Bistability of a slow mechanical oscillator coupled to a laser-driven two-level system

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It has been recently proposed that single molecule spectroscopy could be employed to detect the motion of nanomechanical resonators. Estimates of the coupling constant (g) between the molecular two-level system and the oscillator indicate that it can reach values much larger than the mechanical resonating pulsation (ω_m) and the two-level system linewidth (Γ). Other experimental realizations of the same system are also approaching this strong-coupling regime. In this paper we investigate the behavior of the system in the limit for slow mechanical oscillator $\omega_m \ll \Gamma$. We find that, for sufficiently large coupling, the system undergoes a bistability reminiscent of that observed in optical cavities coupled to mechanical resonators.

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I. INTRODUCTION

The rapid development of nanoelectromechanics in the past decade has seen the proposal and the experimental realization of several systems where in order to detect the displacement of a mechanical resonator it is coupled to a two-level system (TLS). This includes superconducting qubits [1], nitrogen vacancy centers in diamonds [2–9], semiconductor TLS [10–12], spins [13–17], or single molecules [18]. One of the interests in coupling a mechanical resonator to a TLS is that one can reach large couplings constants [2,13,18,19]. According, for instance, to the estimates of Ref. [18], the coupling constant can become larger than the mechanical pulsation ω_m or the TLS linewidth Γ. Increasing the coupling not only improves the detection sensitivity of the mechanical displacement, but allows one to reach new regimes, where the dynamics of the TLS and of the mechanical oscillator have to be considered on the same footing. This is well known for the case of mechanical resonators coupled to optical cavities [20–22], for which it is possible to reach the strong-coupling limit since the effective optomechanical coupling constant is proportional to the square root of the number of photons present in the cavity. A striking effect is the onset of a static bistability that was observed long time ago [21,23,24]. Reaching such a strong-coupling limit with the bare coupling between the oscillator and the quantum system, in our case the TLS, is difficult, but currently at reach of the present technology.

In this paper we consider the case of a slow oscillator $\omega_m \ll \Gamma$ coupled to a laser-driven TLS and, by exploiting an adiabatic expansion, we obtain a description of the TLS-oscillator system in the strong-coupling limit. We find that a static bistability is also expected, with a behavior similar to that observed for optical cavities. In this case it is induced by the coupling to a single quantum degree of freedom of the TLS instead of the macroscopic condensate of photons. Note also that the presence of the TLS renders the problem intrinsically nonlinear. We calculate the luminescence that is the typical observed quantity in single-molecule spectroscopy experiments [25]. We find that the interplay of the cooling-heating effect with the bistability gives rise to anomalous line shapes. This bistability resembles the one observed in optical cavities, but with the notable difference that the quantum nature of the TLS has to be taken into account.

The paper is organized as follows. In Sec. II we present the model. In Sec. III the Born-Markov equations are derived. In Sec. IV we exploit the separation of time scales to simplify the Bork-Markov equations and obtain a description of the slow degree of freedom. In Sec. V we discuss the effective temperature induced by the coupling to the TLS. In Sec. VI the condition for the appearance of the mechanical bistability is discussed. In Sec. VII the effect of the stochastic fluctuations is considered by solving numerically the Fokker-Planck equation. Section VIII gives our conclusions.

II. SYSTEM

We consider a TLS coupled to a laser and to a mechanical oscillator as described in Ref. [18] (see Fig. 1). We will focus on this system, but the model describes several systems, for instance, the TLSs coupled by the strain to an oscillator [9]. The coupling of the TLS to the mechanical oscillator is due to the Stark effect and the difference of potential that is induced between the oscillator (for instance, a suspended carbon nanotube) and the transparent and conducting substrate over which the molecules are dispersed. In experiments the light emitted by a single molecule is collected and detected as a function of the laser beam frequency and intensity. The system is described by the following Hamiltonian:

$$H_S = -\frac{\Delta}{2}\sigma_z + \hbar\Omega\sigma_x \cos\omega_L t - \hbar g\sigma_z(b+b^\dagger) + \hbar\omega_m b^\dagger b. \tag{1}$$

Here Δ is the TLS splitting, Ω the coupling intensity to the laser, g is the electromechanical coupling, ω_L is the laser frequency, and ω_m is the mechanical pulsation ($\hbar$ is the reduced Planck constant). The operators σ_x and σ_z are Pauli matrices and b and $b^\dagger$ are the destruction and creation operators for the mechanical oscillator excitations. The displacement operator reads $x = x_z(b+b^\dagger)$ with $x_z = \sqrt{\hbar/2m\omega_m}$ the zero point displacement fluctuation amplitude for an oscillator of mass m. The TLS and the mechanical oscillator are coupled to the

