useful in many branches of modern physics starting from Wigner's classification of "canonical" random matrix ensembles for the description of statistics of nuclear levels spacing up to quantum chaos, many-body physics, and quantum statistical mechanics. The MRQ driving studied in this paper is a further example.

Let us represent an M-dimensional complex and Hermitian matrix H as point $H = (h_1, h_2, \ldots, h_d)$ in a d-dimensional real space $\mathbb{R}^d$, where $d = M^2$ is the number of real and independent parameters specifying the matrix H. In the following, we consider an ensemble of matrices with *random* parameters h_i and the probability distribution

$$\Pr(H \in D) = \int_D dH \varrho(H) \tag{2}$$

that $H = (h_1, h_2, \ldots, h_d) \in D \subset \mathbb{R}^d$, where $\varrho(H) = \varrho(h_1, h_2, \ldots, h_d)$ is a probability density function (pdf) and $dH = dh_1 dh_2 \ldots dh_d$. We restrict our reasoning to the distribution $\varrho(H)$, which we call a matrix-pdf, supported on the bounded probability space $\mathcal{P} \subset \mathbb{R}^d$ and normalized in such a way that

$$\int_{\mathcal{P}} dH \varrho(H) = 1\,. \tag{3}$$

Let $\mathcal{H}_N$ be an ordered set of random and statistically independent matrices

$$\mathcal{H}_N = (H_1, H_2, \ldots, H_N), \tag{4}$$

with the joint pdf given by the product of individual distributions ensuring statistical independence,

$$\rho(\mathcal{H}_N) = \varrho_1(H_1)\varrho_2(H_2)\ldots\varrho_N(H_N), \tag{5}$$

where the pdf $\varrho_k(H_k) = \varrho_k(h_1^{(k)}, h_2^{(k)}, \ldots, h_d^{(k)})$ for $k = 1, 2, \ldots, N$. For any matrix U depending on the set $\mathcal{H}_N$ one can define the first statistical moment $\langle U\rangle$ as an average of the elements $[\langle U\rangle]_{\alpha\beta} = \langle [U]_{\alpha\beta}\rangle$, where

$$\langle [U]_{\alpha\beta}\rangle = \int_{\mathcal{P}} d\mathcal{H}_N\, [U]_{\alpha\beta}\, \rho(\mathcal{H}_N). \tag{6}$$

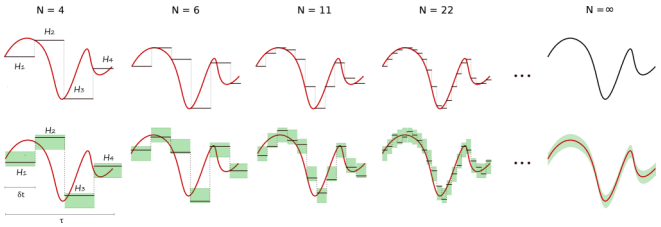

FIG. 1. Schematic visualization of the process of "integration" of arbitrary time-dependent driving into the unitary time-evolution operator. Solid red lines represent changes in time of an arbitrary finite-dimensional, time-dependent Hamiltonian. The deterministic regime in the upper panel shows an increasing number of partially constant Hamiltonians (solid black lines) forming a steplike realization which, in the limit of an infinite number of quenches, converges into the time-evolution operator of a continuous driving (red line). The stochastic realization in the lower panel is a sketch of the same idea of an "integration" into a unitary operator, however, in this case for each partially constant Hamiltonian we take a statistically independent random matrix with some dispersion (indicated by green boxes). In this case solid black lines represent a particular realization of this stochastic process and the red line represents the mean value of the time-dependent Hamiltonian.

Here, $[\cdot]_{\alpha\beta}$ denotes a matrix element, $d\mathcal{H}_N = \prod_{k=1}^N dH_k$, and $dH_k = dh_1^{(k)} dh_2^{(k)} \ldots dh_d^{(k)}$. We define *per analogiam* the variance matrix $\mathrm{Var}(U)$ as the matrix of variances, i.e., $[\mathrm{Var}(U)]_{\alpha\beta} = \mathrm{Var}([U]_{\alpha\beta})$ with

$$\mathrm{Var}([U]_{\alpha\beta}) = \langle |[U]_{\alpha\beta}|^2\rangle - |\,\langle [U]_{\alpha\beta}\rangle\,|^2. \tag{7}$$

In the following we use a Frobenius matrix norm $\|\cdot\|$ defined by

$$\|U\|^2 = \mathrm{Tr}[UU^\dagger] \tag{8}$$

which is known to be submultiplicative, i.e., $\|AB\| \leqslant \|A\|\|B\|$ for any matrices A and B.

III. SUDDEN QUENCH EVOLUTION

In this section, using the random matrix terminology, we formulate the time evolution of quantum systems subjected to random quenches. For completeness we start with the more intuitive case of deterministic dynamics, which can be considered as a limiting case of more general dynamics, which is our primary object of investigation.

A. Deterministic case

We consider a quantum system driven by a deterministic time-dependent Hamiltonian $H(t)$ in the time interval $\mathcal{T} = [0, \tau)$, where τ is fixed. The unitary evolution of the system is determined by the operator

$$U(\tau, 0) = \mathcal{T} e^{-i\int_0^\tau dt H(t)}, \tag{9}$$

where $\mathcal{T}$ is the time-ordering (chronological) operator. This evolution can be approximated by an N-step-like Hamiltonian $H_N(t)$ consisting of partially constant Hamiltonians $H_1, H_2, \ldots, H_N$ in equal time intervals $\mathcal{T}_k = [(k-1)\tau/N, k\tau/N)$ of length $\delta t = \tau/N$ (see Fig. 1). For any time

$t \in \mathcal{T} = \cup_{k=1}^{N} \mathcal{T}_k$ we define

$$H_N(t) = H_k \quad \text{for} \quad t \in \mathcal{T}_k, \tag{10}$$

where

$$H_k = H(t_k), \quad t_k = \frac{k-1}{N}\tau, \quad k = 1, 2, \ldots, N. \tag{11}$$

The corresponding evolution operator takes the form

$$U_N(\tau, 0) = \mathcal{T} e^{-i\int_0^\tau dt H_N(t)} = \prod_{k=1}^{N} e^{-\frac{i}{N} H_k \tau}. \tag{12}$$

The last equality follows from the composition property

$$U(\tau, 0) = U(t_{N+1}, t_N) U(t_N, t_{N-1}) \ldots U(t_3, t_2) \times U(t_2, t_1) \tag{13}$$

and means that the evolution operator with burdensome time ordering reduces to the product of unitary operators generated by time-independent Hamiltonians.

