

Conditional limit measure of a one-dimensional quantum walk with an absorbing sink

Mohamed Sabri and Etsuo Segawa

Graduate School of Information Sciences, Tohoku University, Aoba, Sendai 980-8579, Japan

Martin Štefaňák

Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Břehová 7, 115 19 Praha 1–Staré Město, Czech Republic

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We consider a two-state quantum walk on a line where after the first step an absorbing sink is placed at the origin. The probability of finding the walker at position j , conditioned on that it has not returned to the origin, is investigated in the asymptotic limit. We prove a limit theorem for the conditional probability distribution and show that it is given by the Konno's density function modified by a prefactor ensuring that the distribution vanishes at the origin. In addition, we discuss the relation to the problem of recurrence of a quantum walk and determine the Pólya number. Our approach is based on path counting and the stationary phase approximation.

DOI: [10.1103/PhysRevA.98.012136](https://doi.org/10.1103/PhysRevA.98.012136)**I. INTRODUCTION**

Quantum walks [1–3] emerged during the 1990s as quantum mechanical extensions of classical random walks on a graph or a lattice, although similar ideas had appeared already in 1960s in the works of Feynman and Hibbs on discretization of the Dirac equation [4] and in the 1980s in the work of Gudder on quantum graphic dynamics [5,6]. Since then they have attracted considerable attention due to their potential applications in quantum information [7], e.g., in quantum search algorithms [8] and testing graph isomorphism [9].

Properties of quantum walks on infinite lattices are usually investigated in the asymptotic limit of large number of steps n . For homogeneous quantum walks on a line one can prove weak-limit theorems which show their ballistic spreading. The proofs are based on either a path-counting and combinatorial approach [10–12] or Fourier transformation [13]. The latter method allows for straightforward extensions to quantum walks with larger internal degrees of freedom [14,15] (i.e., larger coin space) and higher-dimensional lattices [16–19]. For quantum walks without translational invariance it is significantly more difficult to derive the explicit shape of the limit distribution; however, analytical treatment is still tractable in some cases. For example, the so-called CGMV method [20], which utilizes matrix-valued orthogonal Laurent polynomials [21], can be applied to quantum walks on a line [22] or a half-line [23] with position-dependent coins. Moreover, in terms of scattering theories, if the perturbation on the one-dimensional lattice is in the trace class, the convergence in law can be shown [24].

In the present paper we consider a discrete-time quantum walk on a line where an absorbing sink is placed at the origin after the first step. The sink is modeled by a projection operator which sets the amplitude at the origin to zero. Hence, the overall evolution is not unitary. Quantum walks with sinks on various graphs were studied in the literature, focusing on the absorption probability [25,26] and its time dependence [27–29]. We investigate the part of the wave function which

survives the absorption in the limit of infinite number of steps. It is shown that the conditional probability distribution to find the walker at position j , conditioned on the fact that the walker was not absorbed at the origin, converges weakly to a limit measure. Our proof is based on the path-counting approach [30] and stationary phase approximation [31].

The rest of the paper is organized as follows: we review the usual unitary two-state quantum walk on a line in Sec. II and set the notation. The main results of our paper are presented in Sec. III, where we state the limit theorem for the quantum walk with an absorbing sink at the origin. The properties of the conditional limit measure are illustrated by several examples. The detailed proof of the theorem is left for the Appendix. In Sec. IV we discuss the relation of our results to the problem of recurrence and show that the conditional limit measure can be used to determine the Pólya number of a quantum walk. We conclude in Sec. V.

II. PRELIMINARY

In this section we set the notation and review the usual unitary quantum walk. We consider a two-state coined quantum walk on an integer lattice \mathbb{Z} . The Hilbert space of the walk is given by the tensor product of $\ell^2(\mathbb{Z})$ corresponding to the position of the walker on the lattice and \mathbb{C}^2 describing the internal coin state:

$$\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 = \{\psi : \mathbb{Z} \rightarrow \mathbb{C}^2 \mid \|\psi\|_{\mathcal{H}} < \infty\}.$$

The scalar product in \mathcal{H} is defined by

$$\langle \varphi, \phi \rangle_{\mathcal{H}} = \sum_{j \in \mathbb{Z}} \langle \varphi(j), \phi(j) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{C}^2 . The standard basis of \mathbb{C}^2 corresponds to the coin states L and R , which determines the direction of the hopping of the walker to the left or to the right. Index j labels the position of the walker on the lattice \mathbb{Z} . The two-component complex vector

