Probability density of relativistic spinless particles

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In this paper, a conserved current for Klein–Gordon equation is derived. It is shown, for $(1+1)$ dimensions, the first component of this current is non-negative and reduces to $|\phi|^2$ in nonrelativistic limit. Therefore, it can be interpreted as the probability density of spinless particles. In addition, main issues pertaining to localization in relativistic quantum theory are discussed, with a demonstration on how this definition of probability density can overcome such obstacles. Our numerical study indicates that the probability density deviates significantly from $|\phi|^2$ only when the uncertainty in momentum is greater than m_0c.

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I. INTRODUCTION

The Born interpretation of $|\psi|^2$ as position probability density is one of the most fundamental axioms of quantum mechanics because it provides a link between the mathematical formalism and empirical results [1]. This axiom has been incredibly successful in predicting position probability density in nonrelativistic quantum mechanics. Although in the relativistic regime, the quadratic relation between position probability density and wave function has been confirmed by recent high-accuracy single-photon multislit experiments [2–6], a satisfactory mathematical expression for position probability density of relativistic bosons has not yet been found. In the simplest case, finding a well-defined position probability density for the free spinless particles is a long-standing problem (see, e.g., Refs. [7,8]): The time component of the well-known Klein–Gordon conserved current, $J^{\mu}_{KG}{=}i(\phi^*\partial^\mu\phi-\phi\partial^\mu\phi^*)$, may be negative on some regions of spacetime and cannot be interpreted as position probability density [9]. One may suggest to use the $|\phi|^2$ as probability density, similar to the nonrelativistic theory [10–12], in which case it is easy to see that the Klein–Gordon equation,

$$\Box\phi+m^2\phi=0, \tag{1}$$

leads to the following continuity equation for $|\phi|^2$ [10]:

$$\partial_t\rho_B+\nabla\cdot\mathbf{J}_B=0, \tag{2}$$

where

$$\rho_B=|\phi(x)|^2=N\int\tilde{\phi}(p)\tilde{\phi}^*(k)e^{i(p-k)\cdot x}\,d^4p\,d^4k, \tag{3}$$

$$\mathbf{J}_B=N\int\tilde{\phi}(p)\tilde{\phi}^*(k)\,e^{i(p-k)\cdot x}\,\mathbf{u}(p,k)\,d^4p\,d^4k, \tag{4}$$

$$\mathbf{u}(p,k)=\frac{\mathbf{p}+\mathbf{k}}{p_0+k_0}, \tag{5}$$

and $\tilde{\phi}(p)$ is the Fourier-transformation of the wave function. It should be noted that, despite the fact that $|\phi|^2$ is non-negative and conserved, it cannot be considered as position probability density: because of the Lorentz length contraction, the probability density cannot be a scalar [13]. In other words, $J^\mu_B=(\rho_B,\mathbf{J}_B)$ is not a four-vector and therefore cannot be interpreted as a relativistic probability current density [12]. In addition, Born's probability density leads to faster-than-light particle propagation [14,15]. In principle, a reasonable probability current must satisfy the following conditions:

(I) Lorentz transformation: $J^{\mu'}=\Lambda^{\mu'}_{\mu}J^\mu$;
(II) probability conservation: $\partial_\mu J^\mu=0$;
(III) future-orientation: $J^0\geqslant 0$;
(IV) causal propagation: $J^\mu J_\mu\geqslant 0$.

The last condition is necessary because it ensures the causal propagation of particles. In fact, there are several other currents which have been suggested for the Klein–Gordon equation [16,17] all of which do not satisfy, at least, one of the above conditions. The aim of this paper is to propose a proper expression for relativistic probability current that satisfies all of the aforementioned conditions.

