Editors' Suggestion

Wigner function of noninteracting trapped fermions

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We study analytically the Wigner function $W_N(\mathbf{x},\mathbf{p})$ of N noninteracting fermions trapped in a smooth confining potential $V(\mathbf{x})$ in d dimensions. At zero temperature, $W_N(\mathbf{x},\mathbf{p})$ is constant over a finite support in the phase space $(\mathbf{x},\mathbf{p})$ and vanishes outside. Near the edge of this support, we find a universal scaling behavior of $W_N(\mathbf{x},\mathbf{p})$ for large N. The associated scaling function is independent of the precise shape of the potential as well as the spatial dimension d. We further generalize our results to finite temperature $T > 0$. We show that there exists a low-temperature regime $T \sim e_N/b$, where e_N is an energy scale that depends on N and the confining potential $V(\mathbf{x})$, where the Wigner function at the edge again takes a universal scaling form with a b-dependent scaling function. This temperature-dependent scaling function is also independent of the potential as well as the dimension d. Our results generalize to any $d \geqslant 1$ and $T \geqslant 0$ the $d = 1$ and $T = 0$ results obtained by Bettelheim and Wiegman [Phys. Rev. B **84**, 085102 (2011)] (see also the earlier paper by Balazs and Zipfel [Ann. Phys. (NY) **77**, 139 (1973)]).

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I. INTRODUCTION

A. Background

The Heisenberg uncertainty principle, the basic cornerstone of quantum mechanics, tells us that the position and the momentum of a single quantum particle cannot be measured simultaneously. In position space, the squared wave function $|\psi(x)|^2$ is the probability density. Similarly, $|\hat{\psi}(p)|^2$ [where $\hat{\psi}(p)$ is the Fourier transform of $\psi(x)$] gives the probability density in momentum space. Although the joint probability density function (PDF) cannot be defined in phase space (x,p), the closest object to such a joint PDF is the celebrated single-particle "Wigner function" [1]

$$W_1(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dy\, e^{ipy/\hbar}\psi^*\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right). \tag{1}$$

By integrating W_1 over p one recovers the spatial PDF, $|\psi(x)|^2$, and similarly, by integrating W_1 over x, one recovers the momentum PDF, $|\hat{\psi}(p)|^2$. However, in general $W_1(x,p)$ need not be positive, and hence does not have the interpretation of a joint PDF. Nevertheless, the Wigner function has been useful in numerous contexts [2,3], including quantum chaos and semiclassical physics [4,5], in quantum optics [6], in the modeling of optical devices [3], and in quantum information [7]. It has also been used in the context of many-body (fermionic and bosonic) systems. For one-dimensional (1D) fermions at zero temperature and in the presence of a confining potential, the Wigner function was first studied by Balazs and Zipfel [8]. Interestingly, the same results were derived recently by using a completely different method by Bettelheim and Wiegmann, in the context of nonequilibrium dynamics of a perturbed Fermi gas [9]. These results were found to be useful in the context of quantum mirror curves [10]. The Wigner function has also been used in the implementation of numerical methods to study fermions [11] and also in the study of Bose-Einstein condensates [12].

As the Wigner function contains information both about the positions and the momenta of particles, its experimental measurement can be useful to probe subtle questions concerning the nature of the quantum state of the system. However, measuring the Wigner function experimentally is a considerable technical challenge. In most cold atom experiments, only either the position or the momentum distributions can be measured, but not both simultaneously due to the uncertainty principle. In certain special cases, e.g., for a Fermi gas trapped in a two-dimensional box potential, measuring the momentum distribution can indirectly provide the measure of the Wigner function [13], but this is not generic. However, the development of quantum tomography has paved the way for measuring the full Wigner function (and not just its position and momentum marginals) in a number of recent contexts, e.g., in quantum optics [14], in trapped atom setups [15–17], and for electrons in ballistic conductors [18].

Given that most experiments are performed at finite temperature, it is also important to provide a finite temperature generalization of the $T = 0$ Wigner function introduced in (1). In fact, one of the main goals of this paper is to characterize this temperature dependence of the Wigner function for noninteracting fermions trapped in a confining potential in d dimensions.

