Correlations in local measurements and entanglement in many-body systems

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While entanglement plays an important role in characterizing quantum many-body systems, it is hardly possible to directly access many-body entanglement in real experiments. In this paper, we study how bipartite entanglement of many-body states is manifested in the correlation of local measurement outcomes. In particular, we consider a measure of correlation defined as the statistical distance between the joint probability distribution of local measurement outcomes and the product of its marginal distributions. Various bounds of this measure are obtained and several examples of many-body states are considered as a testbed for the measure. We also generalize the framework to the case of imprecise measurement and argue that the considered measure is related to the concept of quantum macroscopicity.

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I. INTRODUCTION

Entanglement [1] is a distinctive feature of quantum mechanics, which exposes fundamental differences between quantum and classical physics [2–4] and can be exploited as a resource for quantum information processing [5]. Entanglement is also a useful tool for characterizing quantum states in many-body systems [6,7]. For example, ground states of gapped Hamiltonians typically follow an area law [7–10], whereas random states follow a volume law of entanglement [11,12]. Amid experimental developments in engineering many-body quantum systems [13–16], a great deal of interest has been generated in examining such features of many-body entanglement in real experiments. For example, there have been several proposals for measuring Rényi $\alpha = 2$ entanglement entropies [17–19] and their experimental realizations [20,21]. Generally speaking, however, it is very hard to directly measure the entanglement as it is a nonlinear function of the state itself, not an observable. In order to measure the entanglement, one needs to obtain the density matrix through a quantum state tomography or find the appropriate relations to other measurable quantities, which are nontrivial in many-body systems.

In this paper, we study the many-body entanglement in terms of the correlation in local measurements. To be specific, we consider a bipartite separation of many-body spin states and positive-operator valued measures (POVMs) acting on each party separately. We then investigate the correlation in such local POVM measurements, which is quantified by the statistical distance (total variation distance) between the joint probability distribution of the measurement outcome and the product of its marginal distributions. Formally, given a quantum state ρ_{AB} of a composite system $A \otimes B$ and local POVMs $\{M_i\}$ and $\{N_j\}$ acting on the subsystems A and B, respectively, we consider

$$\Delta_D(\{M_i\},\{N_j\}) \equiv \frac{1}{2}\sum_{i,j} |\operatorname{Tr}[M_i \otimes N_j(\rho_{AB} - \rho_A \otimes \rho_B)]|, \tag{1}$$

where $\rho_A = \operatorname{Tr}_B \rho_{AB}$ and $\rho_B = \operatorname{Tr}_A \rho_{AB}$. Letting $P_A(i) = \operatorname{Tr}[(M_i \otimes \mathbb{1}_B)\rho_{AB}]$, $P_B(j) = \operatorname{Tr}[(\mathbb{1}_A \otimes N_j)\rho_{AB}]$, and $P_{AB}(i,j) = \operatorname{Tr}[(M_i \otimes N_j)\rho_{AB}]$, this quantity can be written more straightforwardly as

$$\Delta_D(\{M_i\},\{N_j\}) = \frac{1}{2}\sum_{i,j} |P_{AB}(i,j) - P_A(i)P_B(j)|. \tag{2}$$

For convenience, we will call this quantity a correlation in local measurements (CLM) throughout the paper.

Apparently, for general mixed state ρ_{AB}, the CLM does not necessarily capture the entanglement between A and B. On the other hand, if the state ρ_{AB} is guaranteed to be pure, the CLM should be nonzero for properly chosen POVMs if and only if ρ_{AB} is an entangled state. Our aim is to study such relation between the CLM and the entanglement in a *quantitative* manner under the condition that ρ_{AB} is a pure many-body spin state. Note that, by definition, the CLM has a direct relevance to real experimental situations. Note also that the CLM is different from conventional correlation functions of two local operators like $\operatorname{Tr}[O_A \otimes O_B(\rho_{AB} - \rho_A \otimes \rho_B)]$ as the CLM is defined by the probability distribution of the measurement outcome, not by the expectation values of general operators. There have been earlier works that studied correlation measures involving local measurements [22–25]. However, the main focus of them was on investigating quantum correlations that are not captured by local measurements. Our focus, on the other hand, is on how far one can access the quantum correlation only using local POVM measurements, especially, in many-body systems. There also exists a previous study of Eq. (2) [26], but it only dealt with two-qubit states.

In Sec. II, we investigate the relation between the CLM and other correlation and entanglement measures that have been studied before [25,27,28]. We then examine, in Sec. III, the CLM for several examples—Haar random states, spin squeezed states, and the ground state of the Heisenberg XXZ model—under the restriction that local measurements are performed in the basis of a collective spin operator. In Sec. IV,

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we generalize the CLM to the case of imprecise measurement and find its relation to the concept of quantum macroscopicity [29–31]. We further investigate in Sec. V how the imprecise measurement affects Bell's inequalities and conclude the paper in Sec. VI.

II. GENERAL PROPERTIES OF THE CLM

Before proceeding, it is worthwhile to mention the relation between Δ_D and another type of correlation measure defined as

$$\mathrm{cov}(A:B) = \max_{M_A,M_B} \frac{|\operatorname{Tr}[M_A \otimes M_B(\rho_{AB} - \rho_A \otimes \rho_B)]|}{\|M_A\|\|M_B\|}, \quad (3)$$

where the maximization is carried over all operators M_A and M_B acting on subsystems A and B, respectively. Here, $\|O\|$ is the operator norm of O given by the maximum eigenvalue of $\sqrt{O^\dagger O}$. The correlation measure $\mathrm{cov}(A:B)$ has been investigated in various contexts [8–10,32,33]. The detailed relation between $\mathrm{cov}(A:B)$ and Δ_D is not clear. However, when we restrict the maximization in Eq. (3) only to Hermitian operators, it is simple to show that $2\max_{\{M_i\},\{N_j\}} \Delta_D(\{M_i\},\{N_j\})$ upper bounds $\mathrm{cov}(A:B)$.

Let us first investigate the relation between Δ_D and quantum mutual information $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, where $S(\rho) = -\operatorname{Tr}[\rho \log \rho]$ is the von Neumann entropy. Throughout the paper, all logarithms will be taken to base 2.

Proposition 1. For a bipartite quantum state ρ_{AB}, the following inequality holds for any POVMs:

$$\Delta_D \leqslant \mathcal{T}(\rho_{AB}) \leqslant \min\left\{\sqrt{\frac{I(A:B)}{2\log e}}, \sqrt{1-2^{-I(A:B)}}\right\}, \quad (4)$$

where $\mathcal{T}(\rho_{AB}) = \operatorname{Tr}|\rho_{AB} - \rho_A \otimes \rho_B|/2$ is the total correlation [34,35] measured using the trace distance [27,28].