II. POSITION DISTRIBUTION

According to Eqs. (3) and (4), we suggest the following expression as the relativistic probability current [18]:

$$J^\mu{=}\int\tilde{\phi}(p)\,\tilde{\phi}^*(k)\,e^{i(p-k)\cdot x}\,u^\mu(p,k)\,d^4p\,d^4k, \tag{6}$$

where $u^\mu(p,k)$ is an unknown function that must be determined by theoretical constrains. In this regard, the condition (I) implies that the $u^\mu(p,k)$ is a four-vector. The general form of a four-vector made by p and k is given by

$$u^\mu(p,k)=\alpha(p^\mu+k^\mu)+\beta(p^\mu-k^\mu), \tag{7}$$

where α and β are scalar coefficients. Next, the conservation condition (II) leads to $\beta=0$. Also, in principle, the coefficient α should be determined using conditions (III) and (IV). This procedure in $(1+1)$ dimensions is straightforward and a

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possible choice is (see Appendix A)

$$\alpha(p,k) = \frac{\xi}{\sqrt{(p+k)^2}}, \tag{8}$$

where $\xi = \frac{1}{2}(\frac{k^0}{|k^0|} + \frac{p^0}{|p^0|})$. So, finally, we get the $u^\mu(p,k)$ as follows:

$$u^\mu(p,k) = \xi \frac{p^\mu + k^\mu}{\sqrt{(p+k)^2}}. \tag{9}$$

Equation (9) can be rewritten as $u^\mu = |\xi|\gamma(1,\mathbf{u})$, in which $\mathbf{u}$ is the velocity vector defined in Eq. (5) and $\gamma = (1-\mathbf{u}^2)^{-1/2}$ is the corresponding Lorentz coefficient. In fact, the expression (6) is the simplest covariant generalization of Eqs. (3) and (4). The only difference between this expression and the Born probability current J_B^μ is the factor $|\xi|\gamma$:

$$\rho(x) = N\int |\xi|\gamma\tilde{\phi}(p)\tilde{\phi}^*(k)\, e^{i(p-k)\cdot x}\, d^4p\, d^4k, \tag{10}$$

$$\mathbf{J}(x) = N\int |\xi|\gamma\mathbf{u}\, \tilde{\phi}(p)\tilde{\phi}^*(k) e^{i(p-k)\cdot x}\, d^4p\, d^4k. \tag{11}$$

The factor γ comes naturally in accordance with Lorentz contraction and the factor $|\xi|$ prohibits the occurrence of *Zitterbewegung* behavior [19,20]. In Appendix A, it is shown that, for massive particles in $(1+1)$ dimensions, Eqs. (10) and (11) can be rewritten in position representation as follows:

$$\rho = |\mathcal{D}^+\phi_+|^2 + |\mathcal{D}^-\phi_+|^2 + |\mathcal{D}^+\phi_-|^2 + |\mathcal{D}^-\phi_-|^2, \tag{12}$$

$$J = (|\mathcal{D}^+\phi_+|^2 - |\mathcal{D}^-\phi_+|^2 + |\mathcal{D}^+\phi_-|^2 - |\mathcal{D}^-\phi_-|^2)c, \tag{13}$$

where $\phi_\pm$ are positive- and negative-frequency components of wave function, $\phi = \phi_+ + \phi_-$, and $\mathcal{D}^\pm$ are pseudo-differential operators which are defined as follows:

$$\mathcal{D}^\pm \equiv \sqrt{\frac{1}{2}\left(\sqrt{1-\lambda_c^2\frac{d^2}{dx^2}} \mp i\lambda_c\frac{d}{dx}\right)}, \tag{14}$$

in which $\lambda_c \equiv \hbar/mc$ is the Compton wavelength. From Eqs. (12) and (13) it is clear that, the probability density is unambiguously positive definite and $|J/\rho| \leqslant c$.

It is clear that when the wave function only has a positive-energy part [21], $\phi = \phi_+$, the Klein Gordon equation leads to

$$i\hbar\frac{\partial\phi}{\partial t} = \sqrt{-\nabla^2 + m^2}\phi, \tag{15}$$

and Eqs. (12) and (13) reduce to the following simpler forms:

$$\rho = |\mathcal{D}^+\phi|^2 + |\mathcal{D}^-\phi|^2, \tag{16}$$

$$J = (|\mathcal{D}^+\phi|^2 - |\mathcal{D}^-\phi|^2)c. \tag{17}$$

In this case, in the nonrelativistic regime ($c \to \infty$), Eq. (15) reduces to the nonrelativistic Schrödinger equation, also Eqs. (16) and (17) reduce to nonrelativistic probability density $|\phi|^2$ and the conventional Schrödinger probability current, $(\hbar/m)\mathrm{Im}(\phi^*\partial_x\phi)$, respectively.