Recently, there has been considerable interest in trapped Fermi gases, both theoretically [19] and in cold atom experiments [20]. Even in the noninteracting limit this system displays rich and universal quantum and thermal fluctuations, as was demonstrated recently [21–28]. The case of the harmonic trap played a fundamental role because it is solvable and makes an important connection between trapped noninteracting fermions and the eigenvalues of a random matrix. Indeed, in one dimension ($d = 1$) and at zero temperature $T = 0$, the positions of the fermions are in one-to-one correspondence with the eigenvalues of the Gaussian unitary ensemble (GUE) of random matrix theory (RMT) [22,25,27,28]. Consequently, at $T = 0$, the quantum fluctuations of N fermions, characterized by the squared many-body ground-state wave function, $|\Psi_0(x_1, \ldots, x_N)|^2$, was shown to be identical to the joint PDF of the eigenvalues of a GUE random matrix. Similarly, the joint distribution of the momenta is given by $|\hat{\Psi}_0(p_1, \ldots, p_N)|^2$, where $\hat{\Psi}_0$ is the N-variable Fourier transform of Ψ_0. In the case of the harmonic trap, because of the symmetry $x \leftrightarrow p$ (in scaled units), it is identical to the joint PDF of the positions, i.e., $\hat{\Psi}_0 = \Psi_0$.

The squared many-body wave function $|\Psi_0(x_1, \ldots, x_N)|^2$ (in real space) or its Fourier counterpart $|\hat{\Psi}_0(p_1, \ldots, p_N)|^2$ (in momentum space) encodes information about quantum fluctuations. For instance, by integrating $|\Psi_0(x_1, \ldots, x_N)|^2$ [respectively $|\hat{\Psi}_0(p_1, \ldots, p_N)|^2$] over $N - 1$ positions (respectively momenta), one obtains the average density of fermions in real space (respectively in momentum space). In the $N \to +\infty$ limit, from the mapping to the GUE, it is given (in scaled units and normalized to unity) by the Wigner semicircle law of RMT, $\rho_W(y) = \pi^{-1}\sqrt{2 - y^2}$, with $|y| \leqslant \sqrt{2}$ and zero elsewhere. Near the soft edge $y = \sqrt{2}$ (and similarly around $y = -\sqrt{2}$), the density gets smeared over a width $w_N \sim N^{-1/6}$, which defines the edge regime, and the density profile is described by a nontrivial scaling function F_1, known in RMT [29,30]. These results extend to all n-point correlation functions, either in position or momentum space. In particular, the scaled PDF of the position of the rightmost fermion, $x_{\max} = \max_{1 \leqslant i \leqslant N} x_i$, is given [23,25] by the Tracy-Widom (TW) distribution of the GUE [31]. Interestingly, the (scaled) largest fermion momentum $p_{\max} = \max_{1 \leqslant i \leqslant N} p_i$, measurable in time of flight experiments [32], is also distributed with the same TW distribution. This analysis has been recently extended to any spatial dimension d [24,26], to finite temperature, and beyond the harmonic oscillator for more general smooth potentials [25].

It is thus natural to ask which of these universal properties extend to the Wigner function for N noninteracting trapped fermions, to gain insight on the quantum fluctuations in the phase space. The N body Hamiltonian is $\hat{\mathcal{H}}_N = \sum_{i=1}^N \hat{H}(\hat{\mathbf{x}}_i, \hat{\mathbf{p}}_i)$, where the single-particle Hamiltonian for spinless fermions of mass m is given by

$$\hat{H} = \hat{H}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}), \tag{2}$$

with $V(\mathbf{x}) = \frac{1}{2} m\omega^2 \mathbf{x}^2$ in the case of the harmonic oscillator. The many-body Wigner function is defined at $T = 0$ as a generalization for any N and d of (1)

$$W_N(\mathbf{x}, \mathbf{p}) = \frac{N}{(2\pi\hbar)^d} \int_{-\infty}^{+\infty} d\mathbf{y}\, d\mathbf{x}_2 \ldots d\mathbf{x}_N\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \times \Psi_0^*\left(\mathbf{x} + \frac{\mathbf{y}}{2}, \mathbf{x}_2, \ldots, \mathbf{x}_N\right) \Psi_0\left(\mathbf{x} - \frac{\mathbf{y}}{2}, \mathbf{x}_2, \ldots, \mathbf{x}_N\right), \tag{3}$$

which by construction satisfies

$$\begin{aligned} \int_{-\infty}^{+\infty} d\mathbf{p}\, W_N(\mathbf{x}, \mathbf{p}) &= \rho_N(\mathbf{x}), \\ \int_{-\infty}^{+\infty} d\mathbf{x}\, W_N(\mathbf{x}, \mathbf{p}) &= \bar{\rho}_N(\mathbf{p}), \\ \int_{-\infty}^{+\infty} d\mathbf{x}\, d\mathbf{p}\, W_N(\mathbf{x}, \mathbf{p}) &= N, \end{aligned} \tag{4}$$

where $\rho_N(\mathbf{x})$ is the average density of fermions (here normalized to N) and $\bar{\rho}_N(\mathbf{p})$ its counterpart in momentum space.