Proof.

$$\Delta_D(\{M_i\},\{N_j\}) = \frac{1}{2}\sum_{i,j} |\operatorname{Tr}[M_i \otimes N_j(\rho_{AB} - \rho_A \otimes \rho_B)]| \leqslant \frac{1}{2}\max_{\{K_m\}} \sum_m |\operatorname{Tr}[K_m(\rho_{AB} - \rho_A \otimes \rho_B)]|, \quad (5)$$

where the maximization is carried over all valid POVMs $\{K_m\}$ for the composite system $A \otimes B$ that satisfy $\sum_m K_m = \mathbb{1}$ and $K_m \geqslant 0$ for all m. The first inequality of the theorem straightforwardly follows from the fact that the last line in Eq. (5) is nothing but the trace distance $D(\rho_{AB},\rho_A \otimes \rho_B)$; hence $\mathcal{T}(\rho_{AB})$, where $D(\rho,\sigma) = \operatorname{Tr}|\rho - \sigma|/2$ [5]. The second inequality consists of two parts. The first part is a well-known Pinsker's inequality, which states $\operatorname{Tr}|\rho_{AB} - \rho_A \otimes \rho_B|/2 \leqslant \sqrt{I(A:B)/2\log 2}$ [36]. The second part comes from the relations between quantum distances. It is known that $D(\rho,\sigma) \leqslant \sqrt{1 - F(\rho,\sigma)^2}$, where $F(\rho,\sigma) = \operatorname{Tr}[\rho^{1/2}\sigma\rho^{1/2}]^{1/2}$ is the fidelity between two quantum states. Using the relations between the affinity $A(\rho,\sigma) = \operatorname{Tr}[\rho^{1/2}\sigma^{1/2}]$ [37] and other quantities, $A(\rho,\sigma) \leqslant F(\rho,\sigma)$ and $-\log A(\rho,\sigma) \leqslant S(\rho||\sigma)/2$ [38], the second inequality is obtained. Here, $S(\rho||\sigma) = \operatorname{Tr}[\rho \log \rho - \rho \log \sigma]$ is the relative entropy between ρ and σ and $S(\rho_{AB}||\rho_A \otimes \rho_B) = I(A:B)$. ■

We note that the Pinsker's inequality is tighter when $I(A:B)$ is smaller, while it is meaningless when $I(A:B) \geqslant 2\log e$. We also note that there is a previous study [26] that investigated the relation between Δ_D and $I(A:B)$ for systems of two qubits.

For pure state $\rho_{AB} = |\psi\rangle\langle\psi|$, $I(A:B) = 2S(\rho_A)$ is twice the entanglement entropy of $|\psi\rangle$, $S(\rho_A) = -\operatorname{Tr}[\rho_A \log \rho_A]$. Thus Proposition 1 implies that Δ_D must be small when the entanglement is small. Let us further investigate the relation between Δ_D and the entanglement for ρ_{AB} being pure. The starting point is a simple proposition.

Proposition 2. A pure quantum state $|\psi\rangle$ is a separable state of two parties (A and B) $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$ if and only if $\Delta_D(\{M_i\},\{N_j\}) = 0$ for any POVMs $\{M_i\}$ and $\{N_j\}$.

The question is what is the lower bound of $\Delta_D(\{M_i\},\{N_j\})$ with an optimal choice of the POVMs when the pure state $|\psi\rangle$ is entangled? The following theorem gives a partial answer.

Theorem 1. For a pure state $|\psi\rangle$, there exist POVMs $\{M_i\},\{N_j\}$ such that $\Delta_D(\{M_i\},\{N_j\}) \geqslant 1 - \mathcal{P}$, where $\mathcal{P} = \operatorname{Tr}[\rho_A^2]$ is the purity of the reduced density matrix.

Proof. We prove this theorem by explicitly constructing the POVMs. Suppose that the Schmidt decomposition of $|\psi\rangle$ is given by $|\psi\rangle = \sum_k \sqrt{\lambda_k}|k_A\rangle|k_B\rangle$ with $\sum_k \lambda_k = 1$, where $\lambda_k \geqslant 0$ are Schmidt coefficients. Using the projective measurements in the Schmidt basis $\{M_i = |i_A\rangle\langle i_A|\}$ and $\{N_j = |j_B\rangle\langle j_B|\}$, the probability outcomes are given by $P_{AB}(i,j) = \lambda_i\delta_{i,j}$, $P_A(i) = \lambda_i$, and $P_B(j) = \lambda_j$. Here, $\delta_{i,j}$ is the Kronecker delta function. Then, for these POVMs,

$$\begin{aligned}\Delta_D &= \frac{1}{2}\sum_{i,j}|P_{AB}(i,j) - P_A(i)P_B(j)| \\ &= \frac{1}{2}\sum_{i,j}|\lambda_i\delta_{i,j} - \lambda_i\lambda_j| \\ &= \frac{1}{2}\left[\sum_i |\lambda_i - \lambda_i^2| + \sum_{i\neq j}\lambda_i\lambda_j\right].\end{aligned} \quad (6)$$

Using $\lambda_i \geqslant \lambda_i^2$ and $1 = \sum_{i,j}\lambda_i\lambda_j = \sum_i \lambda_i^2 + \sum_{i\neq j}\lambda_i\lambda_j$, we obtain

$$\Delta_D = 1 - \sum_i \lambda_i^2 = 1 - \mathcal{P}. \quad (7)$$

■

From the theorem, $\Delta_D(\{M_i\},\{N_j\}) = 0$ for all POVMs implies $\mathcal{P} = 1$, which means ρ_A is pure and hence ρ_{AB} is separable. Note that the lower bound $1 - \mathcal{P}$ is the linear entropy, which has been widely investigated in quantum information theory. The linear entropy is a nice indication of entanglement for pure states, and it is known that its convex roof extension to mixed states is an entanglement monotone [39].

We can also consider a convex roof extension of the CLM. Let us consider the following definition:

$$\mathcal{E}(\rho) = \min_{\{p_k,|\psi_k\rangle\}} \sum_i p_i \max_{M_i,N_j} \Delta_D(|\psi_k\rangle), \quad (8)$$

where the minimization is over all pure-state ensembles $\{p_k,|\psi_k\rangle\}$ that give $\sum_k p_k|\psi_k\rangle\langle\psi_k| = \rho$. Even though it is not an entanglement monotone generally, one can easily show that $\mathcal{E}(\rho)=0$ if and only if ρ is a separable state.

III. CLM FOR COLLECTIVE SPIN MEASUREMENTS

So far, our discussion was general; we did not consider any specific form of POVMs or a system. In this section, we consider several examples of many-body spin systems to investigate the properties of the CLM. To be specific, we consider systems of N $s=1/2$ spins composed of two subsystems A and B, where N is even and the subsystem A (B) contains the first (last) $N/2$ spins. As a natural choice, we consider the case wherein each party performs a collective spin measurement. For subsystems A, the spins are measured in the basis of $S_A(\hat{\alpha})=\hat{\alpha}\cdot\boldsymbol{S}_A$, where $\hat{\alpha}$ is a unit vector and $\boldsymbol{S}_A=\sum_{n\in A}\boldsymbol{\sigma}^{(n)}/2$ is the collective spin operator. Here, $\boldsymbol{\sigma}^{(n)}=\{\sigma_x^{(n)},\sigma_y^{(n)},\sigma_z^{(n)}\}$ is the vector of Pauli spin operators for the nth spin. We can obtain the POVM for $S_A(\hat{\alpha})$ from the decomposition

$$S_A(\hat{\alpha})=\sum_{i=-N/4}^{N/4} i\sum_{\mu_i}|i,\mu_i\rangle\langle i,\mu_i|, \tag{9}$$

where $i\in[-N/4,N/4]$ are possible measurement outcomes and μ_i is the index for the degenerate subspace corresponding to the outcome i. Then the POVM can be written as $M_i(\hat{\alpha})=\sum_{\mu_i}|i,\mu_i\rangle\langle i,\mu_i|$. Likewise, we also define $S_B(\hat{\beta})=\hat{\beta}\cdot\boldsymbol{S}_B$ and the corresponding POVM $\{N_j(\hat{\beta})\}$ such that $S_B(\hat{\beta})=\sum_{j=-N/4}^{N/4} jN_j(\hat{\beta})$ for subsystem B. To simplify the notation, the shorthand expression $\Delta_D(\hat{\alpha},\hat{\beta})$ will be used throughout this section to designate $\Delta_D(\{M_i(\hat{\alpha})\},\{N_j(\hat{\beta})\})$ unless it confuses.