For comparing the relativistic probability density (16) with $|\phi|^2$, in Fig. 1, we plot χ (a measure of deviation from Born probability), which is defined as

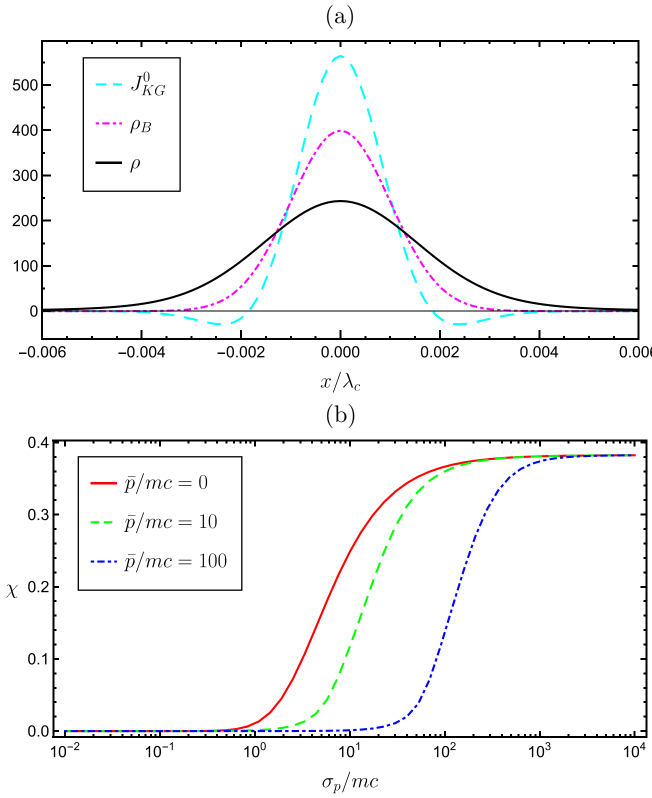

FIG. 1. (a) The first component of Klein–Gordon current J^0_{KG} (dashed line), the Born probability density ρ_B (dash-dotted line), and the relativistic probability density ρ (solid line) referring to the Gaussian wave function (19) with $\sigma_p/mc = 1000$ and $\bar{p}/mc = 0$. (b) Represents the χ for Gaussian wave function (19).

$$\chi = \int_{-\infty}^{\infty} |\rho - |\phi|^2| dx, \tag{18}$$

for this Gaussian wave function

$$\tilde{\phi}(p) = Ne^{-(p-\bar{p})^2/\sigma_p^2}. \tag{19}$$

From Fig. 1(b) it is clear that, when momentum uncertainty is small compared with mc, relativistic probability density deviation from Born probability density is negligible, even assuming that the group velocity of the wave packet is comparable with velocity of light.

Finally, note that, although the expression for probability density in terms of wave function (16) is nonlocal, there is no inconsistency with special relativity. In fact, this nonlocality is essential to introduce a self-consistent relativistic probability density; since the relativistic wave function can propagate outside the light cone, a local relation between wave function and probability density, for instance $\rho = |\phi|^2$, leads to faster-than-light particle propagation [14,15]. In the following section, the relativistic requirements imposed on the definition of probability density is further discussed together with an account of how our suggested expression satisfy them.

III. LOCALIZATION AND CAUSALITY

It is well known there are some problems with the concept of a "localized particle" in relativistic quantum mechanics [22–34]. The notion of localization is closely related to the concept of position probability density. In this section we briefly review these problems and demonstrate how our definition of relativistic probability density can circumvent such obstacles.