B. Main results

In this paper, we compute $W_N(\mathbf{x}, \mathbf{p})$ exactly in the large N limit, both in the bulk and at the edge of a noninteracting Fermi gas trapped by a confining potential $V(\mathbf{x})$. We perform the derivation in arbitrary dimension d, first at $T = 0$ and for the harmonic oscillator and then at finite temperature and for a large class of smooth potentials. Our results generalize, to $d > 1$ and $T > 0$, the results previously obtained for $d = 1$ and $T = 0$ [8,9].

Zero temperature $T = 0$. The result in the bulk is particularly simple,

$$W_N(\mathbf{x}, \mathbf{p}) \simeq \frac{1}{(2\pi\hbar)^d} \Theta(\mu - E(\mathbf{x}, \mathbf{p})), \tag{5}$$

where

$$E(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \tag{6}$$

is the classical energy in the phase space. Here $\Theta(x)$ is the Heaviside unit step function and μ is the Fermi energy which is related to N via the normalization (4). Note that Eq. (5) is valid for large N (equivalently large μ) and for an arbitrary potential $V(\mathbf{x})$. This result, which can be obtained by semiclassical methods such as the local density approximation (see e.g. [33]), is obtained here through a controlled asymptotic analysis of an exact formula. Clearly, the form of $W_N(\mathbf{x}, \mathbf{p})$, given in Eq. (5), vanishes beyond the surface parameterized by $(\mathbf{x}_e, \mathbf{p}_e)$ where

$$\frac{\mathbf{p}_e^2}{2m} + V(\mathbf{x}_e) = \mu. \tag{7}$$

Following Ref. [9], we will call this surface the "Fermi surf;" it is the semiclassical version of the Fermi surface in classical phase space (see Fig. 1).

Note that by integrating (5) over momentum as in (4), one recovers the result for the average number density $\rho_N(\mathbf{x}) = (2\hbar)^{-d}[\mathbf{p}_e(\mathbf{x})]^d/\gamma_d$ with $\gamma_d = \pi^{d/2}\Gamma(1 + d/2)$, where $\mathbf{p}_e(\mathbf{x}) := \sqrt{2m(\mu - V(\mathbf{x}))}$ is the Fermi momentum, which sets the typical inverse interparticle spacing $\propto \hbar(|\mathbf{p}_e(\mathbf{x})|)^{-1}$. A similar result can be obtained for the average momentum density $\bar{\rho}_N(\mathbf{p})$ by integrating (5) over $\mathbf{x}$. Both the position as well as the momentum densities exhibit *marginal* edges $\mathbf{x}_{\rm em}$ (respectively $\mathbf{p}_{\rm em}$) beyond which they vanish. The number density $\rho_N(\mathbf{x})$ vanishes at $\mathbf{x} = \mathbf{x}_{\rm em}$, where $\mathbf{x}_{\rm em}$ satisfies $V(\mathbf{x}_{\rm em}) = \mu$. Similarly, the average momentum density $\bar{\rho}_N(\mathbf{p})$ vanishes at $\mathbf{p} = \mathbf{p}_{\rm em}$, where one can show that $|\mathbf{p}_{\rm em}| = \max_{\mathbf{x}} |\mathbf{p}_e(\mathbf{x})| = \sqrt{\max_{\mathbf{x}} [2m(\mu - V(\mathbf{x}))]}$. In the

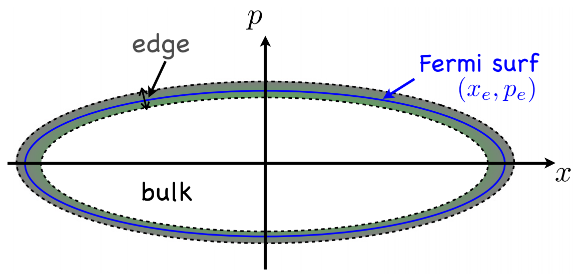

FIG. 1. Schematic representation of the Fermi surf (x_e,p_e) (blue solid line) defined by Eq. (7) in the phase space (x,p). The gray shaded region represents the edge around the Fermi surf, while the white inner region represents the bulk.

case of the harmonic oscillator, $|\mathbf{x}_{\rm em}| = r_e = \sqrt{2\mu/m\omega^2}$, and $\mathbf{p}_{\rm em} = \sqrt{2m\,\mu}$.