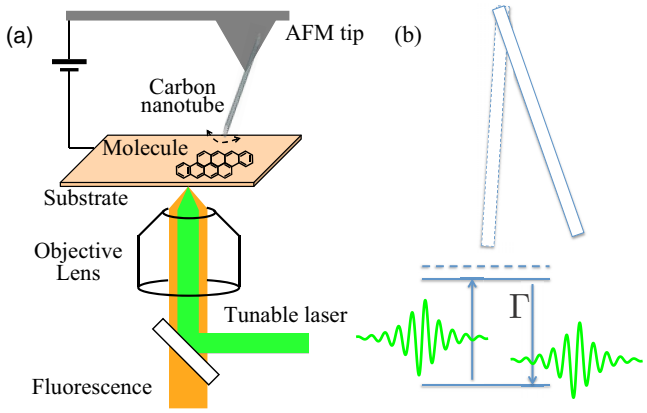

FIG. 1. Left: schematics of the system proposed in Ref. [18] and considered in this paper. Right: simplified view of the two-level system coupled to a mechanical oscillator via the Stark effect and to a laser.

environment that leads to a finite linewidth Γ of the TLS resonance and to a damping rate γ for the oscillator.

III. BORN-MARKOV EQUATIONS

We proceed by assuming a weak coupling with the environment of the system. By standard methods [26] in the Born-Markov approximation the environment can be traced out and an equation for the reduced density matrix $\rho(t)$ for the system (oscillator plus TLS) can be derived:

$$\dot{\rho} = \mathcal{L}\rho. \tag{2}$$

We define now $\rho_{ij}(x,x',t) = \langle x,i|\rho(t)|x',j\rangle$, where $|x,i\rangle$ are the eigenstates of x and σ_z with eigenvalues x and $(\sigma_z)_{ii} = \pm 1$, respectively. It is convenient to introduce the variables $x_+ = (x + x')/2$ and $x_- = x - x'$. As discussed in the Introduction we will consider in this paper the slow oscillator limit. In terms of the above introduced parameters the condition reads $\gamma \ll \omega_m$ and

$$\omega_m, g \ll \Gamma \ll \Delta\,. \tag{3}$$

This implies a time scale separation between the TLS and the mechanical oscillator dynamics. It is thus convenient to write the Born-Markov operator as $\mathcal{L} = \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{s}$ with $\mathcal{L}_\mathrm{f}$ and $\mathcal{L}_\mathrm{s}$ the fast and slow component, respectively. Explicitly the fast component reads

$$\mathcal{L}_\mathrm{f} = \begin{pmatrix} 0 & i\Omega/2 & -i\Omega/2 & \Gamma \\ i\Omega/2 & -i\delta' - \Gamma/2 & 0 & -i\Omega/2 \\ -i\Omega/2 & 0 & i\delta' - \Gamma/2 & i\Omega/2 \\ 0 & -i\Omega/2 & i\Omega/2 & -\Gamma \end{pmatrix}, \tag{4}$$

with $\delta' = \delta - 2gx_+/x_z$ and $\delta = \omega_L - \Delta/\hbar$ the detuning. The components of the density matrix are $\{\rho_{11},\rho_{12},\rho_{21},\rho_{22}\}$. The slow component reads

$$\mathcal{L}_\mathrm{s} = \mathcal{L}_\mathrm{osc} + \frac{x_-}{x_z}\mathcal{L}_-\,, \tag{5}$$

with

$$\mathcal{L}_\mathrm{osc} = \frac{i\hbar}{m}\partial_+\partial_- - i\frac{m\omega_m^2}{\hbar}x_+x_- - 2\gamma x_-\partial_- - Dx_-^2, \tag{6}$$

where $\mathcal{L}_- = ig\,\mathrm{diag}(1,0,0,-1)$, and the last two terms in Eq. (6) describe the coupling of the oscillator to an environment at temperature T. From the fluctuation-dissipation theorem $D = m\gamma(\hbar\omega_m/2)\coth(\hbar\omega_m/2k_BT)$. We use the notation $\partial_\pm = \partial/\partial x_\pm$. The operator $\mathcal{L}_\mathrm{f}$ describes the dynamics of the TLS and implies a fast evolution of the density matrix on a scale of the maximum between Γ and Ω. Thus this evolution is much faster than that induced by the $\mathcal{L}_\mathrm{s}$ term that takes place on the ω_m time scale, for what concerns $\mathcal{L}_\mathrm{osc}$, and even slower for the dissipative part. We have included the term proportional to gx_+ in $\mathcal{L}_\mathrm{f}$ since we want to allow the possibility that this term becomes of the same order of Γ in the strong-coupling limit. In order to check that the approximations are consistent what matters is the fluctuation of the oscillator position $(\Delta x_+)^2 = \langle x_+^2\rangle - \langle x_+\rangle^2$. If $g\Delta x_+/x_z$ is larger than ω_m, then this coupling has to be included in the fast part. On the other side, it is typically correct to regard the term proportional to gx_- as a slow contribution that can be included in $\mathcal{L}_\mathrm{s}$. This second assumption is valid for $g\Delta x_-/x_z \ll \Gamma$. We will come back to these two assumptions after the solution of Eq. (2) has been obtained.