In this approach, the exact starting Hamiltonian $H(t)$ is the limit of the sequence H_N, i.e.,

$$H(t) = \lim_{N\to\infty} H_N(t), \tag{14}$$

and as a consequence,

$$U(\tau, 0) = \lim_{N\to\infty} U_N(\tau, 0). \tag{15}$$

Let us note that any such steplike evolution can be described by an effective Hamiltonian $\tilde{H}_N$ which satisfies the relation

$$U_N(\tau, 0) = \exp[-i\tau \tilde{H}_N]. \tag{16}$$

We should have in mind that τ is fixed. If τ is changed to another value, then the effective Hamiltonian $\tilde{H}_N$ also changes accordingly.

B. Stochastic case

Now, let us consider the probabilistic case for which a system is driven by time-dependent random matrices $H(t)$ in the time interval $\mathcal{T} = [0, \tau)$ with fixed τ. The definition of a stochastic steplike driving of a quantum system, i.e., the MRQ protocol, is analogous to a set of statistically independent Hamiltonians $\mathcal{H}_N = (H_1, H_2, \ldots, H_N)$ which are random matrices of the joint matrix-pdf $\rho(\mathcal{H}_N)$ [cf. Eq. (5)]. Moreover, since all the matrices in $\mathcal{H}_N$ are assumed to be statistically independent, for time-dependent and random driving of the MRQ protocol $H_N(t)$, one can postulate just the time-dependent matrix-pdf $\varrho_t(H)$ defined in a time interval $\mathcal{T}$ in such a way that

$$\Pr(H(t) \in D) = \int_D dH \varrho_t(H). \tag{17}$$

One can represent the distribution $\rho(\mathcal{H}_N)$ in the time domain as

$$\rho(\mathcal{H}_N) = \prod_{k=1}^{N} \varrho_{t_k}(H_k), \quad t_k = \frac{k-1}{N}\tau. \tag{18}$$

The case of a finite number of quenches N, when the evolution is driven by the Hamiltonian $H_N(t)$ in Eq. (10), is hereinafter referred to as a multiple random quench, whereas the limiting case for the Hamiltonian $H(t)$ [Eq. (14)] is called a continuous random quench. This continuous limit inherits the condition that for any $t, s \in \mathcal{T}$ the Hamiltonians $H(t)$ and $H(s)$ are statistically independent random matrices. Note that all protocols for an arbitrary number N (including the limiting CRQ case) can be completely specified by the time-dependent pdf $\varrho_t(H)$.

C. Effective Hamiltonian

The effective Hamiltonian defined in Eq. (16) can explicitly be obtained by using relation (12), from which it follows that

$$e^{-i\tau\tilde{H}_N} = \prod_{k=1}^{N} e^{-\frac{i}{N} H_k \tau}. \tag{19}$$

For a given set $\mathcal{H}_N = (H_1, H_2, \ldots, H_N)$, we can calculate the effective Hamiltonian by use of the Baker-Campbell-Hausdorff formula [19] for the operators $A_k = -iH_k\tau$, namely,

$$\prod_{k=1}^{N} e^{\frac{1}{N} A_k} = e^{Z_N}, \tag{20}$$

where Z_N has the following structure:

$$Z_N = \frac{1}{N}\sum_{i_1=1}^{N} A_{i_1} + \frac{1}{N^2}\sum_{i_1,i_2=1}^{N} \alpha_{i_1,i_2}\big[A_{i_1}, A_{i_2}\big] + \frac{1}{N^3}\sum_{i_1,i_2,i_3=1}^{N} \alpha_{i_1,i_2,i_3}\big[A_{i_1}, \big[A_{i_2}, A_{i_3}\big]\big] + \ldots. \tag{21}$$

The parameters $\alpha_{i_1,\ldots,i_k}$ for $k = 2, 3, \ldots$ can in principle be computed. Some effective algorithms for numerical calculations are presented, e.g., in Refs. [20,21]. However, the explicit form of the higher order terms is not straightforward since they involve more general nested commutators, like the commutators $[[A, B], [C, D]]$. They are not present in Dynkin's expansion [22] for two exponentials, as the commutators in Dynkin's form are "segregated to the right," but nevertheless, the expansion has the structure of Lie polynomials; i.e., it consists of commutators multiplied by numbers, which is crucial for the derivation of some of the results presented in this paper.

Henceforth we use an equivalent form of Eq. (21) given by the expansion of the commutators

$$Z_N = \frac{1}{N}\sum_{i_1=1}^{N} A_{i_1} + \frac{1}{N^2}\sum_{i_1,i_2=1}^{N} \beta_{i_1,i_2} A_{i_1} A_{i_2} + \frac{1}{N^3}\sum_{i_1,i_2,i_3=1}^{N} \beta_{i_1,i_2,i_3} A_{i_1} A_{i_2} A_{i_3} + \ldots \tag{22}$$

with a new set of coefficients $\beta_{i_1,\ldots,i_n}$ which can be expressed by the α coefficients in (21).

From the above relations (19)–(22) it follows that the effective Hamiltonian $\tilde{H}_N$ can be represented by a series of polynomials $P_n(\mathcal{H}_N)$ of nth degree of noncommuting matrix

variables, namely,

$$\tilde{H}_N = \frac{i}{\tau}\sum_{n=1}^{\infty} P_n(\mathcal{H}_N), \tag{23}$$

where according to Eq. (22) one gets

$$P_1(\mathcal{H}_N) = \frac{-i\tau}{N}\sum_{i_1=1}^{N} H_{i_1}, \tag{24}$$

$$P_2(\mathcal{H}_N) = \frac{(-i\tau)^2}{N^2}\sum_{i_1,i_2=1}^{N} \beta_{i_1,i_2} H_{i_1} H_{i_2}, \tag{25}$$

$$P_3(\mathcal{H}_N) = \frac{(-i\tau)^3}{N^3}\sum_{i_1,i_2,i_3=1}^{N} \beta_{i_1,i_2,i_3} H_{i_1} H_{i_2} H_{i_3}, \tag{26}$$

and so on. Although an effective Hamiltonian $\tilde{H}_N$ obeying (16) always exists, representation (23) is valid locally in some convergence domain of the series. There are various quantifiers estimating the convergence domain [21,23,24]. However, the generalized case for N exponentials requires a separate treatment (see Appendix A). It is crucial for our further reasoning to represent the mean value of the effective Hamiltonian as

$$\langle \tilde{H}_N \rangle = \frac{i}{\tau}\sum_{n=1}^{\infty} \langle P_n(\mathcal{H}_N)\rangle \tag{27}$$

and the variance matrix as the series

$$\mathrm{Var}(\tilde{H}_N) = \sum_{n,m=1}^{\infty} S_{n,m}(\mathcal{H}_N), \tag{28}$$

where

$$[S_{n,m}(\mathcal{H}_N)]_{\alpha\beta} = \langle [P_n(\mathcal{H}_N)]_{\alpha\beta}[P_m(\mathcal{H}_N)]^*_{\alpha\beta}\rangle - \langle [P_n(\mathcal{H}_N)]_{\alpha\beta}\rangle \langle [P_m(\mathcal{H}_N)]_{\alpha\beta}\rangle^*. \tag{29}$$

To maintain mathematical rigor and to ensure the existence of these averages one can simply assume that the ensemble is contained in the convergence domain.