In this paper, our main results concern the properties of $W_N(\mathbf{x},\mathbf{p})$ near the Fermi surf in the $(\mathbf{x},\mathbf{p})$ plane, both at $T=0$ and $T>0$, in arbitrary dimensions d and for smooth confining potentials $V(\mathbf{x}) \sim |\mathbf{x}|^p$ for large $|\mathbf{x}|$. Let us first state our results for $T=0$. In this case, we first define a dimensionless variable a

$$a = \frac{1}{e_N}(E(\mathbf{x},\mathbf{p})-\mu), \tag{8}$$

where $(\mathbf{x},\mathbf{p})$ is a point in the phase space close to the Fermi surf and e_N is an energy scale given by

$$e_N = \frac{(\hbar)^{2/3}}{(2m)^{1/3}}\left(\frac{1}{m}(\mathbf{p}_e\cdot\nabla)^2 V(\mathbf{x}_e) + |\nabla V(\mathbf{x}_e)|^2\right)^{1/3}. \tag{9}$$

We then show that the Wigner function $W_N(\mathbf{x},\mathbf{p})$, at $T=0$ and in arbitrary d, can be expressed as a universal function of the dimensionless variable a as

$$W_N(\mathbf{x},\mathbf{p}) \simeq \frac{\mathcal{W}(a)}{(2\pi\hbar)^d}, \tag{10}$$

where the scaling function

$$\mathcal{W}(a) = \int_{2^{2/3}a}^{+\infty} \mathrm{Ai}(u)du \tag{11}$$

is independent of the space dimension d. In Eq. (11), $\mathrm{Ai}(u)$ is the Airy function. The function $\mathcal{W}(a)$ has the asymptotic behaviors

$$\mathcal{W}(a) \sim \begin{cases} (8\pi)^{-1/2}\, a^{-3/4}\, \exp\left[-\frac{4}{3}\, a^{3/2}\right], & a\to+\infty,\\ 1, & a\to-\infty. \end{cases} \tag{12}$$

In particular, the limit $\lim_{a\to-\infty}\mathcal{W}(a)=1$ ensures a smooth matching with the bulk result (5). In the inset of Fig. 2 we show a plot of this function $\mathcal{W}(a)$. Note that in $d=1$ our results coincide exactly with the one obtained using semiclassical methods [8,9]. Our results here provide a generalization of the $d=1$ result to arbitrary d. Interestingly, exactly the same scaling function as in Eq. (11) also appeared recently in the context of electron quantum optics [34]. Note that for the case of harmonic oscillator, where $V(\mathbf{x}) = (1/2)m\,\omega^2\mathbf{x}^2$, the energy scale e_N in (9) reduces to

$$e_N = m\omega^2\, r_e\, w_N, \tag{13}$$

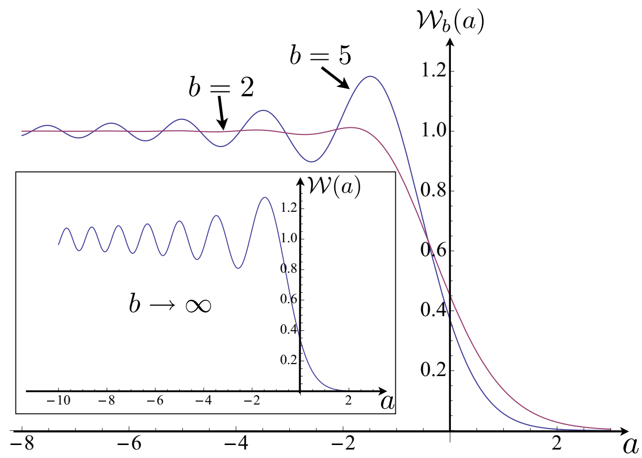

FIG. 2. Plot of the universal scaling function $\mathcal{W}_b(a)$, given in Eq. (22), for two different values of the scaled inverse temperature $b=2$ and $b=5$. In the inset, we show a plot of the zero temperature (i.e., $b\to\infty$) scaling function $\mathcal{W}_\infty(a)\equiv\mathcal{W}(a)$ given in Eq. (11). The oscillations become more pronounced as b increases.

where $r_e = \sqrt{2\mu/(m\omega^2)}$ and

$$w_N = \frac{1}{\alpha\sqrt{2}}(\mu/\hbar\omega)^{-1/6}, \quad \text{with } \alpha = \sqrt{m\omega/\hbar}, \tag{14}$$

represents the width of the edge region in the real space [24,25]. Hence, the energy scale e_N for the harmonic oscillator reads

$$e_N = (\hbar\omega)^{2/3}\,\mu^{1/3}. \tag{15}$$

Furthermore, for the harmonic oscillator, the Fermi energy μ is related to N, for large N, via [24,25]

$$\mu \sim \hbar\omega(N\,\Gamma(d+1))^{1/d}. \tag{16}$$

Consequently, the argument a of the scaling function $\mathcal{W}(a)$ in Eq. (8) reduces, in this case, to

$$a = \frac{1}{w_N}\left(\sqrt{\frac{\mathbf{p}^2}{m^2\omega^2}+\mathbf{x}^2} - r_e\right). \tag{17}$$