Before proceeding, let us first consider simple heuristic examples.

Example 1. Let $|\psi_0\rangle=(|\downarrow\rangle^{\otimes N}+|\uparrow\rangle^{\otimes N})/\sqrt{2}$ and $|\psi_1\rangle=(|\uparrow\rangle^{\otimes N-1}|\downarrow\rangle+|\downarrow\rangle|\uparrow\rangle^{\otimes N-1})/\sqrt{2}$. Then $\Delta_D(\hat{z},\hat{z})=1/2$ for both the states. The possible outcome pairs (i,j) from the measurements are $\{(N/4,N/4),(-N/4,-N/4)\}$ and $\{(N/4,N/4-1),(N/4-1,N/4)\}$, respectively. For the same states, correlation function $\langle S_A(\hat{z})\otimes S_B(\hat{z})\rangle-\langle S_A(\hat{z})\rangle\langle S_B(\hat{z})\rangle$ yields $N^2/16$ and $-1/4$, respectively, which largely differ. This example illustrates a stark difference between the CLM and the correlation function. It also shows that the CLM does not distinguish between bipartite entanglement and genuine multipartite entanglement.

Example 2. Let us consider $|\psi_2\rangle=C\sum_P(P|\downarrow\rangle^{\otimes N/4}|\uparrow\rangle^{\otimes N/4})(P|\downarrow\rangle^{\otimes N/4}|\uparrow\rangle^{\otimes N/4})$, where the summation is over all possible permutations P and the same P is applied on the two subsystems. The normalization constant C is given by $C=\binom{N/2}{N/4}^{-1/2}$. As the whole component states live in the subspace of $S_A(\hat{z})=S_B(\hat{z})=0$, we can see that $P_{AB}(i,j)=\delta_{i,0}\delta_{j,0}$ and $P_A(i)=\delta_{i,0}$, $P_B(j)=\delta_{j,0}$. Therefore, $\Delta_D(\hat{z},\hat{z})=0$. On the other hand, when we compute the entanglement entropy, we get $S=\log\binom{N/2}{N/4}$. Using Stirling's formula, this can be approximated as $S\approx N/2\ln 2+O(\log N)$ for $N\gg 1$, which indicates that the entanglement is extensive. This result illustrates that Δ_D using collective spin measurement cannot capture entanglement of some states.

A. Random states

In this subsection, we investigate the behavior of the CLMs in z direction and optimized over all directions, i.e., $\Delta_D(\hat{z},\hat{z})$ and $\max_{\hat{\alpha},\hat{\beta}}\Delta_D(\hat{\alpha},\hat{\beta})$, for Haar random states. For this, recall Levy's lemma which implies that the values of a Lipschitz continuous function f are all concentrated to its mean value $\langle f\rangle$. Formally it is written as follows.

Theorem 2 (Levy's lemma; see Ref. [40]): Let $f:\mathbb{S}^k\to\mathbb{R}$ be a function with Lipshitz constant η and $\phi\in\mathbb{S}^k$ be a point chosen uniformly at random. Then,

$$\Pr[|f(\phi)-\langle f\rangle|>\epsilon]\leqslant 2\exp(-2C(k+1)\epsilon^2/\eta^2) \tag{10}$$

for a constant $C>0$ that may be chosen as $C=(18\pi^3)^{-1}$.

We now prove the Lipschitz continuity of the CLM for fixed directions.

Theorem 3. For fixed $\hat{\alpha}$ and $\hat{\beta}$, $\Delta_D(\hat{\alpha},\hat{\beta})$ is a Lipschitz continuous function of $|\psi\rangle$ with the Lipschitz constant $\eta\leqslant 3$.

Proof. Let $|\psi\rangle$ and $|\psi'\rangle$ be two different pure states. The difference of Δ_D is given by

$$\Bigg|\frac{1}{2}\sum_{i,j}|\operatorname{Tr}[M_i\otimes N_j(\rho_{AB}-\rho_A\otimes\rho_B)]| -\frac{1}{2}\sum_{i,j}|\operatorname{Tr}[M_i\otimes N_j(\rho'_{AB}-\rho'_A\otimes\rho'_B)]|\Bigg|, \tag{11}$$

where $\rho_{AB}=|\psi\rangle\langle\psi|$ and $\rho'_{AB}=|\psi'\rangle\langle\psi'|$. M_i and N_j are the POVMs in directions $\hat{\alpha}$ and $\hat{\beta}$, respectively. Then we can obtain

$$\leqslant\frac{1}{2}\sum_{i,j}|\operatorname{Tr}[(M_i\otimes N_j)(\rho_{AB}-\rho'_{AB})] -\operatorname{Tr}[(M_i\otimes N_j)(\rho'_A\otimes\rho'_B-\rho_A\otimes\rho_B)]| \tag{12}$$

$$\leqslant\frac{1}{2}[\operatorname{Tr}|\rho_{AB}-\rho'_{AB}|+\operatorname{Tr}|\rho_A\otimes\rho_B-\rho'_A\otimes\rho'_B|], \tag{13}$$

where we have used the reverse triangular inequality $||A|-|B||\leqslant|A-B|$ for the first inequality and $\operatorname{Tr}|\rho-\sigma|=\max_{\{K_m\}}\sum_m|K_m(\rho-\sigma)|$ for the last inequality. As $\operatorname{Tr}|\rho_A\otimes\rho_B-\rho'_A\otimes\rho'_B|\leqslant\operatorname{Tr}|\rho_A\otimes(\rho_B-\rho'_B)|+\operatorname{Tr}|(\rho_A-\rho'_A)\otimes\rho'_B|\leqslant 2\operatorname{Tr}|\rho_{AB}-\rho'_{AB}|$,

$$\begin{aligned}&\leqslant\tfrac{3}{2}\operatorname{Tr}|\rho_{AB}-\rho'_{AB}|\\ &=\tfrac{3}{2}\operatorname{Tr}||\psi\rangle\langle\psi|-|\psi'\rangle\langle\psi'||\\ &=3\sqrt{1-|\langle\psi|\psi'\rangle|^2}\leqslant 3\||\psi\rangle-|\psi'\rangle\|_2.\end{aligned} \tag{14}$$

Therefore, the Lipschitz constant $\eta\leqslant 3$ is obtained. ■

The above two theorems imply that, as $N\to\infty$, the CLMs for Haar random states converge to a certain value with a vanishing variance. We numerically generated Haar random states and obtained the CLM in z direction $\Delta_D(\hat{z},\hat{z})$. The result, averaged over 10^3 random states, is plotted in Fig. 1 along with the linear entropy of a subsystem. It shows that while the linear entropy of a subsystem increases with N and coincides with the analytic result $1-\langle\mathcal{P}\rangle=1-2^{N/2+1}/(2^N+1)$ [41], the CLM in z directions decreases exponentially with N. The collective spin measurement is thus inappropriate to capture the entanglement of random states [11,12].

