Guiding neutral particles endowed with a magnetic moment by an electromagnetic wave carrying orbital angular momentum: Quantum mechanics

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The quantum-mechanical states of neutral particles endowed with the magnetic moment (such as neutrons, light atoms, or even neutrinos, although the effect will be extremely tiny) in the combination of electromagnetic vortex field together with the constant magnetic field are investigated. It is shown that this system of fields is in principle capable of capturing the particle in the perpendicular direction and guiding it along the propagating wave. The quantum evolution is subject to tunneling processes, which can destroy the delicate trapping mechanism. The calculated probability of such processes shows that basically it should be possible to catch and guide the particle for the time corresponding to about $10^5\omega^{-1}$, where ω is the frequency of the guiding wave. This time can be lengthened by the appropriate adjustments of the external magnetic field. Due to the very small values of magnetic moments of available neutral particles and their relatively large masses, this binding mechanism constitutes presently only a theoretical possibility since it would require extremely strong magnetic fields.

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I. INTRODUCTION

Guiding neutral particles, especially atoms or molecules, by light beams has attracted much attention for the past 20 years. It refers to both experimental [1–5] and theoretical investigations [6–13]. A light beam used for transporting particles may be composed, for instance, of evanescent modes propagating along a hollow fiber or exist in free space and form a kind of optical vortex. The mechanism of confining atoms and molecules, which are neutral objects and do not interact directly with the electric field of the wave, is based on the Stark effect. It leads to the emergence of a certain binding potential in the direction perpendicular to that of the wave propagation and allows for guiding atoms along the light beam. Another arrangement exploits the rotating magnetic field for binding both charged and neutral particles [14–17].

In our previous work [18] we proposed another subtle mechanism which allows one to guide or even trap neutral particles due to the interaction of their magnetic moments with the magnetic field of the vortex. This mechanism does nor refer to the particle internal structure and therefore can in principle be applied equally well to composed objects like atoms (with the restrictions spoken of below) and elementary ones like neutrons (their quark structure is irrelevant here) or, although it may seem ridiculous, possibly also neutrinos, provided they are massive and possess magnetic moments, which presently remains unclear [19–21]. The potentially wide future applicability justifies addressing this issue even if the values of the physical quantities needed to capture the particles in question remain beyond the reach of current technological possibilities.

The binding effect of the vortex field can be reinforced by the presence of the external uniform magnetic field pointing along the wave-propagation direction. The *classical* equations of motion of the particle were solved in the quoted work and the trajectories obtained explicitly showed that this configuration of fields is capable of trapping or guiding particles. The true theory governing processes occurring in the microworld has, however, the quantum nature. Therefore, in the present work we address this problem within quantum mechanics.

The electromagnetic wave guiding neutral particles, as proposed in our previous work, can interact with the objects in question via both electric and magnetic fields. As far as the neutron (or hypothetically neutrino) is concerned, one can forget about the former and our proposal leads to the trapping mechanism due to the rotating magnetic field possibly further enhanced by a kind of resonance between the Larmor frequency and that of the vortex wave, as will be explained in this paper. On the other hand, in the case of composed objects like atoms, the electric field of the wave should also be looked at, since this kind of field influences the atomic energy and consequently is responsible for the trapping mechanism based on the Stark effect. Admittedly, for the case of the vanishing atomic polarizability d_{ij} for certain frequencies (corresponding to the so-called tune-out wavelengths), the ac Stark shift proportional to d_{ij}^2 is suppressed by the presence of roots in d_{ij} [22,23], but still the required field is so strong that the atom can hardly be treated as a bound object. For that reason, at the present stage, the invoking of light atoms has only a model meaning.

The Schrödinger-Pauli equation for a neutral particle endowed with magnetic moment μ has the form

$$i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t)=\left(-\frac{\hbar^2\Delta}{2M}-g\boldsymbol{s}\cdot\boldsymbol{B}\right)\Psi(\boldsymbol{r},t), \tag{1}$$

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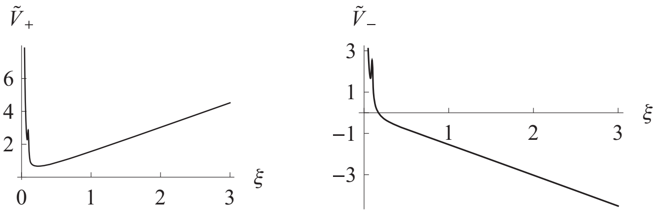

FIG. 1. Behavior of $\tilde{V}_\pm(\xi)$ for the same values of parameters as in Fig. 5 of [18], $\alpha = 3$, $\beta = 0.8$, and $\gamma = 0.01$, as well as $\kappa_z = 0.9$ and $m = 2$.

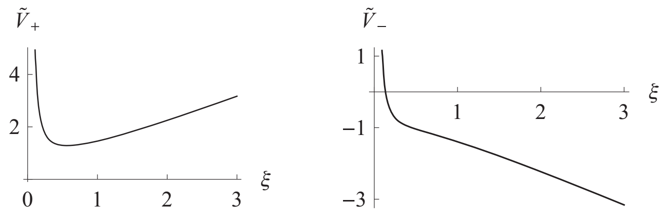

FIG. 2. Behavior of $\tilde{V}_\pm(\xi)$ for the same values of parameters as in the last plot of Fig. 8 of [18], $\alpha = -2$, $\beta = -2$, and $\gamma = -0.02$, as well as $\kappa_z = 0.9$ and $m = 2$.

where g denotes the gyromagnetic ratio (positive or negative, depending on the orientation of the magnetic moment with respect to the spin angular momentum). For spin-$\frac{1}{2}$ particles, such as the neutron, the spin vector $\boldsymbol{s}$ is an operator expressed by Pauli matrices $\boldsymbol{s} = \hbar/2\,\boldsymbol{\sigma}$ and the wave function $\Psi(\boldsymbol{r},t)$ is a two-component one. The same refers to a composed system as the hydrogen atom, for instance (see the Appendix). This Schrödinger equation can be simplified by the separation of time and the variables z and φ (in cylindrical coordinates) and this is done in Sec. II. We are then left with two coupled ordinary differential equations in the radial variable for the upper and lower components of the spinor function.

In Sec. III further simplification is achieved by exploiting the diagonal form of the matrix potential. The obtained form constitutes a convenient starting point for the perturbative calculation. In Sec. IV these results are used to analyze the motion of the particle. This analysis indicates the existence of bound states (in the perpendicular direction). These bound states can become unstable due to the possible tunneling effects characteristic for quantum physics. The possible tunneling channels are considered in Secs. IV C and IV D. Numerical estimations of the tunneling probabilities show that only the process consisting in flipping the spin direction can play an essential role, but the guiding time of order $10^5\omega^{-1}$ can in principle be achieved. This time may be lengthened by appropriately adjusting the value of the external constant magnetic field. A similar effect can also be observed for the classical motion.

The two sets of parameters defined in the captions of Figs. 1 and 2 are used throughout the paper; they are those for which the exemplary stable trajectories in the classical motion were found in [18]. When corresponding to the enormously intense fields, which is a consequence of the very small values of magnetic moments for known neutral particles, they should actually be treated only as some model values, although it is not excluded that some stability regions can also be found for more realistic ones.

In particular, the first set of data (the second set leads to similar values) for the neutron leads to $\omega \approx 1.6 \times 10^{19}$ s^{-1}, which corresponds to x rays. Consequently, for $B_\perp$ one obtains 1.7×10^9 T and $B_z \approx -3.5 \times 10^{11}$ T (the minus referring to its direction) and the resonance is achieved at $B_z \approx -1.7 \times 10^{11}$ T. For the data corresponding to the hydrogen atom, which has a magnetic moment three orders of magnitude larger, the required values of fields would be proportionally reduced by 10^{-3}.