One of the earliest attempts to analyze the notion of localized particle in relativistic quantum mechanics was made by Newton and Wigner. In 1949 they uniquely derived a relativistic position operator and its eigenstates by using some justifiable postulates about the exact localized states [23]. However, the Newton–Wigner position operator, although arising from seemingly reasonable postulates, suffers from the following drawbacks:

(1) A state which is exactly localized in one reference frame, i.e., an eigenstate of the Newton–Wigner position operator, is not localized in other reference frames [23].

(2) The definition of position probability density based on the Newton–Wigner position operator, i.e., $\rho_{NW} = |\psi_{NW}|^2$ [21], leads to faster-than-light particle propagation [24–26].

These difficulties indicate that the Newton–Wigner position operator is not quite acceptable. Moreover, it has been shown that, for a general case, any strict localization leads to superluminal propagation [27–31]. An apparent way out of this problem is to assume that such strict localization is not possible. This implies that a proper relativistic self-adjoint position operator does not exist [26,31], and hence defining the position distribution via the projection-valued measure associated with the position operator is not realizable [35]. A possible treatment is to introduce a reasonable probability density without recourse to a position operator [27], as the one presented in this paper. It must be emphasized that the problem of superluminal propagation is not just the characteristic behavior of the Newton–Wigner probability density. Hegerfeldt proved [27,28,30], on very general grounds and for any reasonable definition of probability density, that a particle initially localized with probability 1 in a finite volume of space, immediately develops infinite "tails." In what follows, we prove a theorem that shows how our probability density keeps the particle from strict localization, which is the main requirement of Hegerfeldt theorem. This is similar to what Thaller proved for the case of Dirac probability density [26].

Theorem. Let ρ be the probability density associated with an arbitrary positive-energy wave function ϕ, presented in Eq. (16). Then

$$\mathrm{Supp}(\rho) = \mathbb{R}, \tag{20}$$

where $\mathrm{Supp}(\rho)$ stands for support of ρ which is defined as

$$\mathrm{Supp}(\rho) \equiv \text{Closure of } \{x \in \mathbb{R} \mid \rho(x) \neq 0\}.$$

Proof. From Eq. (16) it is clear that, for a particle to be strictly localized in a compact subset of $\mathbb{R}$, the supports of $\mathcal{D}^+\phi$ and $\mathcal{D}^-\phi$ should be compact subsets of $\mathbb{R}$. On the other hand, by the Paley–Wiener–Schwartz theorem [36,37], the Fourier transform of a compactly supported function is guaranteed to be analytic anywhere on the complex plane. But the Fourier transform of $\mathcal{D}^+\phi$ and $\mathcal{D}^-\phi$ cannot be simultaneously analytic since they are related to each other by

$$\widetilde{\mathcal{D}^+\phi}(p) = \frac{1}{m}\left(\sqrt{p^2+m^2}+p\right)\widetilde{\mathcal{D}^-\phi}(p). \tag{21}$$

The branch cut in $(p^2+m^2)^{1/2}$ at $p = im$ means both $\widetilde{\mathcal{D}^+\phi}$ and $\widetilde{\mathcal{D}^-\phi}$ cannot be analytic when p is imaginary with magnitude m. Hence, this proves the theorem. ■

The above theorem implies that there is no state for which the probability of finding the particle in a set Δ is 1 unless $\Delta = \mathbb{R}$. Nevertheless, the strict localization of a particle is irrelevant for most practical purposes, and it is quite sufficient to adopt an appropriate notion of localization with adjustable precision. It must be emphasized that, although our probability density has tails extending to infinity, arbitrarily small values of position uncertainty are possible. In fact, for any point of space $a \in \mathbb{R}$, there is a sequence of wave functions $\{\phi_n\}_{n=1}^{\infty}$ whose corresponding probability density sequence $\{\rho_n\}_{n=1}^{\infty}$ approaches $\delta(x-a)$; see Appendix B. This fact indicates that the particle could be localized arbitrarily sharply in the vicinity of any given point. This notion of "arbitrary precise localization" differs from the one introduced by Newton and Wigner; i.e., "exact localization," and was initially employed by Bracken and Melloy for the case of free Dirac electrons [32–34]. It naturally avoids the problems plaguing Newton–Wigner's exact localization; First, the localization defined in this sense has the correct properties under Lorentz transformations because J_μ is a covariant vector [32]; second, since the velocity of probability flow, J/ρ, is less than the speed of light, the propagation of the particle is guaranteed to be causal.