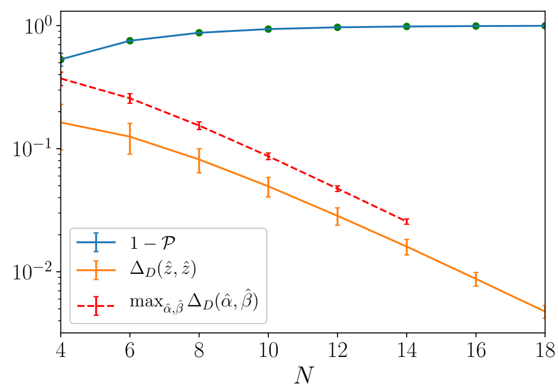

FIG. 1. CLM in z directions, optimal CLM for collective spin measurements, and linear entropy of a subsystem, obtained for Haar random states. For each N, 10^3 random states were taken and the results were averaged. The green dots represent the analytic values of the average linear entropy. The results show that the averaged CLM for random states decreases whereas the averaged linear entropy increases, indicating that for large N, the considered CLM hardly capture the entanglement. The error bars for the linear entropy are invisible as they are too small.

The behavior of $\Delta_D(\hat{z},\hat{z})$ can be understood as follows. Let the basis states in z direction be $\{|\alpha\rangle = |i,\mu_i\rangle\}$ and $\{|\beta\rangle = |j,\mu_j\rangle\}$ for subsystems A and B, respectively, with $-N/4 \leqslant i,j \leqslant N/4$. One can then write the state as $|\psi\rangle = \sum_{\alpha,\beta} A_{\alpha,\beta} |\alpha\rangle |\beta\rangle$. It can be shown that as $N \to \infty$, $|A_{\alpha,\beta}|^2$ approaches $1/2^N$ with a vanishing fluctuation (see Appendix A). In such a limit, $P_{AB}(i,j) = \binom{N/2}{i+N/4}\binom{N/2}{j+N/4}/2^N$ and $P_A(i) = P_B(i) = \binom{N/2}{i+N/4}/2^{N/2}$, leading to $P_{AB}(i,j) = P_A(i)P_B(j)$ and hence vanishing Δ_D.

We also numerically calculated the optimal CLM $\max_{\hat{\alpha},\hat{\beta}} \Delta_D(\hat{\alpha},\hat{\beta})$ for each given random state. The result is shown in Fig. 1 as a dashed curve. We obtained the result only for $N \leqslant 14$ due to the computational cost in the optimization. The result indicates that, for Haar random states, the optimization over the measurement direction is not of much help in identifying the entanglement.

B. Spin squeezed states

In this subsection, we consider one-axis twisted states that are generated by applying a squeezing operator

$$V_\mu = e^{-i\nu S_x} e^{-i\mu S_z^2/2} \tag{15}$$

to the spin coherent state in x direction $|+\rangle^{\otimes N}$, where $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)\sqrt{2}$ [42] (for a review, see Ref. [43]). Here, $S_x = S_A(\hat{x}) + S_B(\hat{x})$ and $S_z = S_A(\hat{z}) + S_B(\hat{z})$. This kind of squeezed state has been experimentally generated in many different setups [44–47]. Note that the squeezing in z direction is followed by the rotation in x direction, making the state have the maximal spin variance in z direction (see below). This scheme is sometimes called a twist-and-turn spin squeezing [48].

As spin coherent states and squeezing operators are symmetric under any permutations between spins, the resulting squeezed states also live in a permutation symmetric subspace

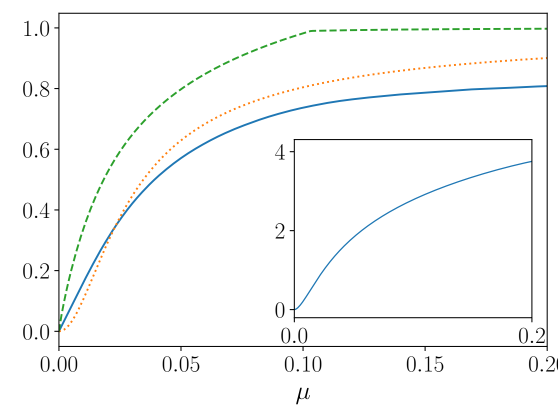

FIG. 2. $\Delta_D(\hat{z},\hat{z})$ (blue curve), linear entropy of a subsystem (red dotted curve), and the upper bound from Proposition 1 (green dashed curve), obtained for spin squeezed states $V_\mu |+\rangle^{\otimes N}$ as a function of the squeezing strength μ. The system size is $N = 200$. The inset shows the entanglement entropy for comparison.

of the total Hilbert space. One may use a vector space spanned by Dicke states to efficiently represent this state. Dicke states are given by

$$|N,k\rangle = \binom{N}{k}^{-1/2} \sum_P P(|\uparrow\rangle^{\otimes k} |\downarrow\rangle^{\otimes N-k}) \tag{16}$$

for $0 \leqslant k \leqslant N$, where the summation runs over all possible permutations. It is easy to show that when we divide a subspace generated by Dicke states into two subsystems of $N/2$ spins, Dicke states in each subsystem ($|N/2,k\rangle$) also become a basis set, i.e., $|N,k\rangle = \sum_{r=0}^{k} C_r |N/2,r\rangle_A |N/2,k-r\rangle_B$. Consequently, the entanglement entropy of any permutation symmetric state is upper bounded by $\log(N/2+1)$.

Expectation values and the variances of spin operators for the spin squeezed state $V_\mu |+\rangle^{\otimes N}$ are calculated in Ref. [42]. It shows

$$\langle S_x \rangle = \frac{N}{2}\cos^{N-1}\frac{\mu}{2}, \quad \langle S_y \rangle = \langle S_z \rangle = 0, \tag{17}$$

$$\langle \Delta S_x^2 \rangle = \frac{N}{4}\left[N\left(1 - \cos^{2(N-1)}\frac{\mu}{2}\right) - \frac{N-1}{2}A\right], \tag{18}$$

$$\langle \Delta S_{y,z}^2 \rangle = \frac{N}{4}\left\{1 + \frac{N-1}{4}[A \pm \sqrt{A^2+B^2}\cos(2\nu+2\delta)]\right\}, \tag{19}$$

where $A = 1 - \cos^{N-2}\mu$, $B = 4\sin\frac{\mu}{2}\cos^{N-2}\frac{\mu}{2}$, and $\delta = \frac{1}{2}\arctan\frac{B}{A}$.