Strangely enough, for the neutrino (assuming $m_\nu \approx 0.05$ eV and $\mu_\nu \approx 10^{-14}\mu_B$) the needed values of physical parameters are not significantly worse. This is owed to its extremely small rest mass. One gets $\omega \approx 8.4 \times 10^8$ s^{-1} (microwaves), $B_\perp \approx 9.6 \times 10^9$ T, $B_z \approx -1.9 \times 10^{11}$ T, and the resonant value at about -9.6×10^{11} T. These values can be still lowered by three orders of magnitude when using the present experimental bounds of $\mu_\nu < 10^{-11}\mu_B$ [19–21,24].

II. SEPARATION OF VARIABLES

Consider the Bessel beam as given in [25] endowed with nonzero orbital angular momentum. It may be labeled with the value of $M - 1$ (this M appears only in this place and should not be confused with the particle mass used throughout the paper). We are particularly interested in the case $M = 2$, dealt with in a series of our previous works [18,25–27], in which case the field bears a vortex topological number equal to 1.

This wave is accompanied by the external uniform magnetic field B_z oriented along the z axis, the proper adjustment of which may help to create a stable trap. The total magnetic field in cylindrical coordinates (ρ,φ,z) is then given by

$$\boldsymbol{B}(\boldsymbol{r},t) = \begin{bmatrix} -2B_\perp[a_+\sin(\zeta_z-\varphi)J_1(k_\perp\rho) + a_-\sin(\zeta_z-3\varphi)J_3(k_\perp\rho)] \\ 2B_\perp[a_+\cos(\zeta_z-\varphi)J_1(k_\perp\rho) - a_-\cos(\zeta_z-3\varphi)J_3(k_\perp\rho)] \\ -4B_\perp\cos(\zeta_z-2\varphi)J_2(k_\perp\rho) + B_z \end{bmatrix}, \tag{2}$$

where J_i denotes the Bessel functions, $k = \sqrt{k_z^2 + k_\perp^2} = \omega/c$ is the wave number, $B_\perp$ measures the strength of the vortex wave, $a_\pm = (\omega/c \pm k_z)/2k_\perp$, and $\zeta_z = \omega t - k_z z$.

In the paraxial approximation, where $k_\perp\rho \ll 1$ and $k_\perp^2 z/k_z \ll 1$, we have

$$k_z \approx k, \quad a_+ \approx 1, \quad a_- \approx 0 \tag{3}$$

and the magnetic field may be written as [25]

$$\boldsymbol{B}(\boldsymbol{r},t)=\begin{bmatrix}B_\perp k(y\cos\zeta - x\sin\zeta)\\ B_\perp k(x\cos\zeta + y\sin\zeta)\\ B_z\end{bmatrix}, \tag{4}$$

where $\zeta=\omega t - kz$. This last form can be now plugged into the Schrödinger equation.

In order to prepare the separation of variables and get rid of the spin rotation we make the substitution exploiting the screw symmetry

$$\Psi(\boldsymbol{r},t)=e^{-i\zeta\sigma_z/2}\tilde{\Psi}(\boldsymbol{r},t), \tag{5}$$

which eliminates the dependence of the Hamiltonian simultaneously on t and z, namely, we obtain

$$i\hbar\frac{\partial}{\partial t}\tilde{\Psi}(\boldsymbol{r},t)=\left[-\frac{\hbar^2\Delta_\perp}{2M}-\frac{\hbar^2(\partial_z+ik\sigma_z/2)^2}{2M}-\frac{\hbar\omega}{2}\sigma_z - \frac{g\hbar B_\perp k}{2}(x\sigma_y+y\sigma_x)-\frac{g\hbar B_z}{2}\sigma_z\right]\tilde{\Psi}(\boldsymbol{r},t), \tag{6}$$

where $\Delta_\perp=\partial_x^2+\partial_y^2$. Following [18] and aimed at simplifying Eq. (6), let us now introduce the dimensionless parameters

$$\alpha=\frac{gB_\perp}{\omega}\sqrt{\frac{Mc^2}{\hbar\omega}},\quad \beta=\left(1+\frac{gB_z}{\omega}\right)\sqrt{\frac{Mc^2}{\hbar\omega}}, \tag{7a}$$

$$\gamma=\frac{gB_\perp}{\omega},\quad \kappa_z=\frac{k_z}{k},\quad \mathcal{E}_\perp=\frac{E_\perp}{\hbar\omega}\sqrt{\frac{Mc^2}{\hbar\omega}}, \tag{7b}$$

together with dimensionless time, space coordinates, and momenta

$$\boldsymbol{\xi}=k\boldsymbol{r},\quad \xi=k\rho, \tag{8a}$$

$$\tau=\omega t\sqrt{\frac{\hbar\omega}{Mc^2}},\quad \boldsymbol{\eta}=\frac{\boldsymbol{p}}{\sqrt{\hbar\omega M}}, \tag{8b}$$

where $\rho=\sqrt{\boldsymbol{r}^2}$ and $\xi=\sqrt{\boldsymbol{\xi}^2}$. The role of these parameters is as follows: α (and also γ at a different scale) is the relative strength (i.e., energy) of the magnetic coupling and the energy of light quanta, β can be treated as deviation from the resonance between the Larmor frequency of the rotating magnetic moment and that of the vortex magnetic field (discussed in more detail at the end of Sec. IV D), and $\mathcal{E}_\perp$ is the energy of the perpendicular motion referred to as the photon energy.

The factor $\sqrt{Mc^2/\hbar\omega}$ appearing in (7) and (8) guarantees the appropriate scale of these parameters. In the problem two main energy scales occur, differing by many orders of magnitude (about six to ten): the particle rest energy and the photon energy. Due to the introduced factor, we avoid the situation where some of the parameters appearing below are extremely tiny or extremely large. With these substitutions we obtain, in place of (6),

$$i\frac{\partial}{\partial\tau}\tilde{\Psi}(\boldsymbol{\xi},\tau)=\mathcal{H}\tilde{\Psi}(\boldsymbol{\xi},\tau), \tag{9}$$

where the transformed Hamiltonian is given by

$$\mathcal{H}=-\frac{\gamma}{2\alpha}\Delta_{\xi_\perp}-\frac{\gamma}{2\alpha}\left(\partial_{\xi_z}+\frac{i}{2}\sigma_z\right)^2-\frac{\beta}{2}\sigma_z -\frac{\alpha}{2}(\xi_x\sigma_y+\xi_y\sigma_x). \tag{10}$$

Apart from (10) there exist two other constants of motion

$$H_1=\boldsymbol{\sigma}^2, \tag{11a}$$

$$H_2=i(\xi_y\partial_{\xi_x}-\xi_x\partial_{\xi_y})-\tfrac{1}{2}\sigma_z, \tag{11b}$$

corresponding to the classical ones found in [18]. The former is simply the spin squared and the latter in polar coordinates reads $H_2=-i\partial_\varphi-\sigma_z/2$, which is the z component of the full angular momentum. The fourth constant known from classical motion has already been exploited in (5) while passing from Ψ to $\tilde{\Psi}$.

One should note that the classical change of variables $(\boldsymbol{x},\boldsymbol{p})\mapsto(\boldsymbol{\xi},\boldsymbol{\eta})$ is not canonical and therefore the quantum-mechanical commutator equals

$$[\xi_m,\eta_n]=i\frac{\gamma}{\alpha}\delta_{mn} \tag{12}$$

and not simply $i\delta_{mn}$. It can be verified by a direct computation that one actually has

$$[H_i,\mathcal{H}]=0\quad \text{for } i=1,2. \tag{13}$$

Exploiting (11b), we can substitute $\tilde{\Psi}$ in the form

$$\tilde{\Psi}(\boldsymbol{\xi},\tau)=e^{-i\mathcal{E}_\perp\tau}e^{-i\gamma\kappa_z^2\tau/2\alpha}e^{i\kappa_z\xi_z}\begin{bmatrix}e^{i(m+1)\varphi}f_+(\xi)\\ ie^{im\varphi}f_-(\xi)\end{bmatrix}, \tag{14}$$

m being an integer, and separate the remaining polar variables. This leads to two coupled ordinary differential equations

$$-\frac{\gamma}{2\alpha}\left[f_+''+\frac{1}{\xi}f_+'-\left(\frac{(m+1)^2}{\xi^2}+\kappa_z+\frac{1}{4}\right)f_+\right] -\left(\mathcal{E}_\perp+\frac{\beta}{2}\right)f_+=\frac{\alpha}{2}\xi f_-, \tag{15a}$$

$$-\frac{\gamma}{2\alpha}\left[f_-''+\frac{1}{\xi}f_-'-\left(\frac{m^2}{\xi^2}-\kappa_z+\frac{1}{4}\right)f_-\right] -\left(\mathcal{E}_\perp-\frac{\beta}{2}\right)f_-=\frac{\alpha}{2}\xi f_+, \tag{15b}$$

where a prime denotes the derivative over ξ.