$\psi(j) \in \mathbb{C}^2$ consists of probability amplitudes of the walker being on the vertex j with the coin states L and R , i.e., $|\psi(j)|^2$ is the probability to find the walker at position j . The discrete-time evolution of the quantum walk is described by the unitary operator U . The operator U can be decomposed into a product of the shift operator S and the coin operator C . The coin operator C does not change the position of the walker, and it acts on the state ψ as

$$(C\psi)(j) = C_j\psi(j),$$

where C_j is a 2×2 unitary matrix which can be position dependent:

$$C_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

Throughout the paper we consider

$$C_j = \begin{cases} C_+ = \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix} & : j > 0, \\ C_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} & : j = 0, \\ C_- = \begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix} & : j < 0, \end{cases}$$

i.e., the coin operator C can act in principle differently at the origin ($j = 0$) and on the positive and negative semiaxis. We assume that the coin is mixing, i.e., $|a_{\pm,0}| \neq 0$ or 1 . The shift S acts nontrivially on the position space $\ell^2(\mathbb{Z})$. Depending on the coin state (L or R) it moves the walker to the left (i.e., from j to $j - 1$) or to the right (from j to $j + 1$). In total, the action of the operator U on the state ψ can be described by

$$(U\psi)(j) = P_{j+1}\psi(j + 1) + Q_{j-1}\psi(j - 1),$$

where

$$P_j = \begin{pmatrix} a_j & b_j \\ 0 & 0 \end{pmatrix}, \quad Q_j = \begin{pmatrix} 0 & 0 \\ c_j & d_j \end{pmatrix}, \quad \text{i.e., } P_j + Q_j = C_j.$$

The time evolution of the quantum walker according to the unitary operator U is given by

$$\psi_n = U\psi_{n-1} = U^n\psi_0.$$

The subscript denotes the number of steps taken by the walker. For the initial state ψ_0 we consider the walker starting from the origin ($j = 0$) with the coin state $\psi_c \in \mathbb{C}^2$:

$$\psi_0(j) = \begin{cases} \psi_c & : j = 0, \\ 0 & : j \neq 0. \end{cases}$$

We write the initial coin state in the form

$$\psi_c = (\alpha, \beta)^T$$

and assume that it satisfies the normalization condition

$$|\alpha|^2 + |\beta|^2 = 1.$$

The subject of interest in quantum walks is the probability $v_n(j)$ to find the walker after n steps at a position j which is given by

$$v_n(j) = \|\psi_n(j)\|^2.$$

The properties of the probability distribution $v_n(j)$ are well understood in the asymptotic limit $n \rightarrow \infty$. In particular, for the homogeneous case where $C_j = C_0$, we find the following limit theorem [10–13]:

Theorem 1. Let X_n be the position of the walker after n steps of the quantum walk, i.e., a random variable following the distribution v_n which satisfies

$$P(X_n \leq m) = \sum_{j \leq m} v_n(j).$$

Then we have

$$\lim_{n \rightarrow \infty} P(X_n/n \leq y) = \int_{-\infty}^y \rho(x) dx$$

for any $y \in \mathbb{R}$, where the limit density $\rho(x)$ is given by

$$\rho(x) = (1 - \lambda x) f_K(x; |a_0|).$$

Here f_K denotes the Konno's density function

$$f_K(x; a) = \frac{\sqrt{1 - a^2}}{\pi(1 - x^2)\sqrt{a^2 - x^2}}, \quad (1)$$

and the parameter λ depends on the initial state ψ_c according to

$$\lambda = |\alpha|^2 - |\beta|^2 + \frac{a_0\alpha\overline{b_0\beta} + \overline{a_0\alpha}b_0\beta}{|a_0|^2}.$$

We note that the limit density can be used to approximate the exact probability distribution according to the relation

$$v_n(j) \approx \frac{2}{n} \rho\left(\frac{j}{n}\right),$$

which is valid for j and n of the same parity. The factor of 2 is introduced artificially to offset the fact that the two-state walk is bipartite, i.e., $v_n(j) = 0$ for j and n of the opposite parity. The limit density captures the overall shape of the exact probability distribution and its main features, such as the positions of the peaks near $\pm|a_0|n$ which correspond to the divergencies of the limit density $\rho(x)$ for $x \rightarrow \pm|a_0|$.

III. QUANTUM WALK WITH AN ABSORBING SINK AT THE ORIGIN

Let us now turn to the central issue of the paper, which is the quantum walk with an absorbing sink at the origin. If the walker returns to the origin, it is annihilated by the sink and the walk ends. The sink sets the amplitude at the origin to zero while leaving the rest of the state untouched. Mathematically, the sink is described by the projection operator Π_0 which acts on the state ψ as

$$(\Pi_0\psi)(j) = \begin{cases} \psi(j) & : j \neq 0, \\ 0 & : j = 0. \end{cases}$$

Let us now consider the evolution of a quantum walk with the absorbing sink at the origin. The state of the walker after n steps, provided that it has survived the action of the sink (i.e., it has not crossed the origin), is given by a non-normalized vector

$$\psi_n^{(\text{sur})} = (U^{(\text{sur})})^n \psi_0.$$

Here we have defined

$$U^{(\text{sur})} = (1 - \Pi_0)U.$$

The norm of the vector $\psi_n^{(\text{sur})}$ is the survival probability up to the n th step

$$P_n^{(\text{sur})} = \|\psi_n^{(\text{sur})}\|_{\mathcal{H}}^2 = \sum_{j \in \mathbb{Z}} \|\psi_n^{(\text{sur})}(j)\|^2. \quad (2)$$

Let us now define the conditional probability distribution, which is normalized by the survival probability, as

$$v_n^{(\text{cond})}(j) = \frac{\|U \psi_{n-1}^{(\text{sur})}(j)\|^2}{P_{n-1}^{(\text{sur})}}.$$

This is a quantum walk's analog of the conditional probability on \mathbb{Z} at time n under the condition that the walker has never returned to the origin.