For the system size $N = 200$, we performed numerical calculations for $\nu = \frac{\pi}{2} - \delta$ that maximize $\langle \Delta S_z^2 \rangle$ and minimize $\langle \Delta S_y^2 \rangle$. In Fig. 2, $\Delta_D(\hat{z},\hat{z})$ and the linear entropy of a subsystem are plotted with respect to the squeezing strength μ. For comparison, the upper bound of the CLM from Proposition 1 and the entanglement entropy are also plotted. All those results show similar functional behaviors, suggesting that the CLM is appropriate to capture the entanglement in this case. One may compare $\Delta_D(\hat{z},\hat{z})$ with the value for the GHZ state ($|\psi_0\rangle$ in

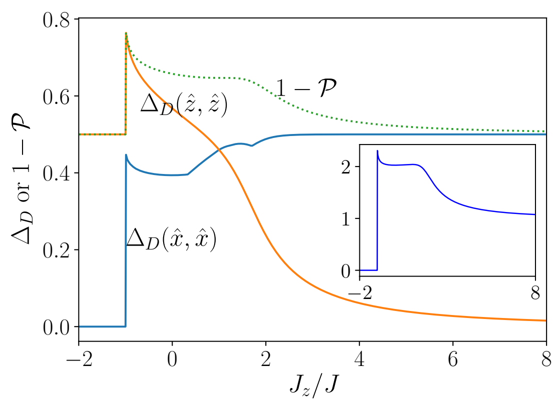

FIG. 3. CLMs in x and z directions and linear entropy $1-\mathcal{P}$ of a subsystem for the ground state of the Heisenberg XXZ model. The inset shows the entanglement entropy for comparison.

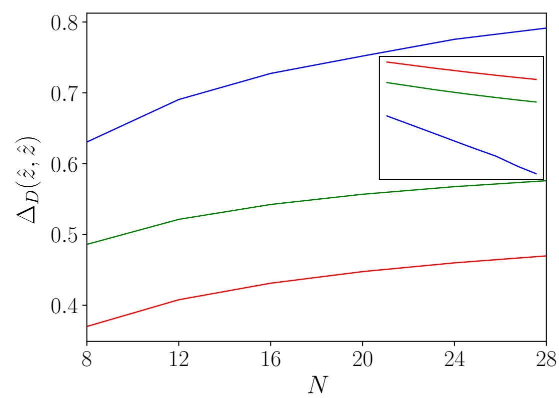

FIG. 4. CLM in z direction for the ground state of the Heisenberg XXZ model for $J_z/J=-1^+$, 0, and 1 (from top to bottom) as a function of N. Inset: log-log plot of N versus $1-\Delta_D$, which suggests $\Delta_D \approx 1-cN^{-\alpha}$ scaling.

Example 1), for which $\Delta_D=0.5$. We find that $\Delta_D(\hat{z},\hat{z})\geqslant 0.5$ for $\mu \gtrsim 0.04$.

C. Ground states of the Heisenberg XXZ model

As a final example, we consider the ground state of the one-dimensional Heisenberg XXZ model. The Hamiltonian of the model is given by

$$H=\sum_{n=1}^{N}\left[J\left(\sigma_x^{(n)}\sigma_x^{(n+1)}+\sigma_y^{(n)}\sigma_y^{(n+1)}\right)+J_z\sigma_z^{(n)}\sigma_z^{(n+1)}\right], \quad (20)$$

where $J>0$ is the interaction strength and J_z/J determines the strength of anisotropy. It is well known that this model is solvable using the Bethe ansatz. For $J>0$, the model is gapless in thermodynamic limit ($N\to\infty$) for $-1<J_z/J\leqslant 1$. When $J_z/J<-1$, two degenerate ground states are $|\uparrow\rangle^{\otimes N}$ and $|\downarrow\rangle^{\otimes N}$. As there is no spontaneous symmetry breaking for finite N, we take $(|\uparrow\rangle^{\otimes N}+|\downarrow\rangle^{\otimes N})/\sqrt{2}$, which is the GHZ state we have studied in Example 1, as the ground state for $J_z/J<-1$. For $J_z/J>1$, the model shows the gapped antiferromagnetic phase [49]. The quantum phase transition at $J_z/J=-1$ is the first order and the infinite order Kosterlitz-Thouless transition occurs at $J_z/J=1$. We note that this Hamiltonian models some real materials [50] and is implementable using engineered systems such as optical lattices [51] and trapped ions [52,53] (see also Ref. [54] which provides the summary of theoretical proposals and experiments of this model).

For the system size $N=24$, we obtained the ground state using the Lanczos method. In Fig. 3, Δ_D in x and z directions and the linear entropy $1-\mathcal{P}$ are plotted for $-2\leqslant J_z/J\leqslant 8$. We have obtained $\Delta_D(\hat{z},\hat{z})\geqslant 0.5$ (Δ_D for the GHZ state) for $-1.0<J_z/J\lesssim 0.66$. The first-order phase transition at $J_z/J=-1$ is directly seen from the sudden changes of Δ_D and $1-\mathcal{P}$. There is a crossing of Δ_Ds in x and z directions at $J_z/J=1$ as the system has a full SU(2) symmetry at that point. Some singular points in $\Delta_D(\hat{x},\hat{x})$ that are nothing to do with a quantum phase transition appear near $J_z/J\approx 0.3$ and $\approx$1.7.

When $J_z/J\gg 1$, the ground state is the superposition of two Néel ordered states $|\uparrow\downarrow\cdots\rangle+|\downarrow\uparrow\cdots\rangle$. The joint probability distribution of the measurement in z direction is given by $P_{AB}(i,j)=\delta_{i,0}\delta_{j,0}$. In this case, $\Delta_D(\hat{z},\hat{z})=0$ is obtained and this is consistent with the result in Fig. 3. By rotating the state, we can also obtain the probability distribution for the measurement in x direction. A simple calculation yields $P_{AB}(i,j)=\binom{N/2}{i+N/4}\binom{N/2}{j+N/4}/2^{N-1}$ when $i+j+N/2$ is even and $P_{AB}(i,j)=0$ otherwise. Using this, $\Delta_D(\hat{x},\hat{x})=1/2$ is obtained, which also agrees with our numerical result.

We also numerically obtained $\Delta_D(\hat{z},\hat{z})$ at $J_z/J=-1^+$, 0, and 1 for the system sizes N that are multiples of 4, which are plotted in Fig. 4. These values of N are used as the ground states are translation invariant, i.e., $T\,|\mathrm{GS}\rangle_N=|\mathrm{GS}\rangle_N$ (for even N that is not a multiple of 4, $T\,|\mathrm{GS}\rangle_N=-|\mathrm{GS}\rangle_N$). The result shows that $\Delta_D(\hat{z},\hat{z})$ is increasing with N. This indicates that a relatively large value of CLM can be obtained for any system size. We also find that this increasing behavior follows a power law that is typical for critical systems.