III. PROPERTIES OF THE MATRIX POTENTIAL

The first derivatives can be eliminated by plugging into (15) the functions $f_\pm(\xi)$ in the form

$$f_\pm(\xi)=\frac{F_\pm(\xi)}{\sqrt{\xi}} \tag{16}$$

and we obtain the equations for $F_\pm(\xi)$,

$$-\frac{\gamma}{2\alpha}F_+''+\frac{\gamma}{2\alpha}\left[\frac{(m+3/2)(m+1/2)}{\xi^2}-\delta_+\right]F_+=\frac{\alpha}{2}\xi F_-, \tag{17a}$$

$$-\frac{\gamma}{2\alpha}F_-''+\frac{\gamma}{2\alpha}\left[\frac{(m+1/2)(m-1/2)}{\xi^2}-\delta_-\right]F_-=\frac{\alpha}{2}\xi F_+, \tag{17b}$$

where

$$\delta_\pm=\frac{\alpha}{\gamma}(2\mathcal{E}_\perp\pm\beta)-\frac{1}{4}\mp\kappa_z. \tag{18}$$

It may be easily verified that for the values of parameters considered in Sec. IV we have $\delta_\pm > 0$.

For the two-component function $F = [F_+, F_-]$ the set of equations (17) may be given the form of a matrical stationary Schrödinger equation

$$-\frac{\gamma}{2\alpha}\partial_\xi^2 F(\xi) + V(\xi)F(\xi) = \mathcal{E}_\perp F(\xi), \tag{19}$$

with the potential

$$V(\xi) = \begin{bmatrix} \frac{\gamma(\kappa_z+1/4)}{2\alpha} - \frac{\beta}{2} + \frac{\gamma}{2\alpha}\frac{(m+3/2)(m+1/2)}{\xi^2} & -\frac{\alpha\xi}{2} \\ -\frac{\alpha\xi}{2} & \frac{\gamma(-\kappa_z+1/4)}{2\alpha} + \frac{\beta}{2} + \frac{\gamma}{2\alpha}\frac{(m+1/2)(m-1/2)}{\xi^2} \end{bmatrix} \tag{20}$$

and eigenenergy $\mathcal{E}_\perp$ to be determined. Due to the transformation (16), the radial character of the variable ξ is lost and the problem has become purely one dimensional. We will show below that this equation exhibits bound states, at least within perturbation theory.

The matrix V is real, symmetric, and has the eigenvalues

$$V_\pm(\xi) = \frac{\gamma}{2\alpha}\left(\frac{(m+1/2)^2}{\xi^2} + \frac{1}{4}\right) \pm \Lambda(\xi), \tag{21}$$

where

$$\Lambda(\xi) = \left[\Theta(\xi)^2 + \frac{\alpha^2\xi^2}{4}\right]^{1/2} \tag{22}$$

and

$$\Theta(\xi) = \frac{\gamma}{2\alpha}\left(\frac{m+1/2}{\xi^2} - \frac{\alpha\beta}{\gamma} + \kappa_z\right). \tag{23}$$

The corresponding eigenvectors are

$$\chi_\pm(\xi) = \begin{bmatrix} \Theta(\xi) \pm \Lambda(\xi) \\ -\frac{\alpha\xi}{2} \end{bmatrix}\sigma_\pm(\xi), \tag{24}$$

where $\sigma_\pm(\xi)$ are certain scalar functions defined below. These eigenvectors $\chi_\pm(\xi)$ satisfy the conditions

$$\chi_+(\xi)^T\chi_+(\xi) = \chi_-(\xi)^T\chi_-(\xi) = 1, \tag{25a}$$

$$\chi_+(\xi)^T\chi_-(\xi) = \chi_-(\xi)^T\chi_+(\xi) = 0 \tag{25b}$$

provided we choose

$$\sigma_\pm(\xi) = [2\Lambda(\xi)(\Lambda(\xi) \pm \Theta(\xi))]^{-1/2}. \tag{26}$$

Let us now substitute into (19) the wave function in the form of $F(\xi) = U(\xi)\Phi(\xi)$, where U is the ξ-dependent transformation matrix for the potential V,

$$U^{-1}VU = V_D = \begin{bmatrix} V_+ & 0 \\ 0 & V_- \end{bmatrix}, \tag{27}$$

and D stands for diagonal. The matrix U is orthogonal and its columns constitute the eigenvectors $\chi_\pm$:

$$U = [\chi_+, \quad \chi_-], \quad U^{-1} = U^T = \begin{bmatrix} \chi_+^T \\ \chi_-^T \end{bmatrix}. \tag{28}$$

The equation for the function Φ can be obtained in a straightforward way,

$$-\frac{\gamma}{2\alpha}U^{-1}\partial_\xi^2 U\Phi + V_D\Phi = \mathcal{E}_\perp\Phi, \tag{29}$$

and is equivalent to

$$-\frac{\gamma}{2\alpha}\partial_\xi^2\Phi + (V_D + W)\Phi(\xi) = \mathcal{E}_\perp\Phi(\xi), \tag{30}$$

where the quantity

$$W = \frac{\gamma}{2\alpha}\left(\partial_\xi^2 - U^{-1}\partial_\xi^2 U\right) \tag{31}$$

may be treated as a perturbation for the diagonal potential V_D. It should be pointed out that the derivative in the expression $U^{-1}\partial_\xi^2 U$ acts on both the U matrix and the wave function in (30). If expressed through the eigenvectors (24), it has the formal form

$$W = \frac{\gamma}{2\alpha}\left(\partial_\xi^2 - \begin{bmatrix} \chi_+^T \\ \chi_-^T \end{bmatrix}\partial_\xi^2[\chi_+, \quad \chi_-]\right)$$
$$= -\frac{\gamma}{2\alpha}\left(\begin{bmatrix} \chi_+^T\chi_+'' & \chi_+^T\chi_-'' \\ \chi_-^T\chi_+'' & \chi_-^T\chi_-'' \end{bmatrix} + 2\begin{bmatrix} \chi_+^T\chi_+' & \chi_+^T\chi_-' \\ \chi_-^T\chi_+' & \chi_+^T\chi_+' \end{bmatrix}\partial_\xi\right). \tag{32}$$

As can be seen, this quantity contains both diagonal and off-diagonal elements and therefore it is more convenient to absorb all the diagonal terms (denoted below by W_D) into the unperturbed potential V_D, defining

$$\tilde{V}_\pm(\xi) = V_\pm(\xi) - \frac{\gamma}{2\alpha}\chi_\pm^T(\xi)\chi_\pm''(\xi). \tag{33}$$

The behavior of $\tilde{V}_\pm(\xi)$ for various values of parameters is shown in Figs. 1 and 2.

The apparently diagonal elements remaining in the second matrix of (32), i.e., the quantities containing $\chi_+^T\chi_+'$ and $\chi_-^T\chi_-'$, identically vanish as a consequence of normalization (25a). The inclusion of W_D into $\tilde{V}_D$ does not change the general form of $V_\pm$, since the additional terms disappear quickly as $\xi \to \infty$ and tend to constants for $\xi \to 0$.