IV. MOMENTUM DISTRIBUTION

Since the position probability density deviates from $|\phi(x)|^2$ in the relativistic regime, one may raise the question of whether momentum probability distribution also deviates from $|\tilde{\phi}(p)|^2$. To answer this question, we note that, based on the quantum theory of measurement, each physical measurement can be described as a position measurement: In principle, the variables that account for the outcome of an experiment are ultimately particle positions [38–41]. This fact has been made clear by Bell [38]:

> In physics the only observations we must consider are position observations, if only the positions of instrument pointers .... If you make axioms, rather than definitions and theorems, about the "measurement" of anything else, then you commit redundancy and risk inconsistency.

In this regard, it is shown in nonrelativistic quantum mechanics that the Born rule for any observable can be derived by considering the Born rule on the position of particles [38–40]. Here we aim to propose a derivation of relativistic momentum probability density from the relativistic position probability density (16). The given argument is based on Feynman's method for initially confined systems; namely, the time-of-flight measurements [39,42–44]. Suppose the wave function is initially confined to a region Δ centered around the origin $x_0 = 0$ and is negligible elsewhere. After allowing the wave function to freely propagate for a considerable amount of time, a measurement of the position x of the particle is effected. The probability of the particle's momentum to lie inside the element dp around the point p at $t = 0$ is equal to probability

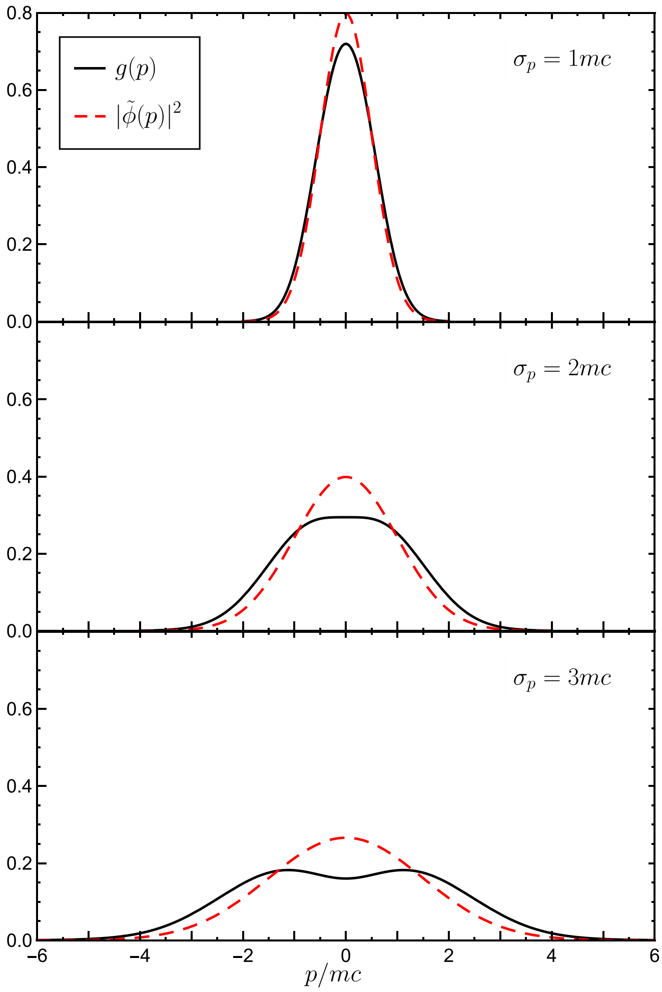

FIG. 2. Plot of the relativistic and nonrelativistic momentum probability density referring to the Gaussian wave function (19) with $\bar{p} = 0$.