D. Commutation in the statistical sense

At the end of this introductory part we define a special condition required in the following proofs that we call commutation in the statistical sense. To this aim, let us note that the mean value of the Hamiltonian $H(t)$ for the CRQ protocol can be expressed as an ensemble average over the distribution $\varrho_t(H)$, namely,

$$\langle H(t)\rangle = \int_{\mathcal{P}} dH\, H \varrho_t(H). \tag{30}$$

We say that two observables O_1 and O_2 commute in the statistical sense if their mean values with respect to the matrix ensemble commute, i.e., $[\langle O_1\rangle, \langle O_2\rangle] = 0$. In our particular case, we say that the whole MRQ protocol, defined solely by the distribution $\varrho_t(H)$, commutes in the statistical sense if

$$[\langle H(t)\rangle, \langle H(s)\rangle] = 0 \tag{31}$$

holds true for any $t, s \in \mathcal{T}$. Note that the above condition also implies that $[\langle H_N(t)\rangle, \langle H_N(s)\rangle] = 0$ for an arbitrary number of quenches N. We stress that commutation in the statistical sense is a weaker condition than standard commutation. In particular, it means that the first moments at different points in time can be simultaneously diagonalized.

IV. SELF-AVERAGING LIMIT

In this section we present two propositions implying explicit conditions for self–averaging, i.e., when a deterministic description can effectively approximate an essentially random system. In the following we use the abbreviation

$$\beta(n) = \max_{i_1,i_2,\ldots,i_n} \left|\beta_{i_1\ldots i_n}\right| \tag{32}$$

and the dimensionless quantity

$$K(n) = \max_{t\in\mathcal{T}} \int_{\mathcal{P}} dH\, \tau^n \|H\|^n \varrho_t(H). \tag{33}$$

In order to simplify the notation we also put $K(n,m) \equiv K(n+m) + K(n)K(m)$.

Now we can state our main result. Let $\mathcal{H}_N$ be a set of random matrices representing a stochastically controlled quantum system via the MRQ protocol. The mean values of the polynomials $P_n(\mathcal{H}_N)$ and $S_{n,m}(\mathcal{H}_N)$, which constitute the expansion of the mean effective Hamiltonian $\langle \tilde{H}_N\rangle$ in Eq. (27) and $\mathrm{Var}(\tilde{H}_N)$ in Eq. (28), respectively, satisfy the following conditions:

Theorem 1. For the MRQ protocol in the time interval $\mathcal{T} = [0,\tau)$ with pdf $\varrho_t(H)$,

$$\|S_{n,m}(\mathcal{H}_N)\| \leqslant R_{n+m}(N)\beta(n)\beta(m)K(n,m) \tag{34}$$

for any $n+m < N$. Moreover, if MRQ commutes in the statistical sense [Eq. (31)], then for $N > n > 1$,

$$\|\langle P_n(\mathcal{H}_N)\rangle\| \leqslant R_n(N)\beta(n)K(n), \tag{35}$$

where

$$R_n(N) = 1 - \frac{N!}{N^n(N-n)!} = O\Big(\frac{1}{N}\Big). \tag{36}$$

Theorem 2. For the MRQ protocol in the time interval $\mathcal{T} = [0,\tau)$ with time-independent distribution $\varrho(H) \equiv \varrho_t(H) = \varrho_s(H)$ for any $t, s \in \mathcal{T}$, the equation

$$\langle P_{2n}(\mathcal{H}_N)\rangle = S_{2n,2m+1}(\mathcal{H}_N) = S_{2n+1,2m}(\mathcal{H}_N) = 0 \tag{37}$$

holds true for any $n, m \in \mathbb{N}$. In addition, if $\varrho(-H) = \varrho(H)$, then $\langle P_n(\mathcal{H}_N)\rangle = 0$.

Note that a rough estimation of $\beta(n)$ shows that $\beta(n) < 1$ and that it is a decreasing function of n. Further, if one additionally assumes the convergence condition to be satisfied, $K(n)$ is an exponentially decreasing function of n. Hence, one concludes that for Hamiltonians and time scales satisfying the convergence, the expected value and variance of the effective Hamiltonian can be approximated by a finite number of terms in Eq. (23) which, for large N, decrease as $O(1/N)$.

A. Variance of the unitary time evolution

For MRQ protocols satisfying the convergence condition (see Appendix A) the variance matrix satisfies

$$\|\mathrm{Var}(\tilde{H}_N)\| \leqslant \sum_{n,m=1}^{\infty} \|S_{n,m}(\mathcal{H}_N)\| = O\left(\frac{1}{N}\right). \tag{38}$$

This condition is sufficient to show not only that the variance of the time-evolution unitary matrix decreases as $O(1/N)$ but also that an arbitrary product of these matrices decreases as $O(1/N)$ (see Appendixes D and F). In other words, a vanishing variance of the local generator (i.e., the effective Hamiltonian) implies the vanishing of the variance of the unitary matrix in a larger domain. Thus, for an arbitrary distribution $\rho_t(H)$ supported on a bounded interval

$$S_N := \|\mathrm{Var}[U_N(\tau,0)]\| = O\left(\frac{1}{N}\right). \tag{39}$$

One infers that the magnitude of the variance matrix for a convergent series, (28), becomes arbitrarily small with an increasing number N of quenches, i.e., for the steplike MRQ protocol one obtains self-averaging of the time evolution of the quantum system. In particular, in the limiting case of control given by the CRQ protocol one obtains

$$\lim_{N\to\infty} S_N = 0 \tag{40}$$

and this implies that U_N (or correspondingly $\tilde{H}_N$) approaches a degenerate random variable, i.e., it almost surely converges to its mean value,

$$\lim_{N\to\infty} Te^{-i\int_0^\tau dt H_N(t)} = \lim_{N\to\infty} \left\langle Te^{-i\int_0^\tau dt H_N(t)} \right\rangle. \tag{41}$$

This result is of interest for potential experimental applications. For general MRQ protocols one expects that the driving of a quantum system given by steplike independent random changes of the Hamiltonian becomes more regular in the limiting control of the CRQ protocol, which can serve as an effective classification scheme of different stochastic time evolutions (convergent to the same self-averaged one).

Note that the commonly defined self-averaging condition, (1), involves scaling of the variance by the square of the mean value. *Per analogiam*, we can scale quantity S_N by the quantity $\| \langle U_N(\tau,0)\rangle \|^2$, however, any unitary matrix is constant in the Frobenius norm, (8), and equal to dimension M of the Hilbert space [see Eq. (8)], thus

$$\frac{\|\mathrm{Var}[U_N(\tau,0)]\|}{\| \langle U_N(\tau,0)\rangle \|^2} = \frac{1}{M} S_N = O\left(\frac{1}{N}\right). \tag{42}$$

Henceforth we just use the quantity S_N.