The main result of our paper is the following limit theorem for the conditional probability distribution:

Theorem 2. Let $X_n^{(\text{cond})}$ be a position of the quantum walker after n steps of the walk with an absorbing sink at the origin, i.e., a random variable following the distribution $v_n^{(\text{cond})}$ and satisfying

$$P(X_n^{(\text{cond})} \leq m) = \sum_{j \leq m} v_n^{(\text{cond})}(j).$$

Then we have

$$\lim_{n \rightarrow \infty} P(X_n^{(\text{cond})}/n \leq y) = \int_{-\infty}^y \rho^{(\text{cond})}(x) dx$$

for any $y \in \mathbb{R}$. The conditional limit density is given by

$$\rho^{(\text{cond})}(x) = H(x)\rho^{(+)}(x) + H(-x)\rho^{(-)}(x),$$

where $H(x)$ is the Heaviside step function and

$$\begin{aligned} \rho^{(+)}(x) &= \frac{P_+}{N(|a_+|)} \frac{4x^2}{1+x} f_K(x; |a_+|), \\ \rho^{(-)}(x) &= \frac{P_-}{N(|a_-|)} \frac{4x^2}{1+|x|} f_K(x; |a_-|). \end{aligned}$$

Here $f_K(x; a)$ is the Konno's density function and

$$\begin{aligned} P_+ &= |\langle v_0^{(+)}, \psi_c \rangle|^2, \quad \langle v_0^{(+)} \rangle = (c_0, d_0), \\ P_- &= |\langle v_0^{(-)}, \psi_c \rangle|^2, \quad \langle v_0^{(-)} \rangle = (a_0, b_0). \end{aligned}$$

By $N(a)$ we have denoted a normalization factor which reads

$$N(a) = \frac{\pi a^2 - 2a\sqrt{1-a^2} + 2(1-2a^2) \arcsin(a)}{\pi(1-a^2)}.$$

We leave the detailed proof of the theorem for the Appendix and turn to the discussion of the results.

First, we see that the conditional limit measure is determined by the Konno's density function as for the walk without the absorbing sink, however, modulated by the factor $\frac{4x^2}{1+|x|}$. This contribution ensures that $\rho^{(\text{cond})}(x)$ tends to zero for $x = 0$ as expected due to the presence of the sink. We illustrate this feature in Fig. 1 where we consider the homogeneous case with the Hadamard coin:

$$C_+ = C_- = C_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

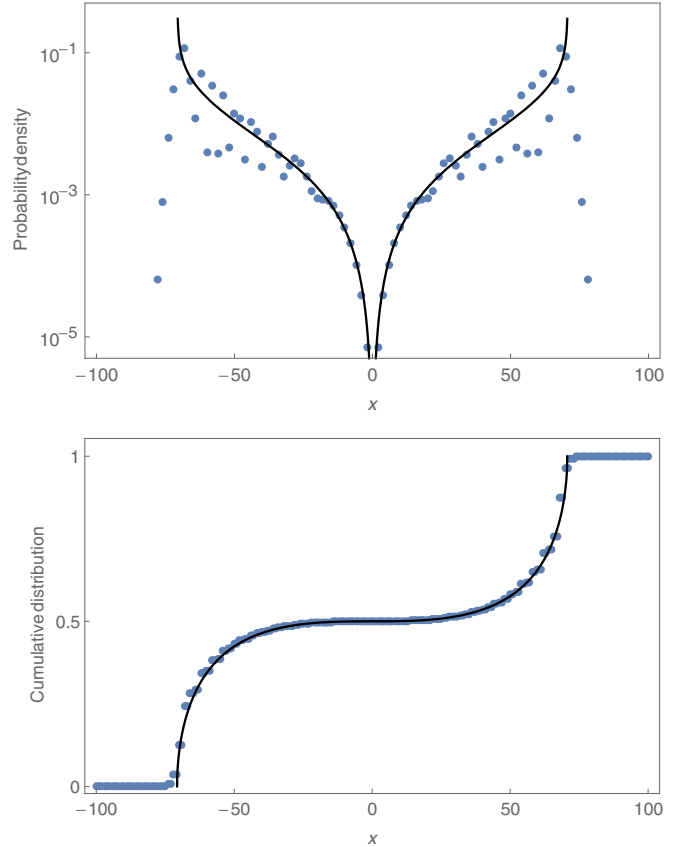


FIG. 1. In the upper plot we show the conditional probability distribution of the Hadamard walk after 100 steps with absorbing sink at the origin. Note that the plot is on the logarithmic scale. Blue dots are obtained from the numerical simulation, black curve is the approximation with the limit density of (3). The initial coin state was chosen as $\psi_c = (1, 0)^T$, which results in a symmetric conditional function where the rapid oscillations of the probability distribution are smeared out. The cumulative distribution function obtained from the limit density perfectly fits the exact numerical data.