IV. ADIABATIC ELIMINATION OF FAST VARIABLES

We now exploit the separation of time scales, $\omega_m \ll \Gamma$, using the method of adiabatic elimination of fast variables (see, for instance, Ref. [27]) to integrate out the TLS fast degrees of freedom and obtain an equation for the mechanical oscillator reduced density matrix. We begin by noting that $\mathcal{L}_\mathrm{f}$ is a function of x_+ only. One can then define the kernel of the operator $\mathcal{L}_\mathrm{f}$ by the equation:

$$\mathcal{L}_\mathrm{f}(x_+)\rho^0(x_+) = 0\,. \tag{7}$$

Physically $\rho^0(x_+)$ is the stationary state of the TLS for a given value of x_+. Since $\mathcal{L}_\mathrm{f}$ is not Hermitian, left and right eigenvectors are different. Let's define w_0 as the left eigenvector of $\mathcal{L}_\mathrm{f}$ with vanishing eigenvalue. One can readily show that it has the form $w_0 = \{1,0,0,1\}$ and when projected on an arbitrary state ρ it gives its trace over the TLS states:

$$(w_0,\rho) = \rho_{11} + \rho_{22} = \mathrm{Tr}\rho\,. \tag{8}$$

Since probability is conserved by the time evolution $0 = d\,\mathrm{Tr}(\rho)/dt = \mathrm{Tr}(\mathcal{L}_\mathrm{f}\rho) = (w_0,\mathcal{L}_\mathrm{f}\rho)$ for any ρ, the relation $w_0^t\mathcal{L}_\mathrm{f} = 0$ holds. We choose now the normalization of $\rho^0(x_+)$ such that $(w_0,\rho^0) = 1$. Any ρ can then be written as a sum of its projection on the kernel of $\mathcal{L}_\mathrm{f}$, the slow component, and on its orthogonal complement, the fast component:

$$\rho_{ij}(x_+,x_-,t) = \rho^0_{ij}(x_+)R(x_+,x_-,t) + \rho^f_{ij}(x_+,x_-,t), \tag{9}$$

where by construction $(w_0,\rho^f) = 0$ and $(w_0,\rho) = R(x_+,x_-)$. Let us define P as the projector on ρ^0 and $Q = 1 - P$ as the projector on the complement. Note that P and Q depend on x_+; they are the projectors on the 4×4 phase space of the TLS. Again, for given x_+, the following properties hold: $P\mathcal{L}_\mathrm{f} = \mathcal{L}_\mathrm{f}P = 0$. We substitute now this expression in the master Eq. (2) and by applying alternatively P and Q we obtain the two equations:

$$\dot{R}\rho^0 = P\mathcal{L}_\mathrm{s}(R\rho^0 + \rho^f), \tag{10}$$

$$Q\dot{\rho^s} = Q\mathcal{L}_\mathrm{f}\rho^f + Q\mathcal{L}_\mathrm{s}(R\rho^0 + \rho^f)\,. \tag{11}$$

Up to now these equations are exact. Let's now use the fact that $\mathcal{L}_f$ is very large to solve the second equation (a systematic expansion could be derived using the Laplace transform; we consider here only the leading term)

$$Q\mathcal{L}_{\rm f}\rho^f = -Q\mathcal{L}_{\rm s}(R\rho^0). \tag{12}$$

Since $\mathcal{L}_{\rm f}$ acts on the subspace orthogonal to the kernel and the solution is sought in the same space $Q\mathcal{L}_{\rm f}Q$ is actually invertible in this subspace (while $\mathcal{L}_{\rm f}$ has one vanishing eigenvalue in the full space of ρ). We thus substitute the solution into Eq. (10) and obtain

$$\dot{R}\rho^0 = P\mathcal{L}_{\rm s}(R\rho^0) - P\mathcal{L}_s(Q\mathcal{L}_f Q)^{-1}Q\mathcal{L}_s(R\rho^0). \tag{13}$$

We begin by evaluating

$$Q\mathcal{L}_s(R\rho^0) = \frac{x_-}{x_z}RQ\mathcal{L}_-\rho^0 + \frac{i\hbar}{m}\partial_- RQ\partial_+\rho^0. \tag{14}$$

By projecting Eq. (13) on w_0 we obtain

$$\dot{R} = \left[\mathcal{L}_{\rm osc} + \frac{x_-}{x_z}\alpha_1 - \frac{i\hbar x_-}{mx_z}\alpha_2\partial_- + \frac{x_-^2}{x_z^2}\alpha_3\right]R, \tag{15}$$

where $\alpha_1 = (w_0, \mathcal{L}_-\rho^0), \alpha_2 = (w_0, \mathcal{L}_-(Q\mathcal{L}_{\rm f}Q)^{-1}Q\partial_+\rho^0)$, and $\alpha_3 = (w_0, \mathcal{L}_-(Q\mathcal{L}_f Q)^{-1}Q\mathcal{L}_-\rho^0)$.