Now the perturbation responsible for the interaction between the two channels has become a purely off-diagonal matrix $\tilde{W}$ of the form

$$\tilde{W} = -\frac{\gamma}{2\alpha}\left(\begin{bmatrix} 0 & \chi_+^T\chi_-'' \\ \chi_-^T\chi_+'' & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & \chi_+^T\chi_-' \\ \chi_-^T\chi_+' & 0 \end{bmatrix}\partial_\xi\right). \tag{34}$$

Upon omission of $\tilde{W}$ in the Schrödinger equation

$$-\frac{\gamma}{2\alpha}\partial_\xi^2\Phi + (\tilde{V}_D + \tilde{W})\Phi(\xi) = \mathcal{E}_\perp\Phi(\xi), \tag{35}$$

the two channels described by the lower and upper components of the wave function Φ decouple from each other, leading to the sector of well-localized bound states existing in the potential

$\tilde{V}_+$ and the sector of scattering states governed by $\tilde{V}_-$. However, in the practical situation of the vortex field, which is certainly limited in the direction perpendicular to the wave propagation, it would be unphysical to expect these potentials to extend to infinity in the unmodified form. We will return to this point in the following section.

IV. ANALYSIS OF THE MOTION

The Schrödinger equation (15), (19), or (35) in its full complexity cannot be solved in an analytic way. Therefore, we consider the behavior of solutions separately for small and large radial distances and apply appropriate approximations.

A. Small distances

Consider now the distances close to the vortex core. By close we mean $\xi \ll 1$, which roughly corresponds to $\rho \lesssim \lambda$ (where λ is the wavelength). In this region Eqs. (15) can be used directly. The term in square brackets in (15a) and analogously (15b) dominates over $\alpha\xi$ on the right-hand sides. We are then allowed to consider the simplified equations

$$f_+'' + \frac{1}{\xi} f_+' + \left[\delta_+ - \frac{(m+1)^2}{\xi^2}\right] f_+ = 0, \tag{36a}$$

$$f_-'' + \frac{1}{\xi} f_-' + \left[\delta_- - \frac{m^2}{\xi^2}\right] f_- = 0. \tag{36b}$$

After the appropriate rescaling of the variable ξ, one can recognize in the above the ordinary and modified Bessel equations correspondingly. Therefore, their solutions can be immediately written out:

$$f_+(\xi) = C_1 J_{m+1}(\sqrt{\delta_+}\,\xi) + C_2 Y_{m+1}(\sqrt{\delta_+}\,\xi), \tag{37a}$$

$$f_-(\xi) = D_1 J_m(\sqrt{\delta_-}\,\xi) + D_2 Y_m(\sqrt{\delta_-}\,\xi). \tag{37b}$$

To guarantee the correct behavior of the wave function Ψ at the origin, we must reject the term containing the Neumann function Y_{m+1} in f_+ and Y_m in f_-, by setting $C_2 = D_2 = 0$. It is then possible to choose the solution of (6), which is deprived of any ambiguities at the origin and is perfectly square integrable as $\xi \to 0$. However, the small-ξ behavior does not rule on the trapping efficiency.

B. Perturbative equations

For larger perpendicular directions the perturbative analysis based on Eq. (35) is necessary. If $\tilde{W} = 0$ there are two unperturbed one-dimensional Schrödinger equations constituting the starting point for the calculation, corresponding to the choice of either $\Phi_0 = [u, 0]$ or $\Phi_0 = [0, v]$ and the potential $\tilde{V}_+$ or $\tilde{V}_-$, respectively:

$$-\frac{\gamma}{2\alpha}\partial_\xi^2 u(\xi) + \tilde{V}_+(\xi)u(\xi) = \mathcal{E}_{\perp 0}u(\xi), \tag{38a}$$

$$-\frac{\gamma}{2\alpha}\partial_\xi^2 v(\xi) + \tilde{V}_-(\xi)v(\xi) = \mathcal{E}_{\perp 0}v(\xi). \tag{38b}$$

Later the solutions of these equations will also be labeled with the value of energy $\mathcal{E}_{\perp 0}$, i.e., they will be denoted by $u_{\mathcal{E}_{\perp 0}}$ and $v_{\mathcal{E}_{\perp 0}}$. The first potential possesses a bound state for sure at least in the interesting range of parameters for which Figs. 1 and 2 are sketched. In contrast, if one takes $\tilde{V}_-$ instead of $\tilde{V}_+$, no bound states exist. This can be easily seen by inspecting (33), from which one can infer that $V_+(\xi)$ $(V_-(\xi))$ grows (declines) linearly as $\xi \to \infty$, as well as by looking at Figs. 1 and 2.

However, bound states are not stationary states of the full Hamiltonian. There are two sources of possible tunneling, which are considered below. One results from the spatial size of the vortex (in the perpendicular direction), which creates an opportunity for tunneling through the barrier. If the extension of the light beam is limited in the radial direction, $\tilde{V}_+(\xi)$ vanishes beyond a certain value ξ_w (where $\xi_w = k\rho_w$, the latter roughly corresponding to the waist of the beam), creating an ordinary potential barrier through which the tunneling is possible even in the zeroth order of the perturbation calculation (i.e., for $\tilde{W} = 0$). The probability of this process is negligible in comparison with the other as estimated in Sec. IV C.

The second possibility is connected to the fact that the theory possesses two channels and there is no binding in the second channel at all. Thus, another way of tunneling emerges, that into the second channel. This phenomenon should be much more important. For such a process to occur it is necessary to turn on the off-diagonal terms of the Hamiltonian represented by $\tilde{W}$. This case is dealt with in Sec. IV D in a perturbative manner.

Let us turn below to the operator $\tilde{W}$ and verify whether this kind of calculation will be justified. The quantity $\gamma/\alpha = \sqrt{\hbar\omega/Mc^2}$ is very small (see also Figs. 1 and 2). Below we estimate $\tilde{W}$ by taking the above into account. From (22)–(24) and (26) for $\gamma/\alpha \ll 1$ we find

$$\Theta = -\frac{\beta}{2} + O\left(\frac{\gamma}{\alpha}\right), \tag{39a}$$

$$\Lambda = \frac{1}{2}\sqrt{\beta^2 + \alpha^2\xi^2} + O\left(\frac{\gamma}{\alpha}\right), \tag{39b}$$

$$\sigma_\pm = \sqrt{2}[\sqrt{\beta^2 + \alpha^2\xi^2}(\sqrt{\beta^2 + \alpha^2\xi^2} \mp \beta)]^{-1/2} + O\left(\frac{\gamma}{\alpha}\right), \tag{39c}$$

$$\chi_\pm = \pm[2\sqrt{\beta^2 + \alpha^2\xi^2}(\sqrt{\beta^2 + \alpha^2\xi^2} \mp \beta)]^{-1/2} \times \begin{bmatrix} \sqrt{\beta^2 + \alpha^2\xi^2} \mp \beta \\ \mp\alpha\xi \end{bmatrix} + O\left(\frac{\gamma}{\alpha}\right). \tag{39d}$$

Collecting all terms of the above approximations, after some laborious calculations omitted here, we get, up to $(\gamma/\alpha)^2$,

$$\tilde{W} = \frac{|\gamma|\beta}{2(\beta^2 + \alpha^2\xi^2)}\left(\frac{\alpha^2\xi}{\beta^2 + \alpha^2\xi^2} - \partial_\xi\right)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \tilde{W}_0(\xi)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{40}$$

which can also be written in the explicitly Hermitian form

$$\tilde{W} = -\frac{|\gamma|\beta}{2}\frac{1}{\sqrt{\beta^2 + \alpha^2\xi^2}}\partial_\xi\frac{1}{\sqrt{\beta^2 + \alpha^2\xi^2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{41}$$

Due to the coefficient γ, this quantity is small and may be treated as a perturbation. Since $\tilde{W}$ is purely off-diagonal, it does not contribute to the first-order perturbative correction to the bound-state energy $\mathcal{E}_{\perp 0}$. Hence $\mathcal{E}_\perp - \mathcal{E}_{\perp 0} = O(\gamma^2)$ and this correction may be neglected within our present approach. The existence of bound states of Eq. (19) at least in the perturbative sense will then be indubitable if the rate of eventual tunneling is sufficiently small.

C. Tunneling through the barrier

While considering the tunneling through the barrier, we cannot limit ourselves to the paraxial approximation where the magnetic field given by the formula (4). Both the height and the thickness of the barrier are important and we are condemned to use the full form of the magnetic field (2). The correction $\tilde{W}$ does not play any role here since it is off-diagonal, so the tunneling probability can be estimated from Eq. (38a).