As for the initial state we consider $\psi_c = (1, 0)^T$, where the walker in the first step jumps to the left or to the right with equal probability: $P_+ = P_- = 1/2$. Hence, the conditional limit measure is symmetric and has the form

$$\rho^{(\text{cond})}(x) = \frac{1}{2 - \frac{4}{\pi}} \frac{4x^2}{1+|x|} f_K\left(x; \frac{1}{\sqrt{2}}\right). \quad (3)$$

We note that P_{\pm} also gives the asymptotic conditional probability of finding the walker on the positive or negative half-line:

$$\begin{aligned} P_+ &= \int_0^{|a_+|} \rho^{(\text{cond})}(x) dx, \\ P_- &= \int_{-|a_-|}^0 \rho^{(\text{cond})}(x) dx. \end{aligned}$$

As the next example we consider a case with different coins for the origin and the positive and negative half-line. The sink at the origin decouples the two half-lines. Hence, the evolution of the walker on the positive (negative) semiaxis is

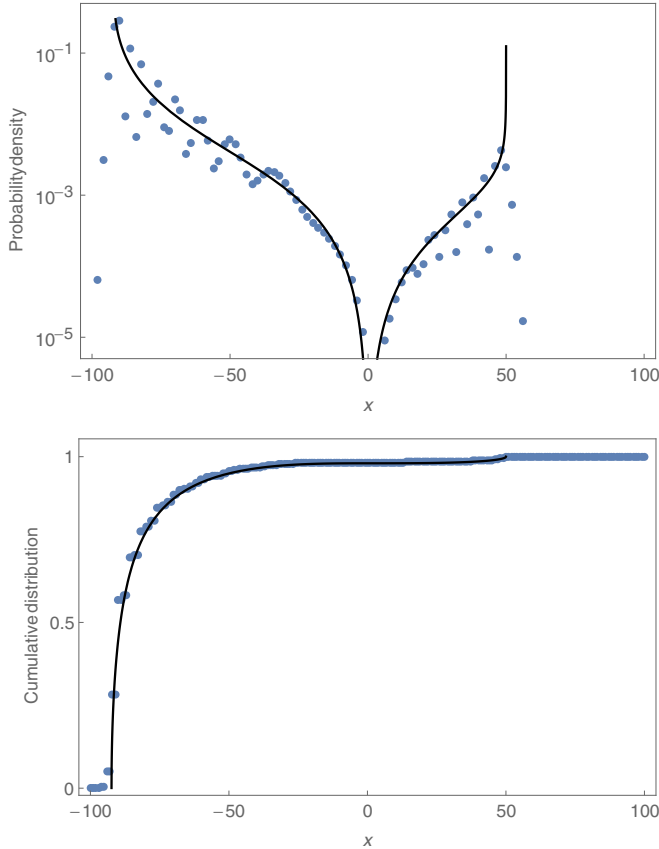


FIG. 2. Conditional probability distribution on the log scale (up) and the cumulative distribution function (down) of the walk with absorbing sink at the origin. The initial coin state was $\psi_c = (0.6, 0.8)^T$. The coins were chosen according to (4). Different coins for $j > 0$ and $j < 0$ result in asymmetric conditional distribution. In this case the walk spreads faster on the negative half-line than on the positive half-line. In addition, due to the choice of the initial coin state, the particle has a higher probability to be on the left side of the absorbing sink.

determined purely by C_+ (C_-). In Fig. 2 we have chosen the coins according to

$$\begin{aligned} C_+ &= \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \\ C_0 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ C_- &= \begin{pmatrix} \cos(\frac{\pi}{8}) & \sin(\frac{\pi}{8}) \\ -\sin(\frac{\pi}{8}) & \cos(\frac{\pi}{8}) \end{pmatrix}. \end{aligned} \quad (4)$$

Since $a_- = \cos(\frac{\pi}{8}) > a_+ = \frac{1}{2}$, the walk spreads faster on the negative half-line than on the positive. Due to the choice of the initial state $\psi_c = (0.6, 0.8)^T$ the conditional distribution is heavily biased towards left. Indeed, we find that $P_- = 0.98$ and $P_+ = 0.02$, which is also clearly visible from the plot of the cumulative distribution function.

Finally, we consider the case where the walker jumps in the first step with probability 1 to one side of the line, i.e., $P_+ = 1$ or $P_- = 1$. This depends on the relation between the coin at the origin C_0 and the initial state ψ_c . From the unitarity of the