Using the explicit form of $\mathcal{L}_{\rm f}$ one can readily find the three matrix elements:

$$\beta_1 = -\frac{i\alpha_1}{g} = \frac{\Gamma^2 + 4\delta'^2}{\Gamma^2 + 4\delta'^2 + 2\Omega^2}, \tag{16}$$

$$\beta_2 = \frac{ix_z\Gamma^2\alpha_2}{2g^2} = -\frac{32\delta'\Gamma\Omega^2(2\Gamma^2 + \Omega^2)}{(\Gamma^2 + 4\delta'^2 + 2\Omega^2)^3}, \tag{17}$$

$$\beta_3 = \frac{\alpha_3\Gamma}{g^2} = \frac{4\Omega^2(\Gamma^2 + 4\delta'^2)(2\Gamma^2 + \Omega^2)}{(\Gamma^2 + 4\delta'^2 + 2\Omega^2)^3}. \tag{18}$$

Substituting these expressions into Eq. (15) and introducing the Wigner transform $W(x_+, p) = \int dx_- e^{-ipx_-}R(x_+, x_-)$, we have $x_- \to i\hbar\partial_p$ and $\partial_- \to ip/\hbar$ in Eq. (13). We thus find for W the following equation:

$$\dot{W} = \left[-\frac{p}{m}\partial_+ + \left(m\omega_m^2 x_+ - F\right)\partial_p + 2\gamma_t\partial_p p + D_t\partial_p^2\right]W, \tag{19}$$

where F is the average force acting on the oscillator,

$$F = \frac{\hbar g}{x_z}\langle\sigma_z\rangle = -\frac{g\beta_1}{x_z}, \tag{20}$$

and $\gamma_t = \gamma + \gamma_o$ and $D_t = D + D_o$ are the total dissipation and diffusion coefficients, respectively, with $\gamma_o = \hbar g^2\beta_2/mx_z^2\Gamma^2$ and $D_o = \hbar^2 g^2\beta_3/x_z^2\Gamma$ the dissipation and the diffusion coefficients induced by the coupling to the driven TLS.

Equation (19) with the relations (11) and (9) allow one to obtain the behavior of the mechanical oscillator and the optical response of the TLS. In practice ρ^f is very important for the derivation of the equation of motion of R, but its contribution to ρ is small (of the order ω_m/Γ), and it can be neglected in the calculation of the averages in the following.

V. EFFECTIVE TEMPERATURE

We begin the study of Eq. (19) by defining an effective temperature in analogy with the fluctuation-dissipation relation:

$$\coth\left(\frac{\hbar\omega_m}{2k_B T_{\rm eff}}\right) = \frac{2D_t}{\gamma_t\hbar\omega_m}. \tag{21}$$

In the case of $\gamma \ll \gamma_0$ one finds

$$\coth\left(\frac{\hbar\omega_m}{2k_B T_{\rm eff}}\right) = -\frac{\Gamma^2 + \delta'^2}{4\delta'\omega_m}. \tag{22}$$

For positive values of δ' the system is unstable (negative damping term). For negative values of δ' the function in Eq. (22) has a minimum value of $\Gamma/4\omega_m \gg 1$ at $\delta' = -\Gamma$. This means that, when the coupling to the TLS dominates over the coupling to the environment, the system reaches a classical stationary state ($k_B T_{\rm eff} \gg \hbar\omega_m$). Thus for $\gamma = 0$ one can write

$$k_B T_{\rm eff} = \hbar\frac{\Gamma^2 + \delta'^2}{8\delta'}. \tag{23}$$

We note that the form of the effective temperature is the same that is found for an oscillator coupled to a cavity [28]. This follows for the similarity of the spectrum of fluctuation of σ_z and that of the number of photons in a cavity. The result implies clearly that it is not possible to use the TLS in the $\Gamma \gg \omega_m$ limit to cool the oscillator in the quantum regime [18,29].

We are now in the position to check the conditions on Δx_+ and Δx_- assumed for the solution of the problem. From the equipartition theorem $m\omega_m^2(\Delta x_+)^2 = k_B T_{\rm eff}$ and $\Delta x_- \sim \hbar/\Delta p$ with $(\Delta p)^2/m = k_B T_{\rm eff}$. The condition on Δx_- reads then

$$g \ll \Gamma\left(\frac{k_B T_{\rm eff}}{\hbar\omega_m}\right)^{1/2}. \tag{24}$$

Since $k_B T_{\rm eff} > \hbar\Gamma$, a sufficient condition for the validity of the approximations is $g \ll \Gamma(\Gamma/\omega_m)^{1/2}$, that gives a large window of validity of the theory.

For finite value of γ a part of the region for which $\delta' > 0$ becomes stable, but before discussing this point we need to take into account the fact that $\delta' = \delta - 2gx_+/x_z$ and that this induces a mechanical bistability.

VI. MECHANICAL BISTABILITY

The value of x_+ entering δ' is a stochastic variable whose statistics can be obtained from the solution of Eq. (19). In general the distribution function is strongly peaked around the equilibrium position x_e that satisfies the equation

$$m\omega_m^2 x_e = F(x_e). \tag{25}$$

We thus begin by solving Eq. (25). The equation can be studied more conveniently by eliminating

$$x_e = (\delta - \delta')x_z/2g \tag{26}$$

from the definition of $\delta'(x_e)$. The equation then reads

$$\delta - \delta' = \lambda\Gamma\beta_1(\delta'), \tag{27}$$

where we defined $\lambda = 4g^2/\omega_m\Gamma$, that is the relevant dimensionless coupling constant also known as cooperativity.