of finding the particle's position in the element dx around the point $x = vt$ provided the limit $t \to \infty$ is taken in order to discard the effect of uncertainty in initial position. So we have

$$g(p)dp = \lim_{t\to\infty} [\rho(x,t)dx]_{x=vt}, \tag{22}$$

where $g(p)$ represents the momentum probability density and $v = p/E$. Using the relativistic position probability density (16), Eq. (22) leads to

$$g(p) = \frac{m^2}{E^3} \lim_{t\to\infty} t(|\mathcal{D}^+\phi|^2 + |\mathcal{D}^-\phi|^2)_{x=pt/E}. \tag{23}$$

In the nonrelativistic regime ($c \to \infty$) Eq. (23) leads to

$$g(p) = \frac{1}{m} \lim_{t\to\infty} [t|\phi(x,t)|^2]_{x=pt/m}. \tag{24}$$

In this case, the Schrödinger equation for an initially confined wave function leads to $\phi(pt/m,t) \sim t^{-1/2}\tilde{\phi}(p)$ at $t \to \infty$ and so Eq. (24) reduces to the Born rule in momentum space, $g(p) = |\tilde{\phi}(p)|^2$ [39,42]. But finding an explicit expression for momentum probability density in the relativistic regime is not straightforward, so a numerical calculation of $g(p)$ for the Gaussian wave packet (19) is presented in Fig. 2. This numerical study indicates that the relativistic momentum probability density deviates significantly from the Born rule only when the width of the wave function in momentum space is greater than mc.

V. CONCLUSION AND OUTLOOK

In this paper, in a simple case of single free spinless particle in $(1+1)$ dimensions, we have extracted a "reasonable" probability density current. By "reasonable" we mean that the current (i) is manifestly covariant, (ii) is conserved, (iii) has a nonnegative first component, (iv) does not lead to faster-than-light particle propagation, and (v) reduces to the Born probability current density in the nonrelativistic limit. These conditions naturally give rise to the given probability density current. Therefore, at least in $(1+1)$ dimensions, a probabilistic interpretation of relativistic spinless wave function is possible. Extending this study to $(3+1)$-dimensional interacting particle systems will be the next step. Such systems should be described by quantum field theory. The state of a system in quantum field theory is an arbitrary vector in the appropriate Fock space and may well involve a superposition of states of different particle numbers; namely, $|\Psi\rangle = \sum_n \int \tilde{\phi}_n(p_1,\ldots,p_n)\,|p_1,\ldots,p_n\rangle$. It evolves according to the appropriate Schrödinger equation $i\partial_t|\Psi\rangle = \mathrm{H}|\Psi\rangle$, where H is the Hamiltonian operator in the Schrödinger picture. In the presence of interaction this equation leads to a system of coupled integro-differential equations for multiparticle wave functions, ϕ_n; a recurrent procedure in the literature of light-front quantization [45]. In future works, we aim to find a probabilistic interpretation for these wave equations in position space.

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APPENDIX A

In this Appendix, we derive Eqs. (8), (12), and (13) in $(1+1)$ dimensions. Without loss of generality, the wave function can be expanded as a linear combination of plane waves:

$$\phi(x) = \sum_n A_n e^{ip_n\cdot x}. \tag{A1}$$

Plugging this into Eq. (6) and using Eq. (7) yields

$$J^0 \pm J^1 = \sum_{n,m} A_n A_m^* \alpha(p_n,p_m)(p_n^\pm + p_m^\pm)e^{i(p_n-p_m)\cdot x}, \tag{A2}$$

where $p_n^\pm = p_n^0 \pm p_n^1$. In $(1+1)$ dimensions, the conditions $J^\mu J_\mu \geqslant 0$ and $J^0 \geqslant 0$ lead to

$$J^0 \pm J^1 \geqslant 0, \tag{A3}$$

for arbitrary wave functions. Therefore, we can consider

$$\alpha(p_n,p_m) = \frac{[F^\pm(p_n)]\,[F^\pm(p_m)]^* + [F^\pm(p_n)]^*\,[F^\pm(p_m)]}{p_n^\pm + p_m^\pm}, \tag{A4}$$