B. Average of the unitary time evolution

Let us now examine the mean value of the effective Hamiltonian. From the second part of Theorem 1, if the additional assumption of commutation in the statistical sense holds true [cf. Eq. (31)], a growing number of quenches N results in a decreasing absolute value of the noncommutative part of series (23). In particular, this implies that

$$\| \langle \tilde{H}_N\rangle - \frac{1}{\tau}\int_0^\tau dt\, \langle H_N(t)\rangle \| = O\left(\frac{1}{N}\right). \tag{43}$$

This condition is sufficient to derive an analogous relation in terms of the unitary operator,

$$D_N := \left\| \left\langle Te^{-i\int_0^\tau dt H_N(t)} \right\rangle - Te^{-i\int_0^\tau dt \langle H_N(t)\rangle} \right\| = O\left(\frac{1}{N}\right), \tag{44}$$

which is valid for an arbitrary distribution $\rho_t(H)$ supported on the bounded interval (see Appendixes E and F). In the limit it gives

$$\lim_{N\to\infty} \left\langle Te^{-i\int_0^\tau dt H_N(t)} \right\rangle = Te^{-i\int_0^\tau dt \langle H(t)\rangle}. \tag{45}$$

Note that in fact, due to condition (31), the time ordering can be dropped here, however, we left it since in the next section we numerically compute the quantity D_N in a more general case.

Surprisingly, the numerical simulation performed for qubits and presented in the next section confirms the validity of formula (45) also in the noncommuting case. This observation, although very particular, suggests the conjecture that Eq. (45) can be valid generally and thus can be successfully applied in practice as an extremely useful tool simplifying very complicated calculations of averaged unitary evolutions.

A special case of Hamiltonians commuting in the statistical sense is exemplified by protocols with time-independent distributions such that $\varrho_t(H) = \varrho_s(H) \equiv \varrho(H)$. In this case the set of $\mathcal{H}_N$ consists of independent and identically distributed random matrices and we refer to this protocol as the IID protocol. Based on Theorem 2 we conclude that only odd terms contribute to the series,

$$\langle \tilde{H}_N\rangle = \frac{i}{\tau}\sum_{n=0}^{\infty} \langle P_{2n+1}(\mathcal{H}_N)\rangle. \tag{46}$$

Moreover, the second part of Theorem 2 also implies that for an even pdf, $\langle \tilde{H}_N\rangle = 0$, or equivalently $\langle U_N(\tau,0)\rangle = \mathbb{1}$, for an arbitrary number of quenches N. Also, one half of the terms in series (23) vanish and the variance matrix, Eq. (28), reduces to the series

$$\mathrm{Var}(\tilde{H}_N) = \sum_{n,m=0}^{\infty} \langle S_{2n,2m}(\mathcal{H}_N) + S_{2n+1,2m+1}(\mathcal{H}_N)\rangle. \tag{47}$$

For this special case the instantaneous first moment $\langle H(t)\rangle$ is time independent and equal to the effective Hamiltonian in the CRQ limit, i.e.,

$$\langle H(t)\rangle = \lim_{N\to\infty} \tilde{H}_N = \int_{\mathcal{P}} dH\, H\varrho(H). \tag{48}$$

V. NUMERICAL TREATMENT

In this section we numerically analyze an MRQ protocol applied to a two-dimensional Hilbert space which describes quantum two-level systems. Despite their simplicity, two-level systems play a crucial role in many branches of theoretical and applied physics. The celebrated nuclear magnetic resonance (NMR) is probably the most spectacular example which is one of the primary stages for dynamical decoupling and averaging schemes [25,26]. Our work, at least partially, goes beyond those studies since here we apply averaging to stochastically driven two-level systems. These can mimic the realistic but

randomly disturbed NMR systems and hence can be of potential applicability not only in theoretical studies of quantum random dynamics but also in magnetic-based imaging ranging from solid state physics to quantum chemistry to medical physics. Moreover, two-level systems, qubits per se, are the basic building blocks for encoding quantum information. Unfortunately, decoherence and uncontrollable fluctuations (both deterministic and random) seem to be one of several obstructions to the effective implementation of the power of quantum information processing and quantum computing. Stochastic averaging is one potential candidate for controlling and correcting errors of a certain type.

For the random evolution of a qubit we represent the time-dependent Hamiltonian in the form

$$H(t) = \tfrac{1}{2}\vec{\alpha}(t)\cdot\vec{\sigma}, \tag{49}$$

where $\vec{\sigma}$ is a vector of Pauli matrices and $\vec{\alpha}(t)$ is a vector of independent random components distributed according to the normal distribution with mean values $\mu_i(t)$ (for $i = 1, 2, 3$) and the same variance s^2 for all components.

We consider three protocols: (i) the time-independent IID protocol, (ii) the time-dependent commuting case, and (iii) the time-dependent noncommuting case. For these three cases, we calculate S_N and D_N with respect to a number of quenches N. For the IID protocol

$$\vec{\mu}_I(t) = \vec{\mu} \tag{50}$$

with magnitude $|\vec{\mu}| = \mu$. Note that if $\mu = 0$ or if the vector $\vec{\mu}$ has only one nonvanishing component, we obtain the Gaussian unitary ensemble [18] for the qubit space. For the time-dependent commuting case we take the single harmonic

$$\vec{\mu}_C(t) = \mu(\sin(\omega t), 0, 0) \tag{51}$$

and for the noncommuting case we assume that

$$\vec{\mu}_N(t) = \mu(\sin(\omega t), \cos(\omega t), 0). \tag{52}$$

In order to calculate S_N and D_N numerically we have performed Monte Carlo simulation: we generate a set of three independent random numbers, distributed according to the corresponding normal distributions. Next, from this set we construct a random matrix representing the Hamiltonian H_k given by Eqs. (11) and (49). Then we calculate its matrix exponential $\exp(-i H_k \tau/N)$ using the scaling-and-squaring method combined with Padé approximation [27] and repeat the same procedure for $k = 1, \ldots, N$. Consequently, we construct the unitary operator $U_N(\tau, 0)$ [Eq. (12)] as a product of corresponding matrix exponentials, and finally, we calculate the statistics of the unitary operator given by the mean value $\langle U_N(\tau, 0)\rangle$ and the variance $\mathrm{Var}[U_N(\tau, 0)]$ from a sample of 10^6 realizations, which gives us the values of S_N and D_N.

Results presented in Figs. 2–4 reveal that the quantities S_N and D_N obey power-law behavior $1/N$ for sufficiently large values of N. Note that in Eqs. (39) and (44) we state only that they behave at least as $O(1/N)$. Surprisingly, the same behavior is also observed for the noncommuting case. According to numerical simulations (Fig. 4), for this particular driving of the two-level system it is shown that the quantity D_N vanishes as 1 over the number of quenches. This result suggests that Eq. (44) could be valid in the more general case. However, this subject needs further studies for higher dimensional systems and other distributions of the MRQ protocol.