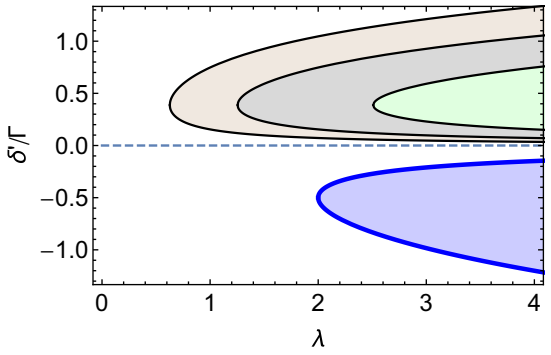

FIG. 2. Region of static (shaded blue region for negative δ') and dynamical (shaded regions for positive δ') instabilities in the plane λ-δ' for $\Omega = \Gamma$ and for $Q\omega_m/\Gamma = 4$ (leftmost brown region), 2 (gray), and 1 (green). The dashed line corresponds to the value $\delta' = 0$, separatrix between cooling and heating in the case $\gamma = 0$ (or $Q = \infty$).

A relevant interpretation of λ for our problem is also the following: When the TLS is in the excited state an additional force $F_o = \hbar g/x_z$ acts on the oscillator modifying its equilibrium position $\Delta x_e = F_0/m\omega_m^2$. Let us call $\epsilon_P = F_o \Delta x_e = \hbar g^2/\omega_m$ the (classical) energy scale that corresponds to (twice) the variation of the potential energy of the oscillator. Comparing it with the relevant energy scale of the TLS $\hbar\Gamma$ we have $\epsilon_P/\hbar\Gamma = \lambda/2$, that measures the relevance of the TLS on the oscillator dynamics. This phenomenon resembles the bistability expected in suspended carbon nanotubes forming a single-electron transistor [30,31]. There the two-level system is the empty or filled suspended quantum dot and the role of the laser driving is played by the electrons entering the quantum dot for transport.

We come now back to Eq. (27). Comparing the derivatives with respect to δ' of the right- and left-hand sides of Eq. (27) one finds that for $\lambda < \lambda_c = 2(\Gamma^2 + 2\Omega^2)^{3/2}/\Omega^2 3\sqrt{3}$ there is a single solution for the equation for any value of δ. For $\lambda > \lambda_c$ three solutions exist, two stable and one unstable. The two stable solutions for δ' correspond to two stable (or metastable) equilibrium positions given by Eq. (26). From the expression of λ_c one can see that λ_c takes the minimum value of 2 for $\Gamma = \Omega$. In terms of the bare coupling g this implies that the minimum value required to observe the bistability is $g_{\min} = (\omega_m \Gamma/2)^{1/2}$, the geometric mean of the mechanical frequency and of the TLS inverse lifetime. Equivalently, the requirement is that the cooperativity $\lambda > 1$.

In order to find the region of bistability as a function of λ and the physical laser detuning δ, we begin by recognizing that the bistable behavior upon increasing λ begins when the $d/d\delta'$ of the left- and right-hand side of Eq. (27) coincide:

$$-1 = \lambda\Gamma \frac{d\beta_1}{d\delta'}. \tag{28}$$

This is an equation for δ' and λ. Its solution is shown in Fig. 2 for $\Omega = \Gamma$. One can see that the bistable region takes place for negative values of δ', for which the TLS generate a standard positive dissipative term. One should however be cautious when converting this result to the externally tunable δ. This can be done by using Eq. (27). The result is shown in Fig. 3.

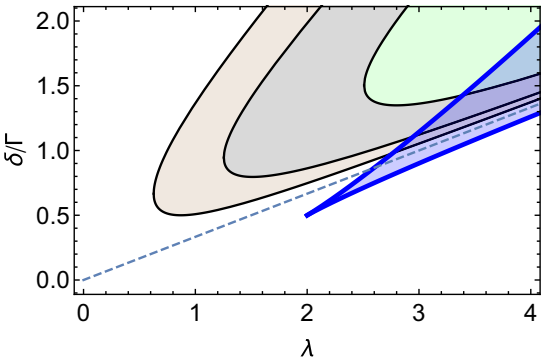

FIG. 3. Same as Fig. 2 in the plane δ-λ. Due to the bistability the contour of the bistability region can be in coincidence with the dynamically unstable region of the phase space.

The dashed line is the region $\delta' = 0$. As one can see a part of the bistable phase is now apparently in the region of dynamical instability ($\delta' > 0$). This is more subtle, since actually the transformation (27) is not bijective; to the same value of δ two values of δ' may be associated. We will discuss later the consequences of this fact in more details.