which leads to the following positive-definite expression for $J^0 \pm J^1$:

$$J^0 \pm J^1 = \left|\sum_n F^{\pm}(p_n)A_n e^{ip_n\cdot x}\right|^2 + \left|\sum_n [F^{\pm}(p_n)]^* A_n e^{ip_n\cdot x}\right|^2, \tag{A5}$$

where $F^{\pm}(p_n)$ is an unknown function which must be determined. Since the only scalar that can be made by p_n is the rest mass, a dimensional analysis leads to $|\alpha(p_n, p_n)| = \frac{1}{2m_0}$; the factor $1/2$ is a convention and can be absorbed in normalization constant. Therefore,

$$F^{\pm}(p_n) = e^{i\lambda^{\pm}(p_n)}\sqrt{p_n^{\pm}/2m_0}. \tag{A6}$$

Whether one substitutes F^+ or F^-, the resulting α is the same. This fact can be used to determine phase of $F^{\pm}$ as $\lambda^{\pm}(p_n) = \pm l\pi$, where l is an integer number. Then we have

$$\alpha(p_n, p_m) = \frac{[\sqrt{p_n^{\pm}}]\,[\sqrt{p_m^{\pm}}]^* + [\sqrt{p_n^{\pm}}]^*\,[\sqrt{p_m^{\pm}}]}{2m_0(p_n^{\pm} + p_m^{\pm})}. \tag{A7}$$

A straightforward but tedious calculation shows that $\alpha(p_n, p_m)$ can be rewritten as Eq. (8), which ensures that $\alpha(p_n, p_m)$ is a scalar. Also from Eqs. (A5) and (A6), it is clear that

$$\begin{aligned} J^0 = &\left|\sum_n \sqrt{\frac{p_n^+}{4m_0}} A_n e^{ip_n\cdot x}|^2 + |\sum_n \left[\sqrt{\frac{p_n^+}{4m_0}}\right]^* A_n e^{ip_n\cdot x}\right|^2 \\ &+ \left|\sum_n \sqrt{\frac{p_n^-}{4m_0}} A_n e^{ip_n\cdot x}|^2 + |\sum_n \left[\sqrt{\frac{p_n^-}{4m_0}}\right]^* A_n e^{ip_n\cdot x}\right|^2, \end{aligned} \tag{A8}$$

$$\begin{aligned} J^1 = &\left|\sum_n \sqrt{\frac{p_n^+}{4m_0}} A_n e^{ip_n\cdot x}|^2 + |\sum_n \left[\sqrt{\frac{p_n^+}{4m_0}}\right]^* A_n e^{ip_n\cdot x}\right|^2 \\ &- \left|\sum_n \sqrt{\frac{p_n^-}{4m_0}} A_n e^{ip_n\cdot x}|^2 - |\sum_n \left[\sqrt{\frac{p_n^-}{4m_0}}\right]^* A_n e^{ip_n\cdot x}\right|^2. \end{aligned} \tag{A9}$$

Finally, using the definition (14) of $\mathcal{D}^{\pm}$ operators, Eqs. (A8) and (A9) reduce to Eqs. (12) and (13).

APPENDIX B

In this Appendix, we show that there is a sequence of positive-energy wave functions, $\{\phi_n\}_{n=1}^{\infty}$, whose corresponding probability density sequence, $\{\rho_n\}_{n=1}^{\infty}$, approaches $\delta(x-a)$ as a generalized function [46]. This argument is closely similar to that of Bracken and Melloy [32] for the case of a Dirac electron.