IV. EFFECTS OF MEASUREMENT IMPRECISIONS

In practice, any measurement in experiments is imperfect to some degree. Then, the measurement outcomes are not perfectly discriminated and the CLM $\Delta_D(\{M_i(\hat{\alpha})\},\{N_j(\hat{\beta})\})$ is thus poorly defined. This motivates us to consider the cases wherein the collective spin measurement of $S_A(\hat{\alpha})$ and $S_B(\hat{\beta})$ has a finite resolution. For subsystem A, the Kraus operators for this type of measurement can be written as [55]

$$E^{\sigma}(\hat{\alpha};x)=\sum_{i=-N/4}^{N/4}\sqrt{p^{\sigma}(x,i)}E_i(\hat{\alpha}), \quad (21)$$

where $\{E_i(\hat{\alpha})\}$ are the Kraus operators for $M_i(\hat{\alpha})$, given by $M_i(\hat{\alpha})=E_i(\hat{\alpha})^{\dagger}E_i(\hat{\alpha})$. In our case, $E_i(\hat{\alpha})=M_i(\hat{\alpha})$ as $M_i(\hat{\alpha})$ is a projection operator. Here, $p^{\sigma}(x,i)$ is a smoothing function, which is a probability distribution function of continuous variable x, i.e., $\int_{x\in D_A}dx\,p^{\sigma}(x,i)=1$. The probability $p^{\sigma}(x,i)dx$ means the probability to obtain measurement outcomes in $[x,x+dx]$ when the state is actually i. Here, σ is a parameter which determines the resolution of the measurement. The Gaussian (normal) distribution $p^{\sigma}(x,i)=e^{-(x-i)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$ with $x\in\mathbb{R}$ is widely used.

Using the Kraus operators, the POVM of continuous outcomes is defined as $M^\sigma(\hat{\alpha};x) = E^\sigma(\hat{\alpha};x)^\dagger E^\sigma(\hat{\alpha};x)$. For subsystem B, we similarly define the Kraus operator $F^\sigma(\hat{\beta},y) = \sum_{j=-N/4}^{N/4} \sqrt{p^\sigma(y,j)} F_j(\hat{\beta})$ and the corresponding POVM $N^\sigma(\hat{\beta};y) = F^\sigma(\hat{\beta},y)^\dagger F^\sigma(\hat{\beta},y)$. This kind of measurement is also called a coarse-grained measurement [56].

The CLM Δ_D we have used above is defined for measurements with discrete outcomes. We define a continuous version of the CLM as

$$\Delta_C(\{M(x)\},\{N(y)\}) = \frac{1}{2}\int_{x\in D_A}\int_{y\in D_B} dx\, dy |P_{AB}(x,y) - P_A(x)P_B(y)|, \quad (22)$$

where $D_A, D_B \subset \mathbb{R}$ are the domains of the possible measurement outcomes for subsystems A and B, respectively. Here, the probability distribution functions are given by $P_{AB}(x,y) = \mathrm{Tr}[M(x)\otimes N(y)\rho_{AB}]$, $P_A(x) = \mathrm{Tr}[M(x)\rho_A]$, and $P_B(y) = \mathrm{Tr}[N(y)\rho_B]$. We note that the properties of Δ_D derived in Sec. II remain valid for Δ_C as a POVM with continuous outcomes can be reduced to that with discrete outcomes as far as the system is finite dimensional [57].

Example 3. Let us recall $|\psi_0\rangle$ and $|\psi_1\rangle$ from Example 1. A simple calculation yields $\Delta_C(\{M^\sigma(\hat{z};x)\},\{N^\sigma(\hat{z};y)\}) = \mathrm{erf}(N/(4\sqrt{2}\sigma))^2/2$ for $|\psi_0\rangle$ and $\mathrm{erf}(1/(2\sqrt{2}\sigma))^2/2$ for $|\psi_1\rangle$ when we use the Gaussian smoothing function. Here, $\mathrm{erf}(x) = \int_{-x}^{x} e^{-t^2}dt/\sqrt{\pi}$ is the error function. Therefore, for large $N \gg 1$, the correlation of $|\psi_0\rangle$ is detectable even with imprecise measurement but that of $|\psi_1\rangle$ is not. For instance, when $N = 20$ and $\sigma = 2.0$, $\Delta_C \approx 0.488$ for $|\psi_0\rangle$, but $\Delta_C \approx 0.019$ for $|\psi_1\rangle$. We also note that when $\sigma \to 0^+$, $\Delta_C \to 0.5$ for both states, recovering Δ_D in Example 1.

Our main point of this section is that the CLM with coarse-grained measurements is related to the concept of quantum macroscopicity. The following two theorems make the relation more explicit.

Theorem 4 (Correlation disturbance).

$$\Delta_C(\{M^\sigma(x)\},\{N^\sigma(y)\}) \leqslant 1 - \mathcal{F}(|\psi\rangle, \rho'_{AB})^2, \quad (23)$$

where $\mathcal{F}(|\psi\rangle,\rho) = \langle\psi|\rho|\psi\rangle^{1/2}$ is the fidelity between a pure state $|\psi\rangle$ and a mixed state ρ. Here, ρ'_{AB} is the postmeasurement state given by

$$\rho'_{AB} = \int_{D_X} dx \int_{D_Y} dy\, E^\sigma(x)\otimes F^\sigma(y)\,|\psi\rangle\langle\psi|\, E^\sigma(x)\otimes F^\sigma(y). \quad (24)$$

Proof. Using $|f(x)-g(x)| = \max[f(x),g(x)] - \min[f(x),g(x)]$ and $f(x)+g(x) = \max[f(x),g(x)] + \min[f(x),g(x)]$, we obtain $|P_{AB}(x,y) - P_A(x)P_B(y)| = P_{AB}(x,y) + P_A(x)P_B(y) - 2\min\{P_{AB}(x,y),P_A(x)P_B(y)\}$. Integrating both sides, we obtain

$$\Delta_C(\{M^\sigma(x)\},\{N^\sigma(y)\}) \leqslant 1 - \int_{D_X} dx \int_{D_Y} dy\, \min[P_{AB}(x,y),P_A(x)P_B(y)]. \quad (25)$$

The theorem follows from

$$P_{AB}(x,y) = \langle\psi|[E^\sigma(x)\otimes F^\sigma(y)]^2|\psi\rangle \geqslant |\langle\psi|E^\sigma(x)\otimes F^\sigma(y)|\psi\rangle|^2 \quad (26)$$

and

$$P_A(x)P_B(y) = \langle\psi|[E^\sigma(x)\otimes \mathbb{1}]^2|\psi\rangle\langle\psi|[\mathbb{1}\otimes F^\sigma(y)]^2|\psi\rangle \geqslant |\langle\psi|E^\sigma(x)\otimes F^\sigma(y)|\psi\rangle|^2. \quad (27)$$

We have used $\langle\psi|A^2|\psi\rangle \geqslant \langle\psi|A|\psi\rangle^2$ for Hermitian A to obtain the inequality in Eq. (26) and the Cauchy-Schwartz inequality $\langle f|f\rangle\langle g|g\rangle \geqslant |\langle f|g\rangle|^2$ with $|f\rangle = [E^\sigma(x)\otimes\mathbb{1}]|\psi\rangle$ and $|g\rangle = [\mathbb{1}\otimes F^\sigma(y)]|\psi\rangle$ for the inequality in Eq. (27). ■

This theorem states that the CLM with coarse-grained measurements is smaller than the disturbance the measurement has caused. This relation resembles the well-studied relation between quantum state disturbance and information gain [58].