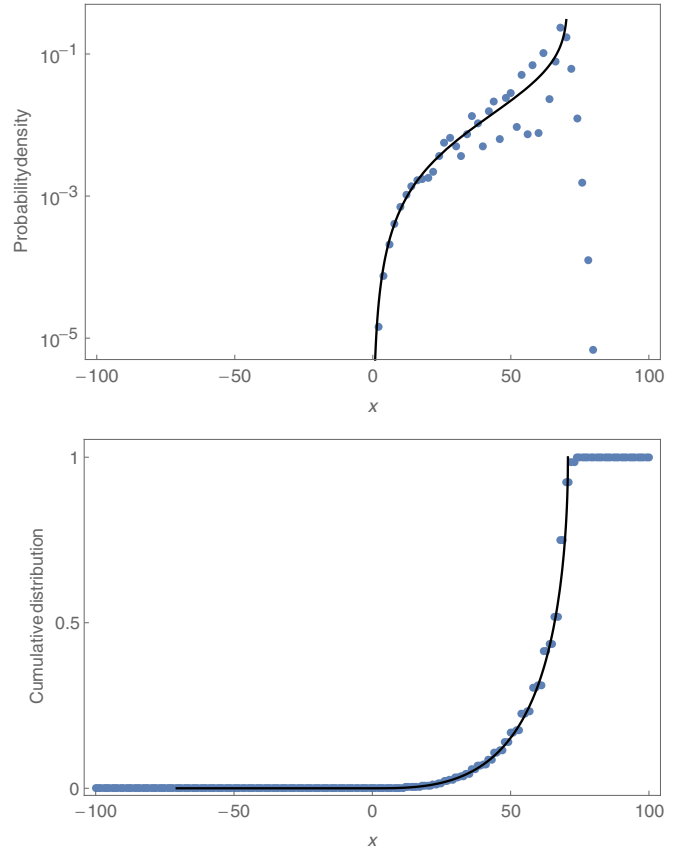


FIG. 3. Conditional probability distribution on the log-scale (up) and the cumulative distribution function (down) of the Hadamard walk with absorbing sink at the origin. The initial coin state was chosen as $\psi_c = \frac{1}{\sqrt{2}}(1, -1)^T$. In such a case the conditional distribution completely vanishes on the left side of the absorbing sink, since in the first step the walker jumps with probability 1 to the right and cannot cross the origin afterwards.

matrix C_0 we find

$$\begin{aligned} P_- = 1 &\iff \psi_c = (\overline{a_0}, \overline{b_0})^T, \\ P_+ = 1 &\iff \psi_c = (\overline{c_0}, \overline{d_0})^T. \end{aligned}$$

Since the walker cannot cross the origin due to the presence of the sink the conditional probability distribution is nonzero only on the positive or negative half-line. We illustrate this feature in Fig. 3 for the case of the Hadamard walk. The initial state was chosen as $\psi_c = \frac{1}{\sqrt{2}}(1, -1)^T$, which results in $P_- = 0$ and $P_+ = 1$. Clearly, the walker can be found only on the positive half-line.

IV. RECURRENCE AND PÓLYA NUMBER

Let us conclude by applying our results to the problem of recurrence [32]. Here we are interested in the probability that the walker returns to the origin, which is called the Pólya number. The walk is said to be recurrent if its Pólya number is equal to one, and transient otherwise. For quantum walk there are two inequivalent definitions of the Pólya number [33,34] depending on the way the return to the origin is observed. In the observation scheme introduced in Ref. [33]

the measurement is performed only once after n steps and the walk is restarted afterwards. The observation scheme in Ref. [34] requires one to monitor the origin continually after every step, and the walk ends if the walker is found. We note that in the classical case the two concepts are equivalent in the sense that recurrence under one definition implies recurrence under the other, and vice versa [35]. However, in the quantum case the two approaches lead to different results already for a quantum walk on a line, where in the restart scheme the walk is recurrent [35], while in the continual scheme it is transient [25,36]. Both observation schemes for recurrence were recently implemented in a photonic quantum walk experiment [37] which has clearly shown the difference between the two schemes arising from the projective nature of quantum measurements.

A quantum walk with the sink at the origin investigated in this paper corresponds to the continual observation scheme for recurrence, where the return to the origin is marked by the absorption of the walker [34]. In this scheme the Pólya number of the walk is given by

$$P = \sum_{k=1}^{\infty} q(0, k),$$

where $q(0, k)$ denotes the probability of the first return of the walker to the origin after k steps. The first return probability is obtained from

$$q(0, k) = \left| \left| (U\psi_{k-1}^{(\text{sur})})(0) \right|^2 \right.$$

The sum

$$P_n^{(\text{abs})} = \sum_{k=1}^n q(0, k)$$

gives the probability that the walker was absorbed by the sink until the step n . Clearly, it is the complement of the survival probability (2):

$$P_n^{(\text{abs})} = 1 - P_n^{(\text{sur})}.$$

Since the Pólya number of the quantum walk is given by the asymptotic value of the absorption probability, we find

$$P = 1 - \lim_{n \rightarrow \infty} P_n^{(\text{sur})}.$$

As we discuss in the Appendix, the limiting value of the survival probability is given by

$$\lim_{n \rightarrow \infty} P_n^{(\text{sur})} = P_+ N(|a_+|) + P_- N(|a_-|).$$

Hence, in the continual measurement scheme the Pólya number for a two-state quantum walk on a line is given by

$$P = 1 - P_+ N(|a_+|) - P_- N(|a_-|).$$

We see that when the coin operator acts differently on the positive and negative half-line the Pólya number depends on the initial coin state ψ_c through the probabilities P_{\pm} . For $|a_+| = |a_-| = a$ the dependence on ψ_c vanishes and we find that the Pólya number is given by

$$P = 2 \frac{a\sqrt{1-a^2} + (1-2a^2)\arccos(a)}{\pi(1-a^2)},$$

which coincides with the value obtained in Ref. [38]. In the particular case of the Hadamard walk corresponding to $a = \frac{1}{\sqrt{2}}$ the Pólya number reduces to $P = \frac{2}{\pi}$ in accordance with the findings of Refs. [25,36].