Finite temperature $T>0$. Next, we generalize our $T=0$ results for the Wigner function to finite temperature T. As in the $T=0$ case, there are two regimes, namely the bulk and the edge. The sharp bulk behavior at $T=0$ in Eq. (5) is smeared out by thermal fluctuations at finite T and is replaced by

$$W_{\tilde\mu}(\mathbf{x},\mathbf{p}) = \frac{1}{1+e^{\beta\left(\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-\tilde\mu\right)}}, \tag{18}$$

where $\beta = 1/T$. The finite temperature chemical potential $\tilde\mu$, in the canonical ensemble, can be determined as a function of β and N via the Fermi relation

$$N = \sum_{\mathbf{k}}\langle n_{\mathbf{k}}\rangle = \sum_{\mathbf{k}}\frac{1}{1+e^{\beta(\epsilon_{\mathbf{k}}-\tilde\mu)}}, \tag{19}$$

where the $\epsilon_{\mathbf{k}}$ denote the single-particle energy levels of the Hamiltonian $\hat H$ in (2). In the limit $T\to0$, $\tilde\mu\to\mu$ from Eq. (19) and Eq. (18) reduces to the $T=0$ result in (5). This semiclassical finite-temperature bulk result in Eq. (18) was also derived by other methods [35].

Near the finite-temperature edge, where $E(\mathbf{x},\mathbf{p}) \to \tilde{\mu}$, we show that the Wigner function has a universal scaling behavior for large N. This universal behavior emerges when the temperature T scales with N (or equivalently with μ) in a particular fashion, namely when temperature $T \sim e_N$, where e_N is the energy scale defined in Eq. (9). Note that e_N just depends on the Fermi energy μ, but not on the temperature. Hence, we set

$$\beta\, e_N = b, \tag{20}$$

where the dimensionless parameter $b = O(1)$ is kept fixed in the limit of large N. For instance, for the harmonic oscillator in 1D, using Eq. (15) and $\mu \sim \hbar\omega\, N$, one gets $b = (\hbar\omega/T)N^{1/3}$. This is the same temperature scale that appears in the analysis of the spatial correlations near the edge in real space [25]. In this temperature regime, one can show that the finite-temperature chemical potential $\tilde{\mu} \sim \mu$, indicating that the finite-temperature edge is the same as the zero-temperature edge. Hence, as in the $T = 0$ case (8), we consider the same dimensionless variable a. We show that, in this temperature regime characterized by the single dimensionless parameter b (20), the Wigner function takes a scaling form

$$W_{\tilde{\mu}}(\mathbf{x},\mathbf{p}) \sim \frac{\mathcal{W}_b(a)}{(2\pi\hbar)^d}, \quad a = \frac{1}{e_N}\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) - \mu\right), \tag{21}$$

with e_N given in Eq. (9). The scaling function $\mathcal{W}_b(a)$ is given by

$$\mathcal{W}_b(a) = \int_{-\infty}^{\infty} dy\, \frac{\mathrm{Ai}(y)}{1 + e^{a\,b} e^{-b\,y 2^{-2/3}}}. \tag{22}$$

In Fig. 2 we show a plot of $\mathcal{W}_b(a)$ for two different values of $b = 2$ and $b = 5$.

In the $T \to 0$ limit, i.e., $b \to \infty$ limit, the function $\mathcal{W}_b(a)$ reduces to $\mathcal{W}(a)$ given in Eq. (11), i.e., $\mathcal{W}_\infty(a) \equiv \mathcal{W}(a)$. The asymptotic behaviors of $\mathcal{W}_b(a)$ are given by

$$\mathcal{W}_b(a) \sim \begin{cases} e^{b^3/12}\, e^{-a\,b}, & a \to +\infty, \\ 1, & a \to -\infty. \end{cases} \tag{23}$$

Note that, for any finite b, the right tail of $\mathcal{W}_b(a)$, as $a \to \infty$, decays exponentially with a. It is only exactly at $T = 0$, i.e., when $b \to \infty$ limit, that the right tail decays faster than exponentially as in Eq. (12). Finally, we note that, as in the $T = 0$ case, this edge scaling function $\mathcal{W}_b(a)$ is completely universal, i.e., independent of the dimension d as well as the confining potential $V(\mathbf{x})$ as long as the potential is nonsingular.