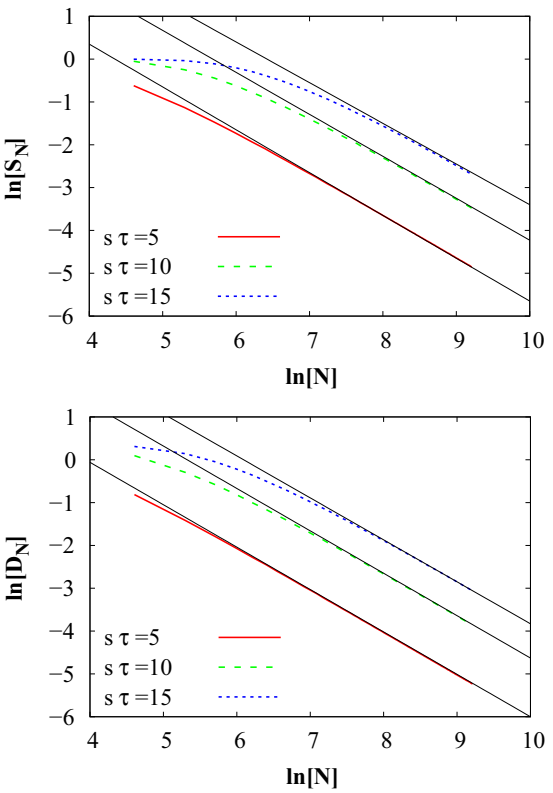

FIG. 2. Graphs of S_N and D_N with respect to the number of quenches N presented in the log-log plot for the IID protocol [Eq. (50)] for different values of the dimensionless parameter $s\tau$ [s is the variance of the random components of the two-level Hamiltonian in Eq. (49)] with the constant value $\mu\tau = 1$ (μ is the mean value of the random components). Graphs apparently show a power law for large values of N. From linear regression we obtain $S_N \propto N^{-1.00}$, $N^{-0.98}$, $N^{-0.96}$ and $D_N \propto N^{-0.99}$, $N^{-0.99}$, $N^{-0.98}$ with respect to an increasing value of the parameter $s\tau = 5, 10, 15$.

VI. SUMMARY

Time-dependent and stochastically manipulated quantum systems are very important for modern applications since they effectively mimic external control applied to gain the desired dynamic properties. In particular, stochastic description becomes unavoidable either if there is a certain degree of uncertainty or randomness affecting the control strategy or if there is disorder essentially present in the system under consideration. Examples are random decoupling schemes for quantum dynamical control and error suppression [28]. In many cases, a proper description requires random operators [9,29], resulting in models which are still very elegant and effective but not easy to analyze. This is why every result simplifying the analysis or serving as a useful tool not only is "theoretically attractive" but also is of great practical importance. Self-averaging is one among such concepts developed to investigate a certain class of stochastically modified quantum systems which still remain challenging not only for mathematical physicists (cf. chap. 3

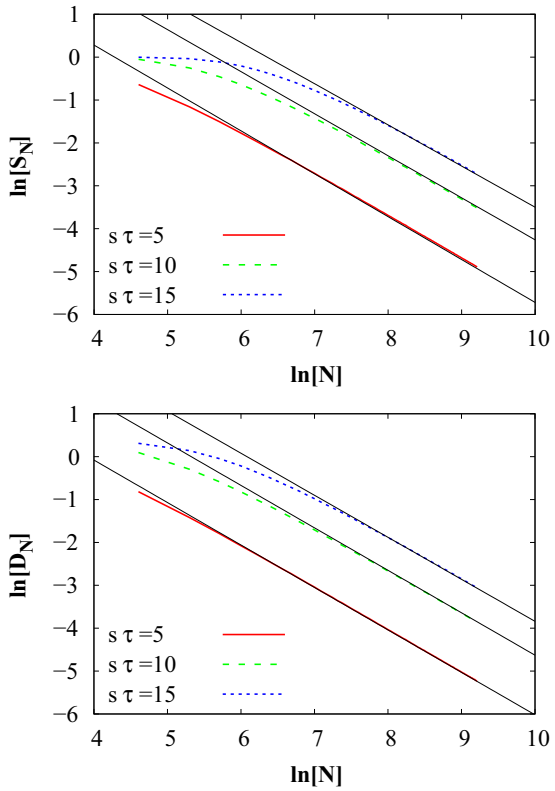

FIG. 3. Graphs of S_N and D_N with respect to the number of quenches N presented in a log-log plot for the statistically commuting case [Eq. (51)] and for selected values of the dimensionless parameter $s\tau$ with the constant value $\mu\tau = 1$ and $\omega\tau = \pi/4$. Graphs apparently show a power law for large values of N. From linear regression we obtain $S_N \propto N^{-1.00}$, $N^{-0.98}$, $N^{-0.95}$ and $D_N \propto N^{-0.99}$, $N^{-0.99}$, $N^{-0.98}$ with respect to an increasing value of the parameter $s\tau = 5, 10, 15$.

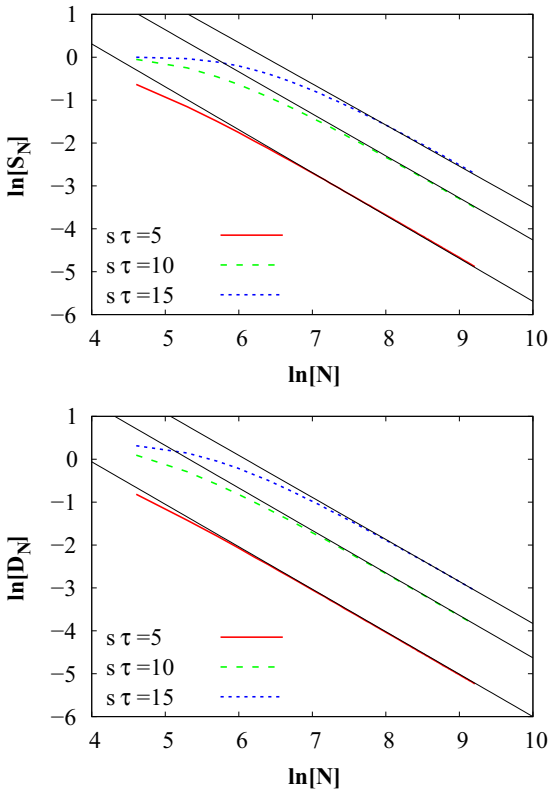

FIG. 4. Graphs of S_N and D_N with respect to the number of quenches N presented in a log-log plot for the statistically noncommuting case [Eq. (52)] and for various values of the dimensionless parameter $s\tau$ with the constant value $\mu\tau = 1$ and $\omega\tau = \pi/4$. Graphs apparently show a power law for large values of N. From linear regression we obtain $S_N \propto N^{-1.00}$, $N^{-0.98}$, $N^{-0.96}$ and $D_N \propto N^{-0.99}$, $N^{-0.99}$, $N^{-0.98}$ with respect to an increasing value of the parameter $s\tau = 5, 10, 15$.

in Ref. [29]) but also for these who want to effectively and credibly simulate quantum dynamics of nontrivial systems.