Let us begin by studying the consequences of the bistability on the luminescence. In order to obtain the luminescence we only need the probability of occupation of the excited state, P_e, as a function of the externally fixed detuning δ. From the solution of $\mathcal{L}_f \rho^0 = 0$ one finds the familiar result:

$$P_e = \frac{\Omega^2}{\Gamma^2 + 2\Omega^2 + 4\delta'^2}. \tag{29}$$

We plot parametrically P_e and $\delta(\delta')$ from Eq. (27) in Fig. 4. One can see that the line shape is strongly modified, with the appearance of a part with three values possible, corresponding to the two stable and one unstable equilibrium states. Clearly the unstable state cannot be realized, but the oscillator spends a sizable part of the time on the two (meta)stable states. In order to evaluate the actual form of the expected luminescence line we need to solve the Fokker-Planck Eq. (19).

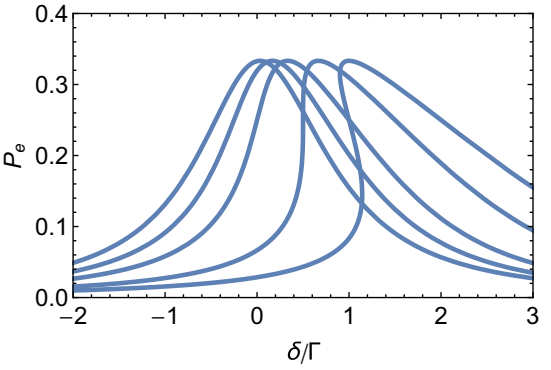

FIG. 4. Probability of occupation of the excited state (P_e proportional to the luminescence) as a function of the detuning δ for $\Omega = \Gamma$ and $\lambda = 0.1,\ 0.5,\ 1,\ 2$, and 3 (from left to right). The bistable behavior begins at $\lambda = 2$.

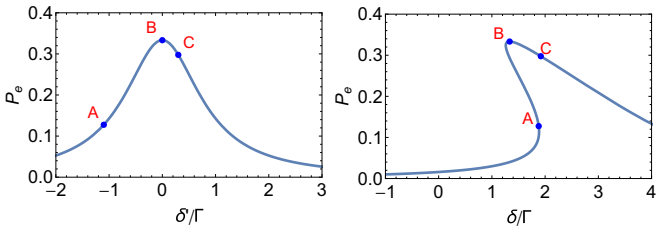

FIG. 5. Mapping of the luminescence line from δ' to δ. The points A, B, and C correspond to cooling, neutral, and heating values, respectively. One can see that for $\delta = 2$ both cooling and heating are possible, depending on value of x that determines if the point A or C is actually occupied.

VII. EFFECT OF FLUCTUATIONS

We need to consider the effect of fluctuations. Since the system may be unstable for positive values of δ', one cannot discard anymore the intrinsic dissipation due to the coupling to the environment at temperature T. The region of dynamical instability is defined by the condition $\gamma_t < 0$, where γ_t is defined after Eq. (20). This gives the equation for the critical line:

$$1 + 2\lambda Q \frac{\omega_m}{\Gamma} \beta_2(\delta') = 0\,, \tag{30}$$

where we introduced the quality factor $Q = \omega_m/\gamma$. We show in Figs. 2 and 3 the regions of dynamical instability as a function of λ and δ' or δ, respectively. It is important to realize that the dynamical instability takes place very close to the static bistability; we will see that this has consequences in the expected line shapes.

Let us now discuss the interplay of the heating and cooling effect and the bistable behavior. In Fig. 5 we show P_e as a function of δ' and δ. One can follow how the states indicated by the letters A, B, and C are mapped in the δ plot. The point A is in the cooling part of the line ($\delta' < 0$) but in the δ space it appears at a value of δ larger than the value of δ corresponding to B ($\delta' = 0$) that defines the border between cooling and heating. The cooling state A is thus bistable with the heating state C. Due to the fluctuations the system will spend some time in both states, with a probability that is determined by Fokker-Planck equation. This also explains the

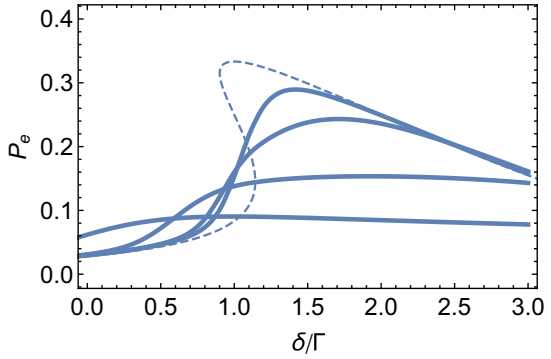

FIG. 6. Luminescence: $\lambda = 2$, $\omega_m/\Gamma = 10^{-3}$, $k_B T/\epsilon_P = 0.01$, $Q = 10$, 10^2, 10^3, and 10^4 for the curves from the steepest to the smoothest, respectively. Increasing Q increases the fluctuations and smoothens the line shape. The parameter $Q\omega_m/\Gamma$ entering Eq. (30) takes thus the values 10^{-2}, 10^{-1}, 1, and 10.