The rest of the paper is organized as follows. In Sec. II, we compute the Wigner function at zero temperature. Section II A contains the exact solution (for any finite N) for the 1D harmonic oscillator, Sec. II B discusses the d-dimensional harmonic oscillator, while in Sec. II C we generalize these results for arbitrary smooth confining potentials. In Sec. III, these results are generalized to finite temperature $T > 0$. Finally, we conclude with a summary and discussion in Sec. IV. Some details are relegated to the Appendixes.

II. WIGNER FUNCTION AT ZERO TEMPERATURE

At $T = 0$, the quantum correlation functions of noninteracting fermions can be written as determinants constructed from a central object, the so-called kernel (see e.g. [25])

$$K_\mu(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{k}} \Theta(\mu - \epsilon_{\mathbf{k}})\psi_{\mathbf{k}}^*(\mathbf{x})\psi_{\mathbf{k}}(\mathbf{x}') \tag{24}$$

in terms of the single-particle eigenfunctions $\psi_{\mathbf{k}}(\mathbf{x})$ of (2) and their associated eigenenergies $\epsilon_{\mathbf{k}}$, labeled by quantum numbers $\mathbf{k}$. In (24) μ is chosen so that the sum contains exactly N levels. It turns out that one can relate the Wigner function (3) to the kernel, using the following formula:

$$K_\mu(\mathbf{x},\mathbf{x}') = N \int_{-\infty}^{+\infty} d\mathbf{x}_2 \dots d\mathbf{x}_N \times \Psi_0^*(\mathbf{x},\mathbf{x}_2,\dots,\mathbf{x}_N)\Psi_0(\mathbf{x}',\mathbf{x}_2,\dots,\mathbf{x}_N), \tag{25}$$

which follows from the property that Ψ_0 is a Slater determinant constructed from the $\psi_{\mathbf{k}}$'s (see the derivation in Appendix A). Comparing (25) and (3) we obtain

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^d}\int_{-\infty}^{+\infty} d\mathbf{y}\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\, K_\mu\left(\mathbf{x} + \frac{\mathbf{y}}{2}, \mathbf{x} - \frac{\mathbf{y}}{2}\right). \tag{26}$$

The scaling behavior of this kernel K_μ has been well studied in the large μ and N limit [25]. One can then use these results in (26) to obtain information about the Wigner function as shown below.

A. Calculation in $d = 1$ for the harmonic oscillator

Let us first present an exact calculation for the $d = 1$ harmonic oscillator, using space, momentum, time, and energy dimensionless units, i.e., in units of

$$x_0 = 1/\alpha, \quad p_0 = \hbar\alpha, \quad t_0 = 1/\omega, \quad e_0 = \hbar\omega. \tag{27}$$

In these scaled units $\mu \simeq N$ for large N. The kernel reads

$$K_\mu(x,x') = \sum_{k=0}^{N-1} \psi_k(x)\psi_k(x'), \tag{28}$$

where $\psi_k(x) = (\frac{1}{\sqrt{\pi}2^k k!})^{\frac{1}{2}} H_k(x) e^{-\frac{1}{2}x^2}$ and H_k is the kth Hermite polynomial. Plugging it in Eq. (26), and specifying $d = 1$, we obtain in dimensionless units and at $T = 0$

$$W_N(x,p) = \frac{1}{2\pi}\sum_{k=0}^{N-1}\int_{-\infty}^{\infty} dy\, e^{ipy}\psi_k\left(x + \frac{y}{2}\right)\psi_k\left(x - \frac{y}{2}\right). \tag{29}$$

Remarkably, the integral over y in Eq. (29) can be performed explicitly using an identity first derived by Groenewold [36]

$$\int_{-\infty}^{+\infty} dy\, e^{ipy}\psi_k\left(x + \frac{y}{2}\right)\psi_k\left(x - \frac{y}{2}\right) = 2(-1)^k L_k(2(x^2 + p^2))e^{-x^2-p^2}, \tag{30}$$

where $L_k(y)$ is the Laguerre polynomial of degree k, defined via its generating function

$$\sum_{k=0}^{\infty} z^k\, L_k(y) = \frac{1}{(1-z)}\, e^{-z\,y/(1-z)}. \tag{31}$$

Substituting (30) in Eq. (29) gives the explicit result

$$W_N(x,p) = \frac{1}{\pi}\sum_{k=0}^{N-1}(-1)^k\, L_k(2(x^2 + p^2))\, e^{-x^2-p^2}. \tag{32}$$