For small values of $k_\perp$ (but still admitting $k_\perp\rho \gtrsim 1$), we can neglect terms containing the coefficient a_-. Moreover, it should be noted that since $|J_2(x)| < 0.5$ [28], then for the values of parameters considered in this work the estimation

$$\left|\frac{4B_\perp\cos(\zeta_z-2\varphi)J_2(k_\perp\rho)}{B_z}\right| < 2\left|\frac{B_\perp}{B_z}\right| = \left|\frac{2\alpha}{\beta-\alpha/\gamma}\right| \approx 2|\gamma| \ll 1 \quad (42)$$

holds, which means that the z component of the magnetic field of the wave may be neglected as compared to the external field (which is strong), and we get

$$\boldsymbol{B}(\boldsymbol{r},t) = \begin{bmatrix} -2B_\perp\sin(\zeta_z-\varphi)J_1(k_\perp\rho) \\ 2B_\perp\cos(\zeta_z-\varphi)J_1(k_\perp\rho) \\ B_z \end{bmatrix}. \quad (43)$$

This form of magnetic field leads to an equation identical to (19) but with the modified matrix potential

$$V(\xi) = \begin{bmatrix} \frac{\gamma(\kappa_z+1/4)}{2\alpha} - \frac{\beta}{2} + \frac{\gamma}{2\alpha}\frac{(m+3/2)(m+1/2)}{\xi^2} & -\alpha J_1(\kappa_z\xi) \\ -\alpha J_1(\kappa_z\xi) & \frac{\gamma(-\kappa_z+1/4)}{2\alpha} + \frac{\beta}{2} + \frac{\gamma}{2\alpha}\frac{(m+1/2)(m-1/2)}{\xi^2} \end{bmatrix}. \quad (44)$$

Now all formulas of Sec. III remain true if we redefine

$$\Lambda(\xi) = [\Theta(\xi)^2 + \alpha^2 J_1(\kappa_z\xi)]^{1/2}, \quad (45)$$

together with

$$\chi_\pm(\xi) = \begin{bmatrix} \Theta(\xi) \pm \Lambda(\xi) \\ -\alpha J_1(\kappa_z\xi) \end{bmatrix}\sigma_\pm(\xi). \quad (46)$$

The behavior of the modified potential $\tilde{V}_+$ for both sets of data (those of Figs. 1 and 2) are drawn in Fig. 3. As it is well known from both the WKB method [29] and Miller and Good's method [30], the value of the tunneling probability is dictated by the typical exponential factor

$$\theta = \exp\left(-2\int_{x_0}^{x_0+d}\sqrt{\frac{2M}{\hbar^2}[V(x)-E]dx}\right), \quad (47)$$

where d denotes the barrier thickness for a given energy. Since both the square root and $V(x)$ are concave functions in this region (see Fig. 3), it is obvious that

$$\theta < \exp\left(-2\int_{x_1}^{x_1+d}\frac{1}{2}\sqrt{\frac{2M}{\hbar^2}(V_{\max}-E)dx}\right) = \exp\left(-d\sqrt{\frac{2M}{\hbar^2}(V_{\max}-E)}\right), \quad (48)$$

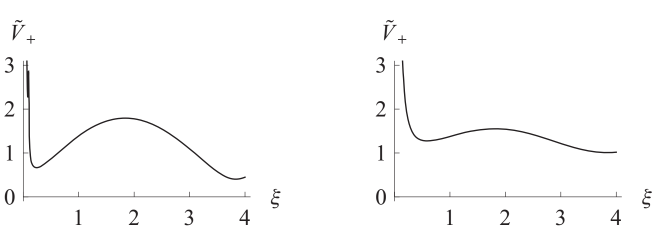

FIG. 3. Behavior of $\tilde{V}_+(\xi)$ derived from (44) for the same values of parameters as in Figs. 1 and 2.

where max refers to the top of the barrier and x_1 denotes the right turning point for the classical motion. It can be rewritten as

$$\theta < \exp\left(-\sqrt{2}\xi_d\sqrt{\frac{\alpha}{\gamma}(\tilde{V}_{+\max}-\mathcal{E}_{\perp 0})}\right), \quad (49)$$

with ξ_d corresponding to d in our dimensionless variables. The numerical prefactors of θ are marginal for the overall value of probability [31].

For the first set of data the ground-state energy can be estimated with the use of the uncertainty principle to be $\mathcal{E}_{\perp 0} \approx 0.67$. The maximal value of the potential is $\tilde{V}_{+\max} \approx 1.79$ and the barrier thickness $\xi_d \approx 3.12$. The barrier is then relatively high and thick, which results in the extremely small value of the factor

$$\theta < 6.7 \times 10^{-36}.$$

The value of θ determines the transmission probability for each individual hitting the potential wall by the particle. To find the probability per unit time it should be multiplied by the number n of hits per unit time. This can be estimated from the bound-state energy as

$$n \lesssim \frac{p_{\max}}{M|x_0-x_1|} \lesssim \frac{\sqrt{2M(E-V_{\min})}}{M|x_0-x_1|}, \quad (50)$$

where x_0 is the left turning point. For the dimensionless variables used throughout the work this reads

$$n \lesssim \frac{\sqrt{2(\gamma/\alpha)^3(\mathcal{E}_{\perp 0}-\tilde{V}_{+\min})}}{|\xi_0-\xi_1|}\omega \approx 3.6\times 10^{-4}\omega, \quad (51)$$

leading to the negligible value

$$\frac{\Gamma}{\omega} \lesssim 2.4\times 10^{-39}. \quad (52)$$

In the second case the potential cavity is shallower and the barrier thinner in a visible way. This is also reflected by the results of our previous work on classical motion [18], where the comparison of Figs. 5 and 8 reveals a much larger size of the trap in the latter case. For these data the trap is less efficient. In quantum theory this should entail a significant increase of the factor θ. We find $\mathcal{E}_{\perp 0} \approx 1.28$ and $\tilde{V}_{+\max} \approx 1.55$. The barrier thickness gets reduced to $\xi_d \approx 2.21$. Consequently, we can estimate (49) to be about

$$\theta < 8.4 \times 10^{-8},$$

which is much larger but still very small. For n we obtain

$$n \lesssim 7.9 \times 10^{-4}\omega, \tag{53}$$

which gives

$$\frac{\Gamma}{\omega} \lesssim 6.6 \times 10^{-11}. \tag{54}$$

As we will see in the following section, the direct tunneling through the barrier (i.e., within the first channel) may be practically neglected. A heavy particle with very short Compton wavelength cannot penetrate a barrier too deep without strongly violating the energy conservation. A much more essential effect is connected with the tunneling into the second channel at which the particle is kicked out of the vortex field.

D. Tunneling to the other channel

In order to consider the tunneling into the second channel we assume that beyond $\xi = \xi_w$ one has $\tilde{V}_-(\xi) = 0$. Otherwise the potential of the scattering sector would be unphysical, leading to the Hamiltonian unbounded below. This aspect was inessential for the evolution of solutions restricted to the first channel but must be taken into account in the case of interchannel transitions. Consistently we modify the formula (21) for $\tilde{V}_-$ by including the factor $\Theta(\xi_w - \xi)$ and henceforth this is the new meaning of that symbol. As regards $\tilde{V}_+$, it is left unmodified as a consequence of the results of the preceding section, which clearly indicate that direct tunneling through the field barrier is improbable and from that point of view the binding potential may be treated as extending to infinity with no essential change of conclusions.