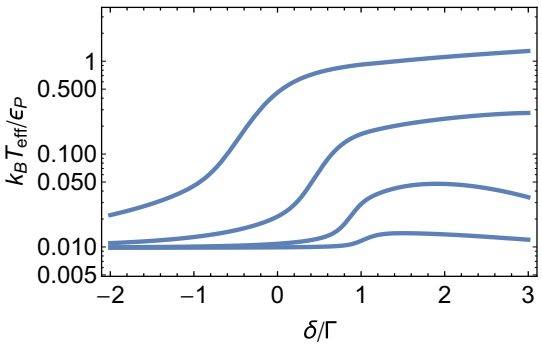

FIG. 7. Effective temperature $k_B T_{\rm eff}/\epsilon_P$ as a function of the detuning δ for the same value of the quality factors of Fig. 4. The higher values of the temperature are of course obtained for the highest Q.

shape of the static bistability region that apparently leaks on the dynamically unstable region, as shown in Fig. 3.

In order to find the contribution of the fluctuations quantitatively we solve numerically the Fokker-Planck Eq. (19) by discretizing the phase space x-p in N_x and N_p points, respectively. The operator entering Eq. (19) and acting on W becomes thus a matrix of dimension $N = N_x N_p$. The stationary solution of Eq. (19) is found by solving the equation with the constraint of the normalization of W. We find that typically $N_x = N_y = 100$ is already sufficient to obtain a solution of the equation in the range of interest of the parameters. The Fokker-Planck equation can be rewritten in terms of the dimensionless variables $\tilde{x} = x/\Delta x_e$, $\tilde{p} = p\omega_m/F_o$, and $\tilde{t} = \omega_m t$. This gives

$$\frac{\partial W}{\partial \tilde{t}} = \left[-\tilde{p}\partial_{\tilde{x}} + [\tilde{x} - \beta_1]\partial_{\tilde{p}} + \tilde{\gamma}_t \partial_{\tilde{p}}\tilde{p} + \tilde{D}_t \partial^2_{\tilde{p}}\right] W, \tag{31}$$

where

$$\tilde{\gamma}_t = \frac{\omega_m}{\Gamma}\left[\frac{\lambda\beta_2}{2} + \frac{\Gamma}{Q\omega_m}\right], \quad \tilde{D}_t = -\frac{\omega_m}{\Gamma}\beta_3 + \frac{k_B T}{\epsilon_P Q}. \tag{32}$$

Results of the numerical solutions for the luminescence are shown in Fig. 6 for $k_B T/\epsilon_P = 0.01$, $\omega_m/\Gamma = 10^{-3}$, $\lambda = 2$ and for different values of Q. For small Q the luminescence follows closely the mean-field result (shown dashed) apart from the bistable region. Increasing Q one enters the region of dynamical instability, as expected from Eq. (30) and Fig. 3, and fluctuations increase dramatically, with the system spending a sizable time on the heating region. This is confirmed by the dependence of the effective temperature on δ for a given value of Q, as shown in Fig. 7. One can note that large values of Q the effective temperature increase for positive values of δ, reaching values of the order of ϵ_P, and thus washing out the bistable behavior. The result is a very smooth and asymmetric line shape.

VIII. CONCLUSIONS

In this paper we studied the behavior of a slow mechanical resonator ($\Gamma \gg \omega_m$) coupled to a laser driven TLS in the strong-coupling regime. We began by the Born-Markov description of the system [cf. Eq. (2)], and then, by eliminating the TLS fast variables, we obtained Eq. (19) for the Wigner function of the mechanical oscillator.

By analyzing this equation we showed that the oscillator effective temperature can be controlled by the laser detuning and the coupling (cooling or heating). We found that, when the coupling of the TLS dominates over the coupling to the environment, the mechanical oscillator can only be in a classical regime ($k_B T_{\rm eff} \gg \hbar\omega_m$).

We showed then that for sufficiently strong coupling the mechanical system can undergo a bistability. The conditions on the cooperativity and detuning for its observation are resumed in Fig. 3. Contrary to optomechanical cavities, the coupling controlling the instability cannot be tuned by the laser intensity (that here is parametrized by Ω). As a rule of thumb, we find that a coupling constant of the order of $(\omega_m\Gamma)^{1/2}$ is necessary for the bistability to take place. The bistability allows the possibility that the two stable states are one in the cooling and the other in the heating regime. This leads to a peculiar shape of the luminescence linewidth (cf. Fig. 6), that is broadened mainly due to the increase of the oscillator fluctuations induced by the heating effect. A promising experimental system where this effect could be observed is the one proposed in Ref. [18] of single molecules coupled to carbon nanotubes. The typical values of the parameters are $m = 10^{-21}$ Kg, $\Gamma > 10^7$ Hz, and $g < 10^9$ Hz, with a mechanical frequency of the carbon nanotube that depending on its length can vary from kHz to MHz. With these numbers it should be possible to reach large values of λ, for instance, for $g = 10^6$ Hz, $\omega_m = 2\pi \times 10^5$ Hz, and $\Gamma = 10^7$ Hz one finds $\lambda \approx 0.5$. For these couplings the effective mechanical quality factor $Q_o = \omega_m/\gamma_o$ induced by the coupling to the oscillator is $(\Gamma/g)^2\beta_2/2 \approx 10^2$. Thus even if the scale of the energy barrier between the two stable states (ϵ_P) is very small, when the coupling to the environment is sufficiently weak ($Q = \omega_m/\gamma \gg Q_o$) the effective temperature is controlled uniquely by the TLS and $k_B T_{\rm eff} \ll \epsilon_P$. This should lead to the observation of a luminescence line similar to the one predicted in Fig. 6.