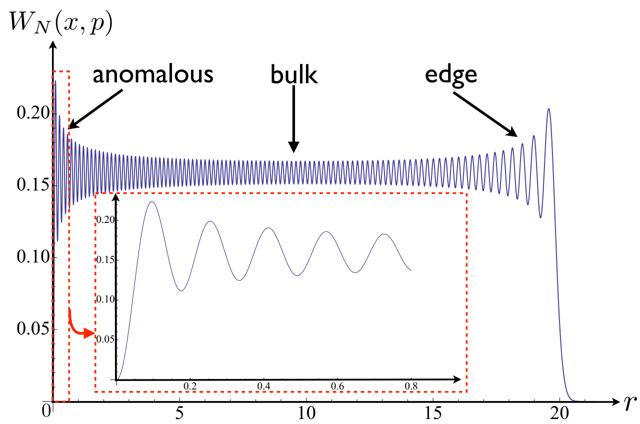

FIG. 3. Plot of $W_N(x,p)$ as a function of $r=\sqrt{x^2+p^2}$ for the 1D harmonic oscillator, as given in Eq. (32) for $N=200$. Inset: zoom on the range $r\in[0,0.8]$. Note in particular that $W_N(0,0)=0$ here as $N=200$ is even [see Eq. (46)].

We have plotted $W_N(x,p)$ as a function of $r=\sqrt{x^2+p^2}$ for $N=200$ in Fig. 3. From this figure one sees that there are three distinct regimes at large N: (i) the bulk regime where $W_N(x,p)$ oscillates around the bulk value $1/(2\pi)=0.159155\ldots$, (ii) the edge regime around $r=\sqrt{2N}$ where the Wigner function vanishes over a width of order $N^{-1/6}$ (see below), and (iii) an "anomalous" regime near $r=0$. This anomalous regime has been pointed out [4] for the single-particle case ($N=1$); here we show that it persists for multiparticle systems. At $r=0$ the Wigner function vanishes exactly for N even, and equals $1/\pi$ for N odd. At small $r\sim 1/\sqrt{N}$ there is a scaling regime describing the Wigner function near $r=0$. We now study these three regimes in detail.

We start by multiplying Eq. (32) by z^N and sum over N. We obtain

$$\sum_{N=1}^{\infty} z^N W_N(x,p)=\frac{1}{\pi}\sum_{N=1}^{\infty} z^N\sum_{k=0}^{N-1}(-1)^k L_k(2r^2)e^{-r^2}, \quad (33)$$

with $r^2=x^2+p^2$. To perform this double sum, we write $z^N=z^{N-k}z^k$ and perform the sums separately over k and $m=N-k$. This gives

$$\sum_{N=1}^{\infty} z^N W_N(x,p)=\frac{1}{\pi}\sum_{m=1}^{\infty} z^m\sum_{k=0}^{\infty}(-z)^k L_k(2r^2)e^{-r^2}. \quad (34)$$

Using the generating function of Laguerre polynomials in Eq. (31), we get

$$\sum_{N=1}^{\infty} z^N W_N(x,p)=\frac{1}{\pi}\frac{z}{1-z^2}e^{-\frac{1-z}{1+z}(x^2+p^2)}. \quad (35)$$

For convenience, we use $z=e^{-s}$ and get

$$\tilde{W}(x,p;s):=\sum_{N=1}^{\infty} W_N(x,p)e^{-Ns}=\frac{e^{-s-(x^2+p^2)\tanh\frac{s}{2}}}{\pi(1-e^{-2s})}. \quad (36)$$

Bulk behavior. To extract the bulk result we consider the limit $s\ll 1$ with $s(x^2+p^2)$ fixed, leading to

$$\tilde{W}(x,p;s)\simeq\frac{e^{-\frac{s}{2}(x^2+p^2)}}{2\pi s}, \quad (37)$$

which yields, after Laplace inversion,

$$W_N(x,p)\simeq\frac{1}{2\pi}\Theta(2N-(x^2+p^2)), \quad (38)$$

in agreement with the general result (5).