In our work we formulated and studied quantum systems undergoing multiple random quench protocols. A more abstract approach to the problems studied in this paper can be found in Ref. [30]. General results put in the framework of convolution semigroups are presented in Ref. [31]. Here, we have investigated the statistical properties of an effective unitary dynamics with an emphasis on the self-averaging property. We recognized that for a broad class of randomly driven systems satisfying relatively nonrestrictive conditions, the self–averaging phenomenon occurs and can be utilized for considerable simplification of the treatment of such systems. Our findings, derived via mathematically rigorous reasoning, are supported by numerical calculations. Such a test not only allows us to verify theoretical and more formal predictions but also helps us to formulate a conjecture that is applicable beyond mathematically proven cases. Our results form a bridge between formal but sometimes highly restricted mathematical treatment and more informal purely numerical modeling, which is applicable to a broad, but so far not precisely defined, class of random systems. We hope that our modest contribution—besides enhancing our understanding of quantum stochastic dynamics—can also serve as a training ground suitable for testing numerical tools: even if one is interested in dynamic properties of systems which do not fulfill the requirements of the propositions stated in this paper, such that numerical treatment becomes unavoidable, one can still verify the credibility of numerics applied to a class of systems described here.

ACKNOWLEDGMENT

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APPENDIX A: ESTIMATION OF THE CONVERGENCE DOMAIN

Let us express the polynomials $P_n(\mathcal{H}_N)$ in Eq. (23) in the associative representation

$$P_n(\mathcal{H}_N) = \frac{(-i\tau)^n}{N^n} \sum_{i_1,\ldots,i_n}^{N} \beta_{i_1,\ldots,i_n} H_{i_1} H_{i_2} \ldots H_{i_n}. \qquad \text{(A1)}$$

From this relation one can estimate

$$\sum_n \|P_n(\mathcal{H}_N)\| \leqslant \sum_n \frac{\tau^n}{N^n} \sum_{i_1,\ldots,i_n}^{N} \left|\beta_{i_1,\ldots,i_n}\right| \left\| H_{i_1} H_{i_2} \ldots H_{i_n} \right\| \leqslant \sum_n \frac{\tau^n}{N^n} \sum_{i_1,\ldots,i_n}^{N} \left\| H_{i_1} \right\| \left\| H_{i_2} \right\| \ldots \left\| H_{i_n} \right\|. \tag{A2}$$

Under the assumption that

$$H_i \in \mathcal{C} = \{H : \|H\| < 1/\tau\} \tag{A3}$$

for $i = 1, 2, \ldots, N$, we infer that there exists a number $M < 1$ such that

$$\sum_n \|P_n(\mathcal{H}_N)\| \leqslant \sum_n \frac{1}{N^n} \sum_{i_1,\ldots,i_n}^{N} M^n \leqslant \sum_n M^n < \infty, \tag{A4}$$

and this proves the absolute convergence of series (23).

APPENDIX B: PROOF OF THEOREM 1

1. First statistical moment

From the Lie representation of the polynomials $P_n(\mathcal{H}_N)$ we conclude that any of them can be represented by a linear combination of the terms

$$H_{i_1} H_{i_2} \ldots H_{i_{p-1}} \left[H_{i_p}, H_{i_{p+1}}\right] H_{i_{p+2}} \ldots H_{i_n}, \tag{B1}$$

where $n > p > 1$. Thus, for a subset of different indices $i_1 \neq i_2 \neq \cdots \neq i_n$ one obtains

$$\langle H_{i_1} H_{i_2} \ldots H_{i_{p-1}} [H_{i_p}, H_{i_{p+1}}] H_{i_{p+2}} \ldots H_{i_n} \rangle = \langle H_{i_1} H_{i_2} \ldots H_{i_{p-1}} \rangle \langle [H_{i_p}, H_{i_{p+1}}] \rangle \langle H_{i_{p+2}} \ldots H_{i_n} \rangle, \tag{B2}$$

where the assumption of statistical independence of the matrices H_i is utilized. Next, under the assumption of commutation of the first moments we get

$$\langle [H_j, H_k] \rangle = \langle H_j \rangle \langle H_k \rangle - \langle H_k \rangle \langle H_j \rangle = [\langle H_j \rangle, \langle H_k \rangle] = 0. \tag{B3}$$

Finally, we show that

$$\sum_{i_1 \neq \cdots \neq i_n}^{N} \beta_{i_1,\ldots,i_n} \left\langle H_{i_1} H_{i_2} \ldots H_{i_n} \right\rangle = 0. \tag{B4}$$

The number of vanishing terms in this sum, if $N > n > 1$, is equal to the number of partial permutations of length n from the set of N elements, i.e., $N!/(N-n)!$. Consequently, the number of all nonzero terms in the sum (A1) is

$$G_{n-1}(N) = N^n - \frac{N!}{(N-n)!}, \tag{B5}$$

where $G_k(N)$ denotes the kth-degree polynomial of the variable N. Further, we estimate that

$$\left\| \left\langle H_{i_1} H_{i_2} \ldots H_{i_n} \right\rangle \right\| \leqslant \left\langle \left\| H_{i_1} H_{i_2} \ldots H_{i_n} \right\| \right\rangle \leqslant \left\langle \left\| H_{i_1} \right\| \left\| H_{i_2} \right\| \ldots \left\| H_{i_n} \right\| \right\rangle \leqslant K(n)/\tau^n, \tag{B6}$$

and as a consequence, only if $N > n > 1$ do we get

$$\| \langle P_n(\mathcal{H}_N) \rangle \| \leqslant \frac{\tau^n}{N^n} \left\| \sum_{i_1,\ldots,i_n}^{N} \beta_{i_1,\ldots,i_n} \left\langle H_{i_1} H_{i_2} \ldots H_{i_n} \right\rangle \right\| \leqslant \frac{G_{n-1}(N)}{N^n} \beta(n) K(n) = R_n(N) \beta(n) K(n). \tag{B7}$$