The only realistic tunneling process consists, therefore, in flipping the magnetic moment [keeping the constant of motion H_2 given by (11b) fixed], after which the particle gets ejected from the vortex field instead of being tunneled through the potential hump. Below we try to estimate the probability per unit time for this kind of a process. Aimed at simplifying the equation, the independent variable ξ (and similarly the parameter ξ_w) will now be rescaled as follows:

$$\mathfrak{z} = \left|\frac{\alpha^2}{\gamma}\right|^{1/3}\xi. \tag{55}$$

If this transformation is applied to Eq. (38), it can be observed that the approximation of large $\mathfrak{z}$ is the same as that for $\gamma/\alpha \ll 1$. Therefore, its asymptotic form may be treated as applicable in the whole domain (possibly except a narrow, and inessential, interval close to the origin)

$$-\partial_{\mathfrak{z}}^2 u(\mathfrak{z}) + \mathfrak{z}u(\mathfrak{z}) = \frac{2\mathcal{E}_{\perp 0}}{|\gamma\alpha|^{1/3}}u(\mathfrak{z}), \tag{56a}$$

$$-\partial_{\mathfrak{z}}^2 v(\mathfrak{z}) - \mathfrak{z}\Theta(\mathfrak{z}_w - \mathfrak{z})v(\mathfrak{z}) = \frac{2\mathcal{E}_{\perp 0}}{|\gamma\alpha|^{1/3}}v(\mathfrak{z}). \tag{56b}$$

First let us deal with Eq. (56a). Upon shifting the independent variable in order to cancel the right-hand side and defining

$$\mathfrak{z}_0 = -2\mathcal{E}_{\perp 0}/(\gamma\alpha)^{1/3}, \tag{57}$$

it becomes the ordinary Airy equation with solutions [28]

$$u(\mathfrak{z}) = C_1\mathrm{Ai}(\mathfrak{z}+\mathfrak{z}_0) + C_2\mathrm{Bi}(\mathfrak{z}+\mathfrak{z}_0). \tag{58}$$

For the bound-state wave function $u_{\mathfrak{z}_0}(\mathfrak{z})$ we require a sufficiently quick decline at infinity. Therefore, $C_2 = 0$ and C_1 may be denoted simply by C. On the other hand, we must have $u_{\mathfrak{z}_0}(0) = 0$, which means that $\mathfrak{z}_0$ is the first zero (as far as the ground state is considered) of the Airy function Ai (i.e., $\mathfrak{z}_0 \approx -2.338$). Using the integral [32]

$$\begin{aligned}
&\int_0^\infty \mathrm{A}i^2(\mathfrak{z}+\mathfrak{z}_0)d\mathfrak{z} \\
&= \int_0^\infty \partial_{\mathfrak{z}}(\mathfrak{z}+\mathfrak{z}_0)\mathrm{A}i^2(\mathfrak{z}+\mathfrak{z}_0)d\mathfrak{z} \\
&= -\mathfrak{z}_0\mathrm{A}i^2(\mathfrak{z}_0) - 2\int_0^\infty (\mathfrak{z}+\mathfrak{z}_0)\mathrm{A}i(\mathfrak{z}+\mathfrak{z}_0)\mathrm{A}i'(\mathfrak{z}+\mathfrak{z}_0)d\mathfrak{z} \\
&= -\mathfrak{z}_0\mathrm{A}i^2(\mathfrak{z}_0) - 2\int_0^\infty \mathrm{A}i''(\mathfrak{z}+\mathfrak{z}_0)\mathrm{A}i'(\mathfrak{z}+\mathfrak{z}_0)d\mathfrak{z} \\
&= -\mathfrak{z}_0\mathrm{A}i^2(\mathfrak{z}_0) - \int_0^\infty \partial_{\mathfrak{z}}\mathrm{A}i'^2(\mathfrak{z}+\mathfrak{z}_0)d\mathfrak{z} \\
&= -\mathfrak{z}_0\mathrm{A}i^2(\mathfrak{z}_0) + \mathrm{A}i'^2(\mathfrak{z}_0),
\end{aligned} \tag{59}$$

the normalization constant C can be found to be

$$C = [-\mathfrak{z}_0\mathrm{Ai}^2(\mathfrak{z}_0) + \mathrm{Ai}'^2(\mathfrak{z}_0)]^{-1/2} = \frac{1}{\mathrm{Ai}'(\mathfrak{z}_0)}, \tag{60}$$

where the absolute value has been omitted as the derivative of the Airy function at $\mathfrak{z} = \mathfrak{z}_0$ is positive.

Now consider the function v satisfying (56b). One can distinguish two characteristic regions: $0 < \mathfrak{z} < \mathfrak{z}_w$ and $\mathfrak{z} > \mathfrak{z}_w$. The well-behaving and continuously differentiable wave function $v(\mathfrak{z})$ corresponding to the quantum number $\mathfrak{z}_0$ can be written as

$$v_{\mathfrak{z}_0}(\mathfrak{z}) = \begin{cases} G_{\mathfrak{z}_0}(\mathfrak{z}) & \text{for } 0 < \mathfrak{z} < \mathfrak{z}_w \\ \frac{1}{q_{\mathfrak{z}_0}}G'_{\mathfrak{z}_0}(\mathfrak{z}_w)\sin q_{\mathfrak{z}_0}(\mathfrak{z}-\mathfrak{z}_w) & \\ \quad + G_{\mathfrak{z}_0}(\mathfrak{z}_w)\cos q_{\mathfrak{z}_0}(\mathfrak{z}-\mathfrak{z}_w) & \text{for } \mathfrak{z} > \mathfrak{z}_w, \end{cases} \tag{61}$$

where $q_{\mathfrak{z}_0} = \sqrt{-\mathfrak{z}_0}$ and

$$G_a(\mathfrak{z}) = D_1\mathrm{Ai}(-\mathfrak{z}+a) + D_2\mathrm{Bi}(-\mathfrak{z}+a). \tag{62}$$

To ensure the nonsingular behavior of $v_{\mathfrak{z}_0}(\mathfrak{z})$ at $\mathfrak{z} = 0$ one has to set $D_1 = D\,\mathrm{Bi}(a)$ and $D_2 = -D\,\mathrm{Ai}(a)$. Above $\mathfrak{z}_w$, where the potential vanishes, we have ordinary trigonometric solutions.

The integral

$$\int_0^\infty v_a(\mathfrak{z})v_b(\mathfrak{z})d\mathfrak{z} = \pi\sqrt{-a}\left(G_a(\mathfrak{z}_w)^2 + \frac{G'_a(\mathfrak{z}_w)^2}{q_a^2}\right)\delta(a-b) \tag{63}$$

can be used to fix D. The δ function on the right-hand side comes from the integration between $\mathfrak{z}_w$ and infinity, where $v(\mathfrak{z})$ is given by the trigonometric functions. The off-diagonal (i.e., for $a \neq b$) integral can be shown to vanish if we use the trick [32]

$$\begin{aligned}\int_0^{\mathfrak{z}_w} G_a(\mathfrak{z})G_b(\mathfrak{z})d\mathfrak{z} &= \int_0^{\mathfrak{z}_w} \frac{G_a''(\mathfrak{z})}{a-\mathfrak{z}}\frac{G_b''(\mathfrak{z})}{b-\mathfrak{z}}d\mathfrak{z}\\ &= \frac{1}{b-a}\int_0^{\mathfrak{z}_w}\left(\frac{G_a''(\mathfrak{z})}{a-\mathfrak{z}}G_b''(\mathfrak{z}) - G_a''(\mathfrak{z})\frac{G_b''(\mathfrak{z})}{b-\mathfrak{z}}\right)d\mathfrak{z}\\ &= \frac{1}{b-a}\int_0^{\mathfrak{z}_w}[G_a(\mathfrak{z})G_b''(\mathfrak{z}) - G_a''(\mathfrak{z})G_b(\mathfrak{z})]d\mathfrak{z}\\ &= \frac{1}{b-a}\int_0^{\mathfrak{z}_w}\partial_{\mathfrak{z}}[G_a(\mathfrak{z})G_b'(\mathfrak{z}) - G_a'(\mathfrak{z})G_b(\mathfrak{z})]d\mathfrak{z}\\ &= \frac{1}{b-a}[G_a(\mathfrak{z}_w)G_b'(\mathfrak{z}_w) - G_a'(\mathfrak{z}_w)G_b(\mathfrak{z}_w)] \qquad (64)\end{aligned}$$

and observe that this expression exactly cancels the terms coming from the integration of trigonometric functions in (61) in the interval $(\mathfrak{z}_w, \infty)$. One can show that the choice

$$D = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{\gamma}\right)^{1/4} \times \frac{|\mathrm{Bi}(\mathfrak{z}_0)|^{-1}}{\sqrt{\mathrm{Ai}(\mathfrak{z}_0-\mathfrak{z}_w)^2 + (\alpha\gamma)^{1/3}/2\mathcal{E}_{\perp 0}\,\mathrm{Ai}'(\mathfrak{z}_0-\mathfrak{z}_w)^2}} \qquad (65)$$

leads to the required normalization of the function $v(\xi)$,

$$\int_0^\infty v_{\mathcal{E}'_{\perp 0}}(\xi)v_{\mathcal{E}_{\perp 0}}(\xi)d\xi = \delta(\mathcal{E}'_{\perp 0} - \mathcal{E}_{\perp 0}). \qquad (66)$$

Complex conjugations are omitted here and below since we are dealing with real functions only.