Concerning the approximations used in the paper, it turns out that one of the main conditions is that the spread of the quantum variable x_- is small: $\Delta x_-/x_z \ll \Gamma/g$. This is necessary to include the term gx_- in the slow component ($\mathcal{L}_{\rm s}$) of the master equation, leaving the fast operator independent on x_-. A less technical way of stating this condition is to say that it fixes a limit on the quantum nature of the mechanical degree of freedom. In terms of the coupling constant the condition is given by Eq. (24). One finds that the scale of validity is set by $g \ll \Gamma(\Gamma/\omega_m)^{1/2}$ [or $\lambda \ll (\Gamma/\omega_m)^2$]. The bistable behavior as described above is thus well inside the limit of validity of the approximation. For $g \gg (\Gamma/\omega_m)^{1/2}$, the coupling is so strong that quantum coherence cannot be neglected anymore between two transition events, even if $\Gamma \gg \omega_m$. This regime is of course relevant for investigations on quantum manipulation of the mechanical states and constitutes an interesting perspective for future work.

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APPENDIX: RELATION TO THE QUANTUM NOISE METHOD

Let us briefly comment on the physical interpretation of the fluctuation and dissipation terms. For weak coupling it is known [32] that the dissipation and fluctuation can be derived directly from the quantum correlation function of the force operator. In our case this is defined as $S_{FF}(t) = (\hbar g/x_z)^2 S_{zz}(t)$ with

$$S_{zz}(t) = \langle\sigma_z(t)\sigma_z(0)\rangle - \langle\sigma_z\rangle^2, \tag{A1}$$

where the time evolution is ruled by only the TLS part of the Hamiltonian ($g = 0$). The quantity S_{zz} can be obtained explicitly:

$$S_{zz}(t > 0) = (w_0, M_z e^{\mathcal{L}_{\rm f} t} M_z \rho^0) - (w_0, M_z\rho^0)^2, \tag{A2}$$

where $M_z = (\mathcal{L}_- + \mathcal{L}_+)/(ig)$ is the superoperator for σ_z and we introduced $\mathcal{L}_+ = ig\,\mathrm{diag}(0,1,-1,0)$ by analogy with the definition of $\mathcal{L}_-$. Introducing the Laplace transform $S_{zz}(s) = \int_0^{+\infty} dt\, e^{st} S_{zz}(t)$ (with $\mathrm{Re}\, s < 0$) we have

$$S_{zz}(s) = (w_0, (M_z)\rho^0)^2/s - (w_0, M_z(s - \mathcal{L}_{\rm f})^{-1} M_z\rho^0). \tag{A3}$$

Using the projectors Q and P one can readily show that

$$S_{zz}(s) = -(w_0, M_z Q(s - Q\mathcal{L}_{\rm f} Q)^{-1} Q M_z \rho^0). \tag{A4}$$

For $s = 0$ the inverse has to be performed in the subspace defined by the projector Q. The power spectrum $S_{zz}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} S_{zz}(t)$ can then be related directly to the Laplace transform by using the property $S^*_{zz}(t) = S_{zz}(-t)$ that leads to

$$S_{zz}(\omega) = 2\,\mathrm{Re}[S_{zz}(s = i\omega - 0^+)]. \tag{A5}$$

We note that $w_0\mathcal{L}_+ = 0$. By explicit calculation one can verify that the term proportional to $\mathcal{L}_+$ coming from the M_z on the right matrix element of Eq. (A4) gives an imaginary term for $s = 0$, and thus it does not contribute to $S_{zz}(\omega \to 0)$. One can thus substitute $\mathcal{L}_-$ into the definition of $M_z = \mathcal{L}_-/(ig)$ in this case. For $\omega_m \ll \Gamma$ we can obtain the diffusion constant of the Fokker-Plack equation from the vanishing frequency value of the force fluctuation spectrum:

$$D_0 = S_{\rm FF}(\omega \to 0)/2 = \hbar^2\alpha_3/x_z^2. \tag{A6}$$

This value coincides with the adiabatic approach result. One can also verify by direct calculation that the derivative with respect to ω of $S_{\rm FF}(\omega)$ gives also correctly the damping term entering Eq. (19). Thus the weak-coupling calculation allows one to find the form of the coefficients entering the Fokker-Plack equation, but it does not allow one to prove the validity of the approach. According to the derivation presented in the main text, actually it is not the weak-coupling condition that allows one to obtain the Fokker-Planck description, but the separation of time scales.

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