Theorem 5. For the Gaussian smoothing $p^\sigma(x,i) = e^{-(x-i)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$,

$$\mathcal{F}(|\psi\rangle,\rho'_{AB})^2 \geqslant \exp\left(-\frac{\mathcal{V}_{|\psi\rangle}(S_A(\hat{\alpha})\otimes\mathbb{1}) + \mathcal{V}_{|\psi\rangle}(\mathbb{1}\otimes S_B(\hat{\beta}))}{4\sigma^2}\right), \quad (28)$$

where $\mathcal{V}_{|\psi\rangle}(A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$ is the variance of operator A for quantum state $|\psi\rangle$.

The proof of the theorem can be found in Appendix B. The steps for the proof are basically the same as those of Theorem 2 in Ref. [59]. We note that $\mathcal{V}_{|\psi\rangle}(S_A(\hat{\alpha})\otimes\mathbb{1}) + \mathcal{V}_{|\psi\rangle}(\mathbb{1}\otimes S_B(\hat{\beta}))$ in the theorem has an obvious relation to the measure of quantum macroscopicity defined as

$$\mathcal{M}(|\psi\rangle) = \max_{A\in S} \mathcal{V}_{|\psi\rangle}(A), \quad (29)$$

where S is the set of collective observables given by

$$S = \left\{\sum_i \hat{\alpha}^{(i)}\cdot\boldsymbol{\sigma}^{(i)} : |\hat{\alpha}^{(i)}| = 1 \quad \text{for all } i \in 1,\ldots,N\right\}. \quad (30)$$

The first definition of this measure appeared in Ref. [29] and the measure has been developed in various contexts [30,31] (see also Ref. [60] for a recent review). As $\mathcal{V}_{|\psi\rangle}(S_A\otimes\mathbb{1}) + \mathcal{V}_{|\psi\rangle}(\mathbb{1}\otimes S_B) \leqslant \max\{\mathcal{V}_{|\psi\rangle}(S_A\otimes\mathbb{1} + \mathbb{1}\otimes S_B), \mathcal{V}_{|\psi\rangle}(S_A\otimes\mathbb{1} - \mathbb{1}\otimes S_B)\}$ and $2(S_A\otimes\mathbb{1} \pm \mathbb{1}\otimes S_B) \in S$, it is evident that $\mathcal{V}_{|\psi\rangle}(S_A\otimes\mathbb{1}) + \mathcal{V}_{|\psi\rangle}(\mathbb{1}\otimes S_B) \leqslant \mathcal{M}(|\psi\rangle)/4$. Using this result, we can rewrite Theorem 4 as

$$\Delta_C(\{M^\sigma(\hat{\alpha};x)\},\{N^\sigma(\hat{\beta};y)\}) \leqslant 1 - \exp\left(-\frac{\mathcal{M}(|\psi\rangle)}{16\sigma^2}\right). \quad (31)$$

Previous studies of quantum macroscopicity in many-body spin systems have shown that a class of quantum states of N spins $|\psi_N\rangle$ can be regarded as a macroscopic superposition if $\mathcal{M}(|\psi\rangle_N) = O(N^2)$, whereas it cannot be if $\mathcal{M}(|\psi\rangle_N) = O(N)$ [29–31]. For example, a product state is not a macroscopic superposition as it gives $\mathcal{M}(|\phi_1\phi_2\ldots\phi_N\rangle) = N$. More recent studies have shown that Haar random states [61,62] and asymptotic states in nonintegrable systems that thermalize also show an $\mathcal{M} = O(N)$ behavior [63]. Our result thus implies that the correlations of those latter states with $\mathcal{M} = O(N)$ cannot be detected if $\sigma \gg \sqrt{N}$. In some literature [64–66], a course-grained measurement with $\sigma \gg \sqrt{N}$ is considered as a classical measurement in the sense that the measurement hardly disturbs the state for large but finite N [55,56,67]. Following this line of arguments, our results suggest that the correlation of pure entangled states $|\psi\rangle$ cannot be captured with classical measurements if $\mathcal{M}(|\psi\rangle) = O(N)$.

V. IMPLICATION TO BELL'S INEQUALITIES

Let us consider nonlocality tests using the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality [3] in our many-body spin setting with imprecise measurements. The Bell-CHSH function is defined as

$$\mathcal{B} = |E(a,b) - E(a,b')| + |E(a',b') + E(a',b)|. \tag{32}$$

Here, $E(a,b)$ is the correlation function of observables with dichotomic outcomes and (a,a') and (b,b') represent two different measurement setups for subsystems A and B, respectively. The Bell theorem states that $\mathcal{B} \leqslant 2$ for local hidden variable theories.

To construct a dichotomy observable in our spin measurement setup, we define the measurement operator for subsystem A as

$$A(a) = \int_{-\infty}^{\infty} dx\, f(x) M^{\sigma}(a;x), \tag{33}$$

where $f(x)$ is an arbitrary function that gives either 1 or -1 according to x. Likewise, we also define $B(b)$ for subsystem B as

$$B(b) = \int_{-\infty}^{\infty} dx\, g(y) N^{\sigma}(b;y). \tag{34}$$

As in the previous section, σ denotes the degree of imprecision. Here, a and b parametrize the directions of collective spin measurements. In this setup, a measurement setting (a,b) can be transformed to others $(a',b),(a,b'),(a',b')$ using local unitary transforms. The correlation function $E(a,b)$ for the Bell-CHSH function is then defined as $E(a,b) = \mathrm{Tr}[\rho_{AB} A(a) \otimes B(b)]$. Under this setting, the following theorem holds.

Theorem 6. The Bell-CHSH function $\mathcal{B}$ for pure state $|\psi\rangle$ is bounded as

$$\mathcal{B} \leqslant 2 + 8\left\{1 - \exp\left(-\frac{\mathcal{M}(|\psi\rangle)}{16\sigma^2}\right)\right\}. \tag{35}$$

Proof. For product state $\rho_A \otimes \rho_B$, let $\widetilde{E}(a,b) = \mathrm{Tr}[(\rho_A \otimes \rho_B) A(a) \otimes B(b)]$ and $\widetilde{\mathcal{B}} = |\widetilde{E}(a,b) - \widetilde{E}(a,b')| + |\widetilde{E}(a',b') + \widetilde{E}(a',b)|$. Then,

$$\begin{aligned}
&|E(a,b) - \widetilde{E}(a,b)| \\
&= \left| \int dx \int dy\, f(x) g(y) M^{\sigma}(x) \otimes N^{\sigma}(y) [\rho_{AB} - \rho_A \otimes \rho_B] \right| \\
&\leqslant \int dx \int dy |M^{\sigma}(x) \otimes N^{\sigma}(y) [\rho_{AB} - \rho_A \otimes \rho_B]| \\
&= 2\Delta_C
\end{aligned} \tag{36}$$

for arbitrary a and b. Moreover, it can be shown that

$$\begin{aligned}
&||E(a,b) - E(a,b')| - |\widetilde{E}(a,b) - \widetilde{E}(a,b')|| \\
&\quad \leqslant |E(a,b) - E(a,b') - \widetilde{E}(a,b) + \widetilde{E}(a,b')| \qquad (37) \\
&\quad \leqslant |E(a,b) - \widetilde{E}(a,b)| + |E(a,b') - \widetilde{E}(a,b')|, \qquad (38)
\end{aligned}$$

where we have used the reserve triangular inequality $||A| - |B|| \leqslant |A - B|$ to obtain the first inequality and the second inequality follows from the triangular inequality. Likewise, we also obtain

$$\begin{aligned}
&||E(a',b') + E(a',b)| - |\widetilde{E}(a',b') + \widetilde{E}(a',b)|| \\
&\quad \leqslant |E(a',b') - \widetilde{E}(a',b')| + |E(a',b) - \widetilde{E}(a',b)|.
\end{aligned} \tag{39}$$