The probability of tunneling per unit time is given by

$$\Gamma = \frac{dP}{dt} = 2\pi\omega\frac{\gamma}{\alpha}\sum_i \delta(\mathcal{E}_{\perp i} - \mathcal{E}_{\perp 0})|I_{i0}|^2, \qquad (67)$$

where i labels the continuum-spectrum states and I_{i0} is the matrix element of the perturbation potential [the off-diagonal element of (41)]. According to what was said above, for the calculation of I_{i0} within the present approximation the quantities $u_{\mathfrak{z}_0}$ and $v_{\mathfrak{z}_0}$ can be used as unperturbed wave functions. Exploiting the δ function in (67), we arrive at

$$\Gamma = \omega\frac{\pi}{2}\frac{\gamma}{\alpha}\left(\frac{\gamma}{\beta}\right)^2\left|\int_0^\infty \partial_{\mathfrak{z}}\left(\frac{u_{\mathfrak{z}_0}(\mathfrak{z})}{\sqrt{1+|\gamma\alpha|^{2/3}/\beta^2\,\mathfrak{z}^2}}\right) \times \frac{v_{\mathfrak{z}_0}(\mathfrak{z})}{\sqrt{1+|\gamma\alpha|^{2/3}/\beta^2\mathfrak{z}^2}}d\mathfrak{z}\right|^2. \qquad (68)$$

The coefficient $\frac{\pi}{2}\frac{\gamma}{\alpha}(\frac{\gamma}{\beta})^2$ is about 10^{-6} for either set of data. It should be noted that the strength of the interchannel potential $\tilde{W}$, together the value of Γ, can be further reduced by the choice of β to be very small. This point will be referred to at the end of this section.

The estimation of the integral can be done numerically assuming, for instance, the value of $\zeta_w \approx 4\pi|\alpha^2/\gamma|^{1/3}$, which roughly corresponds to two wavelengths. For the bound-state energy $\mathcal{E}_{\perp 0}$ the value estimated in Sec. IV C can be used and if instead the potential of Eq. (56a) is taken for this assessment the obtained values turn out to be practically identical. It would be less accurate to use $\mathcal{E}_{\perp 0}$ directly from Eq. (57) since it would be underestimated due to the incorrect behavior of the approximated potential in the vicinity of $\zeta = 0$ (the true potential does not vanish and has a positive minimum, which shifts the energy up). As it was mentioned, Eqs. (38) do not yield the correct form of the wave functions close to the origin. We finally find

$$\frac{\Gamma}{\omega} \approx 5.1 \times 10^{-7} \qquad (69)$$

for the data of Fig. 1 and

$$\frac{\Gamma}{\omega} \approx 7.2 \times 10^{-6} \qquad (70)$$

for those of Fig. 2. These values seem to be relatively large. Nevertheless, they prove that with quantum effects involved it is in principle possible to trap neutral particles through a very delicate mechanism relying on their magnetic moment interacting with the magnetic field of the vortex for a short time (for instance, 10^5 of wave periods $2\pi/\omega$).

The trapping time can be significantly prolonged if the external magnetic field is well tuned so as to make the value of β very small. For instance, with the identical values of parameters as those of Fig. 1, except for β, chosen now to be equal to 0.01, one gets

$$\frac{\Gamma}{\omega} \approx 1.5 \times 10^{-9}. \qquad (71)$$

This effect can be explained in classical terms as follows. The small value of the parameter β corresponds to the tuning of B_z, so the frequency of the Larmor precession of the magnetic moment around that field becomes close to the wave frequency. One could say that the external magnetic field keeps the magnetic moment synchronized in its rotation with the rotating vortex field. In these conditions the flipping of $\boldsymbol{\mu}$ necessary for the tunneling into the second channel becomes less probable.

This conclusion is supported by the observation referring to the classical motion of a particle for decreasing values of the parameter β. Upon precise tuning of the initial particle state, there exist trajectories that become less and less chaotic as $\beta \to 0$ and turn into circles. As seen in Fig. 4 for a value of the Larmor frequency that is extremely well adjusted to that of the vortex field or vice versa, one can obtain a very stable (almost circular) trajectory, which suggests relatively strong binding of the particle by the vortex field.

However, the region of this resonance is presently unreachable. The low value of β would require that the energy of the interaction of the magnetic moment with the constant magnetic field be comparable to the photon energy. As already stated in the Introduction, the required value of B_z for the neutron would be about -1.7×10^{11} T for the parameters of Fig. 1 and -1.6×10^{12} T for those of Fig. 2. In the case of hydrogen, which has approximately the same mass but a magnetic moment three orders of magnitude larger, the required fields should be proportionally weaker. The possible way out of this would be to reduce the wave frequency by several orders of magnitude for the main problem is that the relatively heavy particle endowed with a small magnetic moment precess very slowly and the resonant value of ω must respect it. By increasing B_z one forces it to rotate more quickly

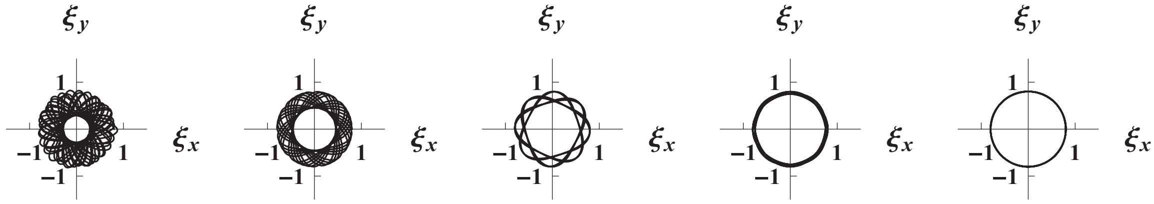

FIG. 4. Exemplary classical trajectory of a particle in a running wave considered in [18] for subsequent decreasing values of the parameter β: $\beta = -0.03, -0.01, -0.005, -0.001$, and -0.0001.

and vice versa. It should be mentioned that the resonance is not required for the trap to operate, but it greatly prolongs the trapping time.

V. SUMMARY

The present work has been concerned with the motion of a neutral quantum particle endowed with a magnetic moment interacting with a certain special configuration of electromagnetic fields: a wave bearing orbital angular momentum (a vortex field) and a constant magnetic field aligned along the direction of propagation of the former (i.e., z axis). It has been shown, by solving the appropriate Schrödinger equation, that this setup leads to the guidance of the particle along the vortex core and trapping it in the perpendicular direction. Due to the relatively weak strength of the interaction between the magnetic moment and the wave magnetic field, the mechanism of capturing is delicate and requires extremely strong fields, which are presently unavailable.

The quantum theory confirms then the results obtained for the same values of parameters within classical mechanics. However, there additionally appears a purely quantum effect of tunneling which can play an important role in the particle being captured.

Two possible sources of tunneling were identified. The first one is connected with the fact that the perpendicular size of the light beam is limited, thus creating a kind of potential hump binding the particle. On the exterior of it the potential vanishes and a free motion is possible. The tunneling probability through this barrier was estimated to be extremely small and practically negligible in comparison with the second possibility. The latter is connected with the spontaneous process of flipping the direction of the magnetic moment to the opposite one. In these conditions the vortex field no longer keeps the particle bound but ejects the particle out of the field. This kind of tunneling narrows the trapping time to $10^5\omega^{-1}$. However, this short time can be prolonged if one accurately tunes the external magnetic field. The idea is to make the Larmor frequency of precession of the magnetic moment around the z axis close to that of the rotating magnetic field of the vortex. These circumstances make the flipping of spin much less probable since it is well synchronized with the rotating field. In consequence, the trapping time can be lengthened by a couple of orders of magnitude.