2. Variance

For the elements of the variance-matrix series $\mathrm{Var}(\tilde{H}_N)$ we have

$$[S_{n,m}(\mathcal{H}_N)]_{\alpha\beta} = \frac{\tau^{n+m}}{N^{n+m}} \sum_{i_1,\ldots,i_n}^{N} \sum_{j_1,\ldots,j_m}^{N} \beta_{i_1\ldots i_n} \beta_{j_1\ldots j_m} \left[S_{i_1,\ldots,i_n,j_1,\ldots,j_m}\right]_{\alpha\beta}, \tag{B8}$$

where

$$\left[S_{i_1,\ldots,i_n,j_1,\ldots,j_m}\right]_{\alpha\beta} = \left\langle \left[H_{i_1} \ldots H_{i_n}\right]_{\alpha\beta} \left[H_{j_1} \ldots H_{j_m}\right]^*_{\alpha\beta} \right\rangle - \left\langle \left[H_{i_1} \ldots H_{i_n}\right]_{\alpha\beta} \right\rangle \left\langle \left[H_{j_1} \ldots H_{j_m}\right]^*_{\alpha\beta} \right\rangle. \tag{B9}$$

In analogy to previous considerations for the subset of indices $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \cdots \neq i_n \neq j_n$, under the assumption of statistical independence, we have

$$\sum_{i_1 \neq j_1 \neq \cdots \neq i_n \neq j_n}^{N} \beta_{i_1\ldots i_n} \beta_{j_1\ldots j_m} \times \left[S_{i_1,\ldots,i_n,j_1,\ldots,j_m}\right]_{\alpha\beta} = 0 \tag{B10}$$

for any $n + m < N$. Thus, the number of all nonzero terms in the sum (B8) at least is

$$G_{n+m-1}(N) = N^{n+m} - \frac{N!}{(N-n-m)!}. \tag{B11}$$

Similarly to the earlier reasoning we estimate that

$$\left\| S_{i_1,\ldots,i_n,j_1,\ldots,j_m} \right\| \leqslant \left\langle \left\| H_{i_1} H_{i_2} \ldots H_{i_n} \right\| \left\| H_{j_1} H_{j_2} \ldots H_{j_m} \right\| \right\rangle + \left\langle \left\| H_{i_1} H_{i_2} \ldots H_{i_n} \right\| \right\rangle \left\langle \left\| H_{j_1} H_{j_2} \ldots H_{j_m} \right\| \right\rangle \leqslant K(n,m)/\tau^{n+m}, \tag{B12}$$

where we use the fact that the matrix defined by the elements $[Z]_{\alpha\beta} = [X]_{\alpha\beta}[Y]_{\alpha\beta}$ satisfies the relation $\|Z\| \leqslant \|X\|\|Y\|$.

Finally, we obtain, in analogy to prior reasoning,

$$\| \langle S_{n,m} \rangle \| \leqslant \frac{G_{n+m-1}(N)}{N^{n+m}} \beta(n)\beta(m) K(n,m) = R_{n+m}(N) \beta(n)\beta(m) K(n,m). \tag{B13}$$

APPENDIX C: PROOF OF THEOREM 2

We define the set $\bar{\mathcal{H}}_N = (H_N, H_{N-1}, \ldots, H_1)$, which is the reverse protocol of $\mathcal{H}_N$. From the identity

$$e^{-i\tilde{H}_N \tau} = e^{-\frac{i}{N} H_N \tau} \ldots e^{-\frac{i}{N} H_2 \tau} e^{-\frac{i}{N} H_1 \tau}, \tag{C1}$$

we can rearrange the order and get

$$e^{i\tilde{H}_N \tau} = e^{\frac{i}{N} H_1 \tau} e^{\frac{i}{N} H_2 \tau} \ldots e^{\frac{i}{N} H_N \tau}, \tag{C2}$$

which implies that for any polynomial $P_n(\mathcal{H}_N)$ of nth degree the relation

$$P_n(-\mathcal{H}) = -P_n(\bar{\mathcal{H}}_N) \tag{C3}$$

is satisfied. What is more, for any $n \in \mathbb{N}$,

$$\begin{aligned} P_{2n}(-\mathcal{H}_N) &= P_{2n}(\mathcal{H}_N), \\ P_{2n+1}(-\mathcal{H}_N) &= -P_{2n+1}(\mathcal{H}_N). \end{aligned} \tag{C4}$$

The joint pdf for i.i.d. matrices $\mathcal{H}_N$ has the form

$$\begin{aligned} \rho(\mathcal{H}_N) &= \varrho(H_1)\varrho(H_2)\ldots\varrho(H_N) \\ &= \varrho(H_N)\ldots\varrho(H_2)\varrho(H_1) = \rho(\bar{\mathcal{H}}_N), \end{aligned} \tag{C5}$$

and this leads to the relation

$$\langle P_n(\mathcal{H}_N)\rangle = \langle P_n(\bar{\mathcal{H}}_N)\rangle . \tag{C6}$$

Finally, taking into consideration Eqs. (C3), (C4), and (C5), we have

$$\langle P_{2n}(\mathcal{H}_N)\rangle = -\langle P_{2n}(\bar{\mathcal{H}}_N)\rangle = -\langle P_{2n}(\mathcal{H}_N)\rangle = 0 \tag{C7}$$

for any $n \in \mathbb{N}$. Similarly to before, we have the relation

$$S_{n,m}(-\mathcal{H}_N) = S_{n,m}(\bar{\mathcal{H}}_N). \tag{C8}$$

Thus, for any $n, m \in \mathbb{N}$ with pdf Eq. (C5), in analogy one can show that

$$S_{2n,2m+1}(\mathcal{H}_N) = S_{2n+1,2m}(\mathcal{H}_N) = 0, \tag{C9}$$

and this proves the first part of Theorem 1.

The proof of the second part is straightforward if one notes that for even pdf $\varrho(H)$ we have

$$\langle P_n(-\mathcal{H})\rangle = \langle P_n(\mathcal{H}_N)\rangle . \tag{C10}$$

APPENDIX D: VARIANCE OF THE UNITARY TIME EVOLUTION

Let us assume that $x_1, x_2, \ldots$ are complex random variables and each of them behaves as

$$\mathrm{Var}(x_k) = O\left(\frac{1}{N}\right). \tag{D1}$$

Then the sum of them

$$\mathrm{Var}\left(\sum_i x_i\right) = \sum_{i,j}\mathrm{Cov}(x_i, x_j) \leqslant \sum_{i,j}\sqrt{\mathrm{Var}(x_i)\mathrm{Var}(x_j)} \tag{D2}$$

behaves as

$$\mathrm{Var}\left(\sum_i x_i\right) = O\left(\frac{1}{N}\right). \tag{D3}$$

Next, we would like to estimate the variance of the product. To this aim, let us define the centered random variable,

$$\delta x_i = x_i - \langle x_i\rangle , \tag{D4}$$

where $\mathrm{Var}(\delta x_i) = \mathrm{Var}(x_i)$ and $\langle \delta x_i\rangle = 0$. Then the product can be expanded as

$$\prod_i x_i = \prod_i \langle x_i\rangle + \sum_k \delta x_k \prod_{i\neq k}\langle x_i\rangle + \ldots . \tag{D5}$$