Then the difference between the two Bell-CHSH functions is bounded as

$$|\mathcal{B} - \widetilde{\mathcal{B}}| \leqslant 8\Delta_C \leqslant 8\left\{1 - \exp\left(-\frac{\mathcal{M}(|\psi\rangle)}{16\sigma^2}\right)\right\},$$

where we have used Eq. (31). This completes the proof as the Bell-CHSH function for a product state is bounded by 2, i.e., $\widetilde{\mathcal{B}} \leqslant 2$. ■

This theorem indicates that, in order to observe a large violation of the Bell-CHSH inequality, $\mathcal{M}(|\psi\rangle)$ should be sufficiently large and/or σ should be sufficiently small. This elucidates why previous studies have used macroscopic quantum superpositions to show a violation of the Bell-CHSH inequality or witness entanglement with imprecise measurements [68–71].

VI. CONCLUSION

We have investigated bipartite entanglement in many-body spin systems in terms of the correlation in local measurements. It turned out that the CLM is upper bounded by a function of quantum mutual information for general mixed states and there exist local POVMs that give a CLM larger than the linear entropy of a subsystem for pure states. As a realistic example, we have considered the case wherein local measurements are performed in the basis of a collective spin operator. Under this restriction, while the CLM with appropriate spin directions properly captures the entanglement of spin squeezed states and the ground state of the Heisenberg XXZ model, it does not capture the correlation of Haar random states. We have also considered the case of imprecise measurement and generalized the definition of the CLM accordingly. It turned out that the measure of quantum macroscopicity gives a bound to the CLM with imprecise measurement and similarly to the Bell-CHSH function. This analysis indicates that in order to observe a large violation of the Bell-CHSH inequality with many-body spin systems, one needs to prepare an entangled state with a large quantum macroscopicity. As a final remark, it would be interesting to investigate if there is a certain class of POVM that reveals entanglement of any pure state. Symmetric, informationally complete POVMs (SIC-POVMs) might be one of such candidates.

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APPENDIX A: FLUCTUATION OF $|A_{\alpha,\beta}|^2$

In Sec. III A, we have argued that the CLM for Haar random states goes to zero as N increases as $|A_{\alpha,\beta}|^2 = |\langle \alpha,\beta|\psi\rangle|^2$ should approach $1/2^N$ with a vanishing fluctuation. This can be shown from the following theorem.

Theorem 7. For any state $|\phi\rangle$,

$$\mathbb{E}_U\left(|\langle\phi|U|0\rangle|^2-\frac{1}{d}\right)^2=O\left(\frac{1}{d^2}\right), \tag{A1}$$

where d is the dimension of the Hilbert space and $\mathbb{E}_U$ indicates the average over all unitary operator U in terms of the Haar measure.

Proof. Using $\mathbb{E}_U UAU^\dagger=\mathrm{Tr}[A]/d$, we obtain

$$\mathbb{E}_U\left(|\langle\phi|U|0\rangle|^2-\frac{1}{d}\right)^2=\mathbb{E}_U|\langle\phi|U|0\rangle|^4-\frac{1}{d^2}. \tag{A2}$$

Therefore, we only need to calculate $\mathbb{E}_U|\langle\phi|U|0\rangle|^4$. As

$$\mathbb{E}_U|\langle\phi|U|0\rangle|^4$$
$$=\langle\phi,\phi|\mathbb{E}_U[U|0\rangle\langle 0|U^\dagger\otimes U|0\rangle\langle 0|U^\dagger]|\phi,\phi\rangle, \tag{A3}$$

the theorem follows from the well-known result (see, e.g., Ref. [72])

$$\mathbb{E}_U U|0\rangle\langle 0|U^\dagger\otimes U|0\rangle\langle 0|U^\dagger=\frac{\mathbb{1}+\mathbb{F}}{d(d+1)}, \tag{A4}$$

where $\mathbb{I}$ and $\mathbb{F}$ are the identity and the swap operators in $(\mathbb{C}^d)^{\otimes 2}$, respectively. ∎

The desired argument follows by choosing $|\phi\rangle=|\alpha,\beta\rangle$ as $U|0\rangle$ is a Haar random state and $d=2^N$.

APPENDIX B: PROOF OF THEOREM 5

Note that

$$\begin{aligned}\mathcal{F}(|\psi\rangle,\rho'_{AB})^2&=\int_{D_X}dx\int_{D_Y}dy\,\langle\psi|E^\sigma(x)\otimes F^\sigma(y)|\psi\rangle\langle\psi|E^\sigma(x)\otimes F^\sigma(y)|\psi\rangle\\&=\sum_{\substack{i,i',j,j',\\ \mu_i,\mu_i'\mu_j,\mu_j'}}\exp\left[-\frac{(i-i')^2+(j-j')^2}{8\sigma^2}\right]|\langle i,\mu_i,j,\mu_j|\psi\rangle|^2|\langle i',\mu_i',j',\mu_j'|\psi\rangle|^2\\&\geqslant\exp\left[\sum_{\substack{i,i',j,j',\\ \mu_i,\mu_i'\mu_j,\mu_j'}}-\frac{(i-i')^2+(j-j')^2}{8\sigma^2}|\langle i,\mu_i,j,\mu_j|\psi\rangle|^2|\langle i',\mu_i',j',\mu_j'|\psi\rangle|^2\right],\end{aligned}$$

where we have used $\int_{-\infty}^{\infty}dx\;p(x,i)^{1/2}p(x,i')^{1/2}=\exp[-(i-i')^2/(8\sigma^2)]$ in the second equality and the Jensen's inequality to obtain the last expression. Then the proof is completed as

$$\sum_{\substack{i,i',j,j',\\ \mu_i,\mu_i'\mu_j,\mu_j'}}(i-i')^2|\langle i,\mu_i,j,\mu_j|\psi\rangle|^2|\langle i',\mu_i',j',\mu_j'|\psi\rangle|^2=2\mathcal{V}_{|\psi\rangle}(S_A\otimes\mathbb{1}),$$

$$\sum_{\substack{i,i',j,j',\\ \mu_i,\mu_i'\mu_j,\mu_j'}}(j-j')^2|\langle i,\mu_i,j,\mu_j|\psi\rangle|^2|\langle i',\mu_i',j',\mu_j'|\psi\rangle|^2=2\mathcal{V}_{|\psi\rangle}(\mathbb{1}\otimes S_B).$$

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