V. CONCLUSIONS

In this paper we have investigated the asymptotic properties of the quantum walk on a line with an absorbing sink at the origin. We have shown that the position of the quantum walker rescaled by the number of steps $X_n^{(\text{cond})}/n$ converges in the limit $n \rightarrow \infty$, similarly like for the unitary quantum walk. The explicit form of the conditional limit measure was derived. Finally, the relation to recurrence of a quantum walker was discussed. In particular, we have derived the Pólya number in dependence on the initial state and the parameters of the quantum coin.

It would be interesting to investigate quantum walks with an absorbing sink on higher-dimensional lattices. For a one-dimensional lattice considered in the present paper the sink at the origin separates the dynamics into two independent quantum walks on a half-line. This is no longer true for higher-dimensional lattices. One can anticipate that the conditional limit density will exhibit more complicated patterns depending also on the initial state. However, this investigation is beyond the scope of the present paper.

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APPENDIX: PROOF OF THEOREM 2

Let $\Omega_n = \{1, -1\}^n$. For $\xi = (\xi_1, \dots, \xi_n) \in \Omega_n$, we define $\sigma_m = \xi_1 + \dots + \xi_m$ ($m \leq n$). We define the weight of path: $w : \Omega_n \rightarrow M_2(\mathbb{C}_2)$ such that

$$w(\xi) = P_{\xi_n, \sigma_{n-1}} \cdots P_{\xi_2, \sigma_1} P_{\xi_1, 0}$$

for $\xi = (\xi_1, \dots, \xi_n)$, where

$$P_{1,x} = Q_{\text{sgn}(x)} \text{ and } P_{-1,x} = P_{\text{sgn}(x)}.$$

We put $\Theta_j(n) := \{\xi \in \Omega_n \mid \sigma_0 = 0, \sigma_1 \neq 0, \dots, \sigma_{n-1} \neq 0, \sigma_n = j\}$, which is the set of all the possible paths from the origin to j without the absorption. In the following we consider only the positive half-line $j \geq 1$, since the proof for the negative half-line is analogous. The weight of the paths avoiding to touch the absorbing sink is denoted by $\Xi_j^{(\text{sur})}(n) = \sum_{\xi \in \Theta_j(n)} w(\xi)$. Remark that the non-normalized amplitude at position j after n steps can be obtained from

$$\psi_n^{(\text{sur})}(j) = \Xi_j^{(\text{sur})}(n)\psi_c.$$

Therefore, the weight $\Xi_j^{(\text{sur})}(n)$ will play an important role. We introduce the generating function of $\Xi_j^{(\text{sur})}(\cdot)$ with respect to

time series n ,

$$\tilde{\Xi}_j^{(\text{sur})}(z) = \sum_{n=0}^{\infty} \Xi_j^{(\text{sur})}(n)z^n,$$

for a complex value z . We obtain the following lemma for the explicit expression of $\tilde{\Xi}_j^{(\text{sur})}(z)$.

Lemma 1. Put $\mathbf{e}^{i\delta} = \det(C_+)$, $\Sigma = \{\theta \in [0, 2\pi) \mid |c_+| \geq |\sin \theta|\}$ and $\Sigma^c = [0, 2\pi] \setminus \Sigma$. Let $\tilde{\Xi}_j^{(\text{sur})}(z)$ be the above. Then we have

$$\tilde{\Xi}_j^{(\text{sur})}(z) = \tilde{\lambda}^{j-1}(z)|\tilde{u}(z)\rangle\langle v_0^{(+)}| \quad (j \geq 1), \quad (\text{A1})$$

where $|\tilde{u}(z)\rangle = (\tilde{\lambda}(z)\tilde{f}(z), z)^T$ and $\langle v_0^{(+)}| = (c_0, d_0)$. Here for $z = \mathbf{e}^{i(\theta-\delta/2)}$, $\tilde{\lambda}(z)$ and $\tilde{f}(z)$ are given by

$$\tilde{f}(\mathbf{e}^{i(\theta-\delta/2)}) = -e^{i\theta} \frac{|c_+|}{c_+} \begin{cases} \mathbf{e}^{i\eta(\theta)} & : \theta \in \Sigma^c, \\ \text{isgn}(\sin \theta) \left(\left| \frac{\sin \theta}{c_+} \right| - \sqrt{\left| \frac{\sin \theta}{c_+} \right|^2 - 1} \right) & : \theta \in \Sigma. \end{cases} \quad (\text{A2})$$