Consider following sequence of positive-energy wave functions:

$$\phi_n(x) = \int \sqrt{\frac{m}{nE(p)}} f\left(\frac{p}{n}\right) e^{ip(x-a)}dp, \tag{B1}$$

in which $E_p = (p^2+m^2)^{1/2}$ and $f(p)$ is a normalized Gaussian function, $\int |f(p)|^2 dp = 1$, defined as

$$f(p) = (1/m\sqrt{\pi})^{\frac{1}{2}} e^{-p^2/2m^2}. \tag{B2}$$

Substituting Eq. (B1) into Eq. (16) gives

$$\begin{aligned} \rho_n(x) = &\frac{1}{n}\left|\int S_+(p)f(p)e^{ip(x-a)}dp\right|^2 \\ &+ \frac{1}{n}\left|\int S_-(p)f(p)e^{ip(x-a)}dp\right|^2, \end{aligned} \tag{B3}$$

where $S_{\pm} = \sqrt{E_p \pm p/2E_p}$. By using the convolution theorem, the Fourier transform of the probability density $\tilde{\rho}_n(p)$ can be written as

$$\tilde{\rho}_n(p) = R_n(p)\frac{e^{-ipa}}{\sqrt{2\pi}}, \tag{B4}$$

in which

$$R_n(p) = \frac{1}{n}\int f\left(\frac{q-p}{n}\right) f\left(\frac{q}{n}\right)\Gamma(q-p,q)dq, \tag{B5}$$

$$\Gamma(q,p) = S_+(q)S_+(p) + S_-(q)S_-(p). \tag{B6}$$

Since the Fourier transform of $\delta(x-a)$ is $e^{-ipa}/\sqrt{2\pi}$, we need to show that $\lim_{n\to\infty} R_n(p) = 1$. For this, we consider $q = nr$ and rewrite $R_n(p)$ as

$$R_n(p) = \int f\left(r - \frac{p}{n}\right) f(r)\Gamma(nr-p, nr)dr. \tag{B7}$$

A straightforward calculation shows that $\Gamma(q,p)$ can be rewritten as

$$\Gamma(q,p) = G_1(q)G_1(p) + G_2(q)G_2(p), \tag{B8}$$

in which

$$G_1(p) = \sqrt{\frac{E_p+m}{2E_p}}, \tag{B9}$$

$$G_2(p) = \frac{p}{\sqrt{2E_p(E_p+m)}}. \tag{B10}$$

By Taylor's theorem, we have

$$f\left(r-\frac{p}{n}\right) = f(r) - \frac{p}{n}f'\left(r - \eta\frac{p}{n}\right), \tag{B11}$$

$$G_i(nr-p) = G_i(nr) - \frac{p}{n}G_i'(nr-\theta p), \tag{B12}$$

where $0 \leqslant \eta \leqslant 1$ and $0 \leqslant \theta \leqslant 1$. Using equations (B11) and (B12), Eq. (B7) reads

$$R_n(p) = A_n(p) + B_n(p) + C_n(p) + D_n(p), \tag{B13}$$

where

$$A_n(p) = \int |f(r)|^2\Gamma(nr,nr)dr, \tag{B14}$$

$$B_n(p) = -\frac{p}{n}\int f'\left(r-\eta\frac{p}{n}\right)f(r)\Gamma(nr,nr)dr, \tag{B15}$$

$$C_n(p) = \frac{p^2}{n}\int f'\left(r-\eta\frac{p}{n}\right)f(r)\Upsilon(nr-\theta p,nr)dr, \tag{B16}$$

$$D_n(p) = -p \int |f(r)|^2 \Upsilon(nr - \theta p, nr) dr, \quad \text{(B17)}$$

$$\Upsilon(q, p) = G_1'(q) G_1(p) + G_2'(q) G_2(p). \quad \text{(B18)}$$

Finally, after tedious calculations we get

$$A_n(p) = 1, \quad \text{(B19)}$$

$$\lim_{n\to\infty} B_n(p) = \lim_{n\to\infty} C_n(p) = \lim_{n\to\infty} D_n(p) = 0, \quad \text{(B20)}$$

which show that the probability density sequence $\{\rho_n(x)\}$ converges to $\delta(x - a)$ as n tends to infinity.

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$$i\partial_t \phi_+ = H\phi_+ \Rightarrow i\partial_t \psi_{NW} = H\psi_{NW}.$$

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