The same effect may be identified in classical mechanics. Naturally, there is no tunneling there, but what is visible is the stabilization of the particle orbit while approaching the resonance. The trajectory becomes much less chaotic and more circular, which corresponds to reducing the tunneling probability in the quantum case.

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APPENDIX: SCHRÖDINGER-PAULI EQUATION FOR THE CENTER-OF-MASS MOTION OF THE HYDROGEN ATOM

Let us start with the classical Hamiltonian of the electron-proton system in the external electromagnetic field

$$H = \frac{1}{2m_e}[\boldsymbol{p}_e - e\boldsymbol{A}(\boldsymbol{r}_e, t)]^2 + \frac{1}{2m_p}[\boldsymbol{p}_p - e\boldsymbol{A}(\boldsymbol{r}_p, t)]^2 - \frac{e^2}{4\pi\varepsilon_0|\boldsymbol{r}_e - \boldsymbol{r}_p|} - \boldsymbol{\mu}_e\boldsymbol{B}, \tag{A1}$$

where indices e and p refer to the electron and proton, respectively. The magnetic moment of the proton has been neglected due to the very small ratio m_e/m_p.

We now follow the procedure elaborated in [33]. Introducing relative and center-of-mass coordinates

$$\boldsymbol{r} = \boldsymbol{r}_e - \boldsymbol{r}_p, \tag{A2a}$$

$$\boldsymbol{R} = \frac{m_e\boldsymbol{r}_e + m_p\boldsymbol{r}_p}{m_e + m_p} \tag{A2b}$$

and momenta

$$\boldsymbol{p} = \frac{m_p\boldsymbol{p}_e - m_e\boldsymbol{p}_p}{m_e + m_p}, \tag{A3a}$$

$$\boldsymbol{R} = \boldsymbol{p}_e + \boldsymbol{p}_p, \tag{A3b}$$

one can rewrite the Hamiltonian in the form

$$H = \frac{1}{2M}[\boldsymbol{P} - e(\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A}(\boldsymbol{R}, t)]^2 + \frac{1}{2M_r}[\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{R}, t)]^2 - e\frac{\Delta M}{M}(\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A}(\boldsymbol{R}, t)]^2 - \frac{e^2}{4\pi\varepsilon_0 r} - \boldsymbol{\mu}_e\boldsymbol{B}, \tag{A4}$$

where $M = m_e + m_p$ is the total mass of the system, $M_r = m_e m_p/(m_e + m_p)$ is the reduced mass, and $\Delta M = m_p - m_e$.

While passing from (A1) to (A4) we assumed that the external field varies slowly on the scale imposed by the size of the hydrogen atom. In the present work we deal with wavelengths which are three or four orders of magnitude larger than this size, so the above approximation is well justified.

Therefore, one can write

$$\boldsymbol{A}(\boldsymbol{r}_e, t) = \boldsymbol{A}\left(\boldsymbol{R} + \frac{m_p}{m_e + m_p}\boldsymbol{r}, t\right) \simeq \boldsymbol{A}(\boldsymbol{R}, t) + \frac{m_p}{m_e + m_p}(\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A}(\boldsymbol{R}, t), \quad \text{(A5a)}$$

$$\boldsymbol{A}(\boldsymbol{r}_p, t) = \boldsymbol{A}\left(\boldsymbol{R} - \frac{m_e}{m_e + m_p}\boldsymbol{r}, t\right) \simeq \boldsymbol{A}(\boldsymbol{R}, t) - \frac{m_e}{m_e + m_p}(\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A}(\boldsymbol{R}, t). \quad \text{(A5b)}$$

In order to express the Hamiltonian in terms of physical fields $\boldsymbol{E}$ and $\boldsymbol{B}$ only, we first perform the Legendre transformation on (A4) and find the appropriate Lagrangian:

$$L = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}M_r\dot{\boldsymbol{r}}^2 + e\dot{\boldsymbol{r}}\boldsymbol{A} + \frac{e^2}{4\pi\varepsilon_0 r} + \boldsymbol{\mu}_e\boldsymbol{B} + e\dot{\boldsymbol{R}}(\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A} + e\frac{\Delta M}{M}\dot{\boldsymbol{r}}(\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A}. \quad \text{(A6)}$$

This Lagrangian may be modified by subtracting from it a total derivative over time:

$$\tilde{L} = L - \frac{d}{dt}\left[e\boldsymbol{r}\boldsymbol{A} + \frac{e}{2}\frac{\Delta M}{M}\dot{\boldsymbol{r}}(\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A}\right]. \quad \text{(A7)}$$

In the radiation gauge, where $A_0 = 0$ and $\nabla\boldsymbol{A} = 0$, we have $\boldsymbol{E} = -\dot{\boldsymbol{A}}$, and (A7) may be given the form

$$\tilde{L} = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}M_r\dot{\boldsymbol{r}}^2 + \frac{e^2}{4\pi\varepsilon_0 r} + \boldsymbol{\mu}_e\boldsymbol{B} + e\boldsymbol{r}\boldsymbol{E} - e\dot{\boldsymbol{R}}(\boldsymbol{r}\times\boldsymbol{B}) - \frac{e}{2}\frac{\Delta M}{M}\dot{\boldsymbol{r}}(\boldsymbol{r}\times\boldsymbol{B}) + \frac{e}{2}\frac{\Delta M}{M}(\boldsymbol{r}\nabla_{\boldsymbol{R}})(\boldsymbol{r}\boldsymbol{E}), \quad \text{(A8)}$$

where we made use of the identity

$$\nabla_{\boldsymbol{R}}(\boldsymbol{r}\boldsymbol{A}) - (\boldsymbol{r}\nabla_{\boldsymbol{R}})\boldsymbol{A} = \boldsymbol{r}\times\boldsymbol{B}. \quad \text{(A9)}$$

Now performing the inverse Legendre transformation, we obtain the modified Hamiltonian (from now on we omit the tilde)

$$H = \frac{1}{2M}(\boldsymbol{P} + e\boldsymbol{r}\times\boldsymbol{B})^2 + \frac{1}{2M_r}\left(\boldsymbol{p} + \frac{e}{2}\frac{\Delta M}{M}\boldsymbol{r}\times\boldsymbol{B}\right)^2 - \frac{e^2}{4\pi\varepsilon_0 r} - \boldsymbol{\mu}_e\boldsymbol{B} - e\boldsymbol{r}\boldsymbol{E} - \frac{e}{2}\frac{\Delta M}{M}(\boldsymbol{r}\nabla_{\boldsymbol{R}})(\boldsymbol{r}\boldsymbol{E}). \quad \text{(A10)}$$

We are not interested in the internal atomic structure but in the motion of a neutral particle, for instance, a hydrogen atom, as a whole. We therefore separate the center-of-mass motion from the internal one. All terms that do not depend on $\boldsymbol{R}$ are then irrelevant. Furthermore, since $r \sim a_0$, where a_0 is the Bohr radius, we can neglect terms which are of order r^2. The same refers to those that are linear in r but are accompanied by the magnetic field $\boldsymbol{B}$. The term $e\boldsymbol{r}\boldsymbol{E}$ is of order $\varepsilon_0 a_0^3 E^2$ and may be omitted too. One should also recall that the external electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, due to the approximations (A5), depend only on $\boldsymbol{R}$. We are then left with a rather obvious form

$$H_{\text{c.m.}} = \frac{\boldsymbol{P}^2}{2M} - \boldsymbol{\mu}_e\boldsymbol{B}, \quad \text{(A11)}$$

which will constitute the starting point for considering the Schrödinger-Pauli equation for the hydrogen atom, as well as for any neutral particle endowed with the magnetic moment, such as a neutron.

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