$$\tilde{\lambda}(\mathbf{e}^{i(\theta-\delta/2)}) = \mathbf{e}^{i\delta/2} \frac{|a_+|}{a_+} \begin{cases} \text{sgn}(\cos \theta) \left(\left| \frac{\cos \theta}{a_+} \right| - \sqrt{\left| \frac{\cos \theta}{a_+} \right|^2 - 1} \right) & : \theta \in \Sigma^c, \\ \mathbf{e}^{i\kappa(\theta)} & : \theta \in \Sigma. \end{cases} \quad (\text{A3})$$

where

$$\begin{aligned} \sin \eta(\theta) &= \frac{\sin \theta}{|c_+|}, \quad \text{sgn}(\cos \eta(\theta)) = \text{sgn}(\cos \theta), \\ \cos \kappa(\theta) &= \frac{\cos \theta}{|a_+|}, \quad \text{sgn}(\sin \kappa(\theta)) = -\text{sgn}(\sin \theta). \end{aligned} \quad (\text{A4})$$

Proof of Lemma 1. Replacing $\tilde{\Xi}_0(z)$ with $\tilde{\Xi}_0^{(\text{sur})}(z) = I$ in Lemma 3.1 in Ref. [39], we obtain (A1). The explicit expressions for \tilde{f} and $\tilde{\lambda}$; (A2) and (A3), respectively, can be referred by (3.25) in Ref. [39].□

We use the notation

$$\tilde{\psi}_j^{(\text{sur})}(z) = \tilde{\Xi}_j^{(\text{sur})}(z)\psi_c.$$

By the Cauchy's coefficient formula, the amplitude at time n and position j without the absorption $\psi_n^{(\text{sur})}(j)$ is expressed by

$$\begin{aligned} \psi_n^{(\text{sur})}(j) &= \mathbf{e}^{in\delta/2} \int_0^{2\pi} \tilde{\psi}_j^{(\text{sur})}(\mathbf{e}^{i(\theta-\delta/2)}) \mathbf{e}^{-in\theta} \frac{d\theta}{2\pi} \\ &= \mathbf{e}^{in\delta/2} [I_j(n) + J_j(n)], \end{aligned}$$

where we have introduced

$$\begin{aligned} I_j(n) &= \int_{\Sigma} \tilde{\psi}_j^{(\text{sur})}(\mathbf{e}^{i(\theta-\delta/2)}) \mathbf{e}^{-in\theta} \frac{d\theta}{2\pi}, \\ J_j(n) &= \int_{\Sigma^c} \tilde{\psi}_j^{(\text{sur})}(\mathbf{e}^{i(\theta-\delta/2)}) \mathbf{e}^{-in\theta} \frac{d\theta}{2\pi}. \end{aligned}$$

Let us first consider $J_j(n)$. By Lemma 1, since $\theta \in \Sigma^c$, the absolute value of $\tilde{\lambda}$ is bounded as $|\tilde{\lambda}| < 1$. Therefore we have

$$\begin{aligned} J_j(n) &= \langle v_0^{(+)}|, \psi_c \rangle \int_{\Sigma^c} \mathbf{e}^{-in\theta} \{\tilde{\lambda}(\mathbf{e}^{i(\theta-\delta/2)})\}^{j-1} \\ &\quad \times \left(-\tilde{\lambda}[\mathbf{e}^{i(\theta-\delta/2)}] \mathbf{e}^{i(\theta+\eta(\theta))} |c_+|/c_+ \right) \frac{d\theta}{2\pi} \xrightarrow{(n \rightarrow \infty)} 0, \end{aligned}$$

which is an exponentially decay to 0. Therefore, $J_j(n)$ does not contribute to the limit density. Let us turn to $I_j(n)$. In this case $\theta \in \Sigma$ and by Lemma 1 $\tilde{\lambda}$ becomes a unit complex value. We find that for fixed $y \in \mathbb{R}_+$ the integral $I_{[ny]}(n)$ is

expressed by

$$I_{[ny]}(n) = \mathbf{e}^{i\sigma} \langle v_0^{(+)}|, \psi_c \rangle \int_{\Sigma} \mathbf{e}^{i[n\gamma\kappa(\theta)-\theta] + \beta_n(\theta)} |\tilde{u}(\mathbf{e}^{i(\theta-\delta/2)})\rangle \frac{d\theta}{2\pi},$$

where $\beta_n(\theta) = ([ny] - ny)\kappa(\theta) = O(1)$ and $\sigma = \delta/2 - \arg(a)$. Here $|\tilde{u}$ is expressed by

$$|\tilde{u}(\mathbf{e}^{i(\theta-\delta/2)})\rangle = \left(\mathbf{e}^{i\omega(\theta)} \left[|\sin \theta/c_+|^2 - \sqrt{|\sin \theta/c_+|^2 - 1} \right], \mathbf{e}^{i(\theta-\delta/2)} \right),$$

where

$$\mathbf{e}^{i\omega(\theta)} = -\mathbf{i} \frac{|a_+||c_+|}{a_+c_+} \mathbf{e}^{i\delta/2} \text{sgn}(\sin \theta) \mathbf{e}^{i[\kappa(\theta)-\theta]}.$$

We set $g(\theta) := \gamma\kappa(\theta) - \theta$. The singular points of $g(\cdot)$ on $[0, 2\pi)$ are described by $\{\theta_*^{(1)}, \dots, \theta_*^{(4)}\}$. Now we can apply the following kind of extended stationary phase method. This lemma is obtained by following Ref. [31] even if we modify the function in the exponent by adding a function $\omega(x)$ with some conditions. We skip the proof in this paper.