Thus up to the leading orders of N we have

$$\mathrm{Var}\left(\prod_i x_i\right) = \sum_{k,m}|\langle x_k\rangle||\langle x_m\rangle| \prod_{i\neq k,m}|\langle x_i\rangle|^2\mathrm{Cov}(x_k, x_m), + \ldots \tag{D6}$$

which implies that

$$\mathrm{Var}\left(\prod_i x_i\right) = O\left(\frac{1}{N}\right). \tag{D7}$$

For the self-averaging effective Hamiltonian we have shown that any of its elements is asymptotically equivalent to functions belonging to $O(1/N)$, and since the unitary matrix can be expressed as the series

$$U_N(\tau, 0) = \sum_{k=1}\frac{(-i\tilde{H}_N\tau)^k}{k!}, \tag{D8}$$

which involves sums and products of the effective Hamiltonian elements, hence we conclude also that

$$\|\mathrm{Var}[U_N(\tau, 0)]\| = O\left(\frac{1}{N}\right). \tag{D9}$$

APPENDIX E: MEAN OF THE UNITARY TIME EVOLUTION

We want to prove that

$$\left\| \left\langle Te^{-i\int_0^\tau dt H_N(t)}\right\rangle - Te^{-i\int_0^\tau dt\langle H_N(t)\rangle}\right\| = O\left(\frac{1}{N}\right) \tag{E1}$$

assuming that

$$\|\langle\tilde{H}_N\rangle - \widehat{H}_N\| = O\left(\frac{1}{N}\right), \tag{E2}$$

where $\widehat{H}_N = \frac{1}{\tau}\int_0^\tau dt\,\langle H_N(t)\rangle$ and

$$[\langle H(t)\rangle , \langle H(s)\rangle] = 0 \tag{E3}$$

for any $t, s \in \mathcal{T}$. First, one can estimate

$$\begin{aligned} &\|\langle Te^{-i\int_0^\tau dt H_N(t)}\rangle - Te^{-i\int_0^\tau dt\langle H_N(t)\rangle}\| \\ &= \|\langle e^{-i\tilde{H}_N\tau} - e^{-i\widehat{H}_N\tau}\rangle\| \leqslant \sum_k\frac{\tau^k}{k!}\left\|\langle\tilde{H}_N^k - \widehat{H}_N^k\rangle\right\|. \end{aligned} \tag{E4}$$

Let us define the matrix

$$\delta H_N = \tilde{H}_N - \widehat{H}_N. \tag{E5}$$

We note that it satisfies

$$\|\langle\delta H_N\rangle\| = \|\langle\tilde{H}_N - \widehat{H}_N\rangle\| = O\left(\frac{1}{N}\right), \tag{E6}$$

and variance

$$\|\mathrm{Var}(\delta H_N)\| = \|\mathrm{Var}(\tilde{H}_N)\| = O\left(\frac{1}{N}\right). \tag{E7}$$

Further,

$$\tilde{H}_N^k - \widehat{H}_N^k = \sum_{m=0}^{k-1}\widehat{H}_N^m\delta H_N\widehat{H}_N^{k-m-1} + \ldots , \tag{E8}$$

where dropped terms involve higher powers of δH_N elements. According to (E6) and (E7), the leading order of the average is then equal to

$$\langle \tilde{H}_N^k - \widehat{H}_N^k \rangle = \sum_{m=0}^{k-1} \widehat{H}_N^m \langle \delta H_N \rangle \widehat{H}_N^{k-m-1} + \dots . \tag{E9}$$

However, due to the submultiplicative condition of the norm we conclude that

$$\left\| \sum_{m=0}^{k-1} \widehat{H}_N^m \langle \delta H_N \rangle \widehat{H}_N^{k-m-1} \right\| \leqslant k \| \langle \delta H_N \rangle \| \| \widehat{H}_N \|^{k-1}, \tag{E10}$$

and this finally implies that

$$\| \langle \tilde{H}_N^k - \widehat{H}_N^k \rangle \| = O\left(\frac{1}{N}\right). \tag{E11}$$

According to (E4), this proves relation (E1).

APPENDIX F: BEYOND THE CONVERGENCE DOMAIN

1. Variance S_N

Let us consider an MRQ evolution

$$U_N(\tau, 0) = \mathcal{T} e^{-i \int_0^\tau dt H_N(t)} = \prod_{k=1}^{N} e^{-\frac{i}{N} H_k \tau}, \tag{F1}$$

where the convergence condition, (A3), is not satisfied. Nevertheless, one can always split the unitary evolution into m products

$$U_N(\tau, 0) = \prod_{k=1}^{m} U_{N/m}(\tau_{k+1}, \tau_k), \tag{F2}$$

where $\tau_k = (k-1)\tau/m$, such that for each term the convergence condition is obeyed. Then if

$$\mathrm{Var}[U_{N/m}(\tau_{k+1}, \tau_k)] = O\left(\frac{1}{N}\right) \tag{F3}$$

according to relations (D3) and (D7), one concludes that also an arbitrary finite product of matrices satisfies

$$\| \mathrm{Var}[U_N(\tau, 0)] \| = \left\| \mathrm{Var}\left[\prod_{k=1}^{m} U_{N/m}(\tau_{k+1}, \tau_k) \right] \right\| = O\left(\frac{1}{N}\right). \tag{F4}$$

2. Distance D_N

Further, let us define a matrix

$$\delta U_{N/m}^{(k)} = U_{N/m}(\tau_{k+1}, \tau_k) - \widehat{U}_{N/m}(\tau_{k+1}, \tau_k), \tag{F5}$$

where

$$\widehat{U}_{N/m}(\tau_{k+1}, \tau_k) = \mathcal{T} e^{-i \int_{\tau_{k+1}}^{\tau_k} dt \langle H_N(t) \rangle}. \tag{F6}$$

If $H(t)$ commute in the statistical sense, then from Eq. (44) we obtain

$$\| \langle \delta U_N^{(k)} \rangle \| = O\left(\frac{1}{N}\right) \tag{F7}$$

and variance

$$\| \mathrm{Var}(\delta U_N^{(k)}) \| = \| \mathrm{Var}[U_{N/m}(\tau_{k+1}, \tau_k)] \| = O\left(\frac{1}{N}\right). \tag{F8}$$

Finally, we can expand the unitary time-evolution matrix up to leading terms of N,

$$\begin{aligned} U_N(\tau, 0) = \widehat{U}_N(\tau, 0) &\\ + \sum_{k=0}^{m-1} \widehat{U}_{kN/m}(\tau, \tau_{m-k+1}) \delta U_{N/m}^{(k)} &\\ \times \widehat{U}_{(m-k-1)N/m}(\tau_{m-k}, 0) + \dots , & \end{aligned} \tag{F9}$$

which, due to (F7), once again gives

$$\| \langle U_N(\tau, 0) \rangle - \widehat{U}_N(\tau, 0) \| = O\left(\frac{1}{N}\right) \tag{F10}$$

and proves Eq. (44).

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