Lemma 2. (An extended stationary phase method) Let $h : [x_*, b] \rightarrow \mathbb{R}$ be a C^2 function and $h'(x_*) = 0, h''(x_*) \neq 0$ and $h'(x) \neq 0$. ($x_* < x \leq b$). Let $\omega : [x_*, b] \times \mathbb{N} \rightarrow \mathbb{R}$ be a C^1 function with respect to $x \in [x_*, b]$, and satisfy the following condition:

$$\begin{aligned} \sup_{x,n} |w(x, n)|, \sup_{x,n} |w'(x, n)| &< c_1, \\ |w(n, x) - w(n, x')| &< c_2|x - x'|, \end{aligned}$$

where c_1 and c_2 are constant, which are independent of x and n . Then for $n \rightarrow \infty$ we find

$$\begin{aligned} \int_{x_*}^b \mathbf{e}^{i[nh(x)+\omega(x,n)]} dx &= \frac{1}{\sqrt{n}} \sqrt{\frac{\pi}{2|h''(x_*)|}} \mathbf{e}^{i\text{sgn}[h''(x_*)\pi/4]} \\ &\quad \times \mathbf{e}^{i[nh(x_*)+\omega(x_*,n)]} + O(1/\sqrt{n}). \end{aligned}$$

In our case Lemma 2 implies

$$e^{-i\sigma} I_{[ny]}(n) / \langle v_0^{(+)} , \psi_c \rangle = \sum_{j=1}^4 \frac{1}{\sqrt{n}} \sqrt{\frac{2\pi}{|y\kappa''(\theta_*^{(j)})|}} e^{i\text{sgn}[\kappa''(\theta_*^{(j)})\pi/4]} \times e^{i[n\kappa(\theta_*^{(j)})+\beta_n(\theta_*^{(j)})]}\tilde{u}(\mathbf{e}^{i(\theta_*-\delta/2)}) \frac{1}{2\pi} + O(1/\sqrt{n}).$$

Hence, for the square norm we find

$$\|I_{[ny]}(n)\|^2 = \frac{1}{n} \left[P_+ \sum_{j=1}^4 \frac{1}{|y\kappa''(\theta_*^{(j)})|} \times \|\tilde{u}(\mathbf{e}^{i(\theta_*^{(j)}-\delta/2)})\|^2 \frac{1}{2\pi} + \text{cross terms} \right] + O(1/n).$$

Since $y\kappa'(\theta_*) - 1 = 0$, using the formula of (A4) we obtain

$$\frac{1}{\pi |y\kappa''(\theta_*^{(j)})|} = y^2 f_K(y; |a_+|),$$

$$\|\tilde{u}(\mathbf{e}^{i(\theta_*^{(j)}-\delta/2)})\|^2 = 1 + |\tilde{f}(\mathbf{e}^{i(\theta_*^{(j)}-\delta/2)})|^2 = \frac{2}{1+y}.$$

Since

$$\int_0^y \|I_{[nx]}(n)\|^2 dx = \frac{1}{n} \left[\sum_{j=1}^{[ny]} \|I_j(n)\|^2 + O(1) \right],$$

and the ‘‘cross terms’’ contain ‘‘ $e^{in(\theta_*^{(l)}-\theta_*^{(m)})+\beta_n(\theta_*^{(l)})-\beta_n(\theta_*^{(m)})}$ ’’, we can use Lemma 2 again and find

$$\sum_{j=1}^{[ny]} \|I_j(n)\|^2 \xrightarrow{(n \rightarrow \infty)} \int_0^y P_+ \frac{4x^2}{1+x} f_K(x; |a_+|) dx.$$

This implies that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \|\psi_n^{(\text{sur})}(j)\|^2 = P_+ N(|a_+|),$$

where

$$N(|a_+|) = \int_0^{|a_+|} \frac{4x^2}{1+x} f_K(x; |a_+|) dx.$$

The conditional limit density for the positive half-line is then given by performing a proper normalization:

$$\rho^{(+)}(x) = \frac{P_+}{N(|a_+|)} \frac{4x^2}{1+x} f_K(x; |a_+|).$$

For the negative half-line we proceed analogously and obtain the conditional limit density $\rho^{(-)}(x)$. We note that the limiting value of the survival probability (2) is given by

$$\lim_{n \rightarrow \infty} P_n^{(\text{sur})} = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \|\psi_n^{(\text{sur})}(j)\|^2 + \lim_{n \rightarrow \infty} \sum_{j=-1}^{-\infty} \|\psi_n^{(\text{sur})}(j)\|^2 = P_+ N(|a_+|) + P_- N(|a_-|).$$

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