Edge behavior. We now show that the exact formula (35) can be used to derive the edge scaling function (10), (11) in $d=1$ for the harmonic oscillator. Our starting point is the exact relation (35). Inverting the generating function using Cauchy's inversion formula,

$$W_N(x,p)=\frac{1}{\pi}\int_{c-i\times\infty}^{c+i\times\infty}\frac{ds}{2i\pi}\frac{e^{s(N-1)}}{(1-e^{-2s})}\exp\left[-(x^2+p^2)\tanh\frac{s}{2}\right], \quad (39)$$

where c is to the right of all singularities in the complex s plane. For large N, the most important contributions come from the vicinity of $s\to 0$. We set, near the edge $\sqrt{x^2+p^2}=\sqrt{2N}$,

$$\sqrt{x^2+p^2}=\sqrt{2N}+\frac{1}{\sqrt{2}}N^{-1/6}a, \quad (40)$$

where a denotes the distance (on the scale of $N^{-1/6}$) from the edge; see (8) in $d=1$ and dimensionless units. Using $\tanh(s/2)=s/2-s^3/24$ as $s\to 0$, we find that the integral in Eq. (40) reduces for large N to

$$W_N(x,p)\approx\frac{1}{2\pi}\int_{c-i\times\infty}^{c+i\times\infty}\frac{ds}{2i\pi s}\exp\left[-s\,N^{1/3}\,a+s^3\,\frac{N}{12}\right]. \quad (41)$$

Rescaling further by setting $t=s\,N^{1/3}$, we finally get, near the edge,

$$W_N(x,p)\approx\frac{1}{2\pi}\mathcal{W}(a), \quad (42)$$

where

$$a=\sqrt{2}\,N^{1/6}(\sqrt{x^2+p^2}-\sqrt{2N}), \quad (43)$$

and the scaling function $\mathcal{W}(a)$ is given exactly by

$$\mathcal{W}(a)=\int_{c-i\times\infty}^{c+i\times\infty}\frac{dt}{2i\pi t}\exp\left[-t\,a+\frac{t^3}{12}\right]. \quad (44)$$

Using the integral representation of the Airy function

$$\mathrm{Ai}(x)=\int_{c-i\times\infty}^{c+i\times\infty}\frac{d\tau}{2i\pi}\exp\left[-\tau\,x+\frac{\tau^3}{3}\right], \quad (45)$$

we find that this scaling function $\mathcal{W}(a)$ is indeed given by formula (11). One can easily check that it has the asymptotic behaviors given in Eq. (12).

Anomalous behavior near $r=\sqrt{x^2+p^2}=0$. It is easy to see that the Wigner function vanishes at $r=0$ for N even.

From the definition (29),

$$W_N(x=0,p=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \sum_{k=0}^{N-1} \psi_k^*\left(\frac{y}{2}\right)\psi_k\left(-\frac{y}{2}\right)$$
$$= \frac{1}{\pi}\sum_{k=0}^{N-1}(-1)^k = \begin{cases} \frac{1}{\pi}, & N \text{ odd,} \\ 0, & N \text{ even,} \end{cases} \quad (46)$$

where we used (i) the orthonormality of the single-particle wave functions $\psi_k(x)$ and (ii) the fact that for the harmonic-oscillator potential $\psi_k(-x) = (-1)^k\psi_k(x)$. Note the property (46) extends to an arbitrary even potential $V(x) = V(-x)$ in $d = 1$.

As we show in Appendix D, near $r = 0$ in a regime where $r \sim 1/\sqrt{N}$, the Wigner function has the following scaling behavior for large N:

$$W_N(x,p) \sim \frac{1}{2\pi} - (-1)^N F(\sqrt{N(x^2+p^2)}),$$
$$F(z) = \frac{1}{2\pi}J_0(2\sqrt{2}z), \quad (47)$$

where $J_\nu(x)$ is the Bessel function with index ν. Interestingly, this parity dependence in Eq. (47) persists even for large N. We have verified the scaling behavior in (47) by numerically evaluating $W_N(x,p)$ in Eq. (32).

B. d-dimensional harmonic oscillator

In dimension $d > 1$ it is more convenient to use a different method using the quantum propagator to calculate the kernel, and in turn the Wigner function via (26). In addition, as we show later this method is more versatile as one can treat more general potentials and demonstrate universal properties of the Wigner function. The method relies on the following representation of the kernel in d dimensions for arbitrary single-particle Hamiltonian [25]

$$K_\mu(\mathbf{x},\mathbf{x}') = \int_C \frac{dt}{2\pi i t} e^{\mu t/\hbar} G(\mathbf{x},\mathbf{x}';t), \quad (48)$$

where C is the Bromwich contour in the complex plane and $G(\mathbf{x},\mathbf{x}';t) = \langle \mathbf{x}'|e^{-\hat{H}t/\hbar}|\mathbf{x}\rangle$ is the one-particle Euclidean quantum propagator associated to the Hamiltonian $\hat{H}$ in (2). Let us apply this relation to the case of the harmonic oscillator in dimension d for which the exact propagator is known [37]. We work again here in the aforementioned dimensionless units in which the propagator reads

$$G(\mathbf{x},\mathbf{x}';t) = \frac{1}{(2\pi\ \sinh t)^{d/2}} \times \exp\left[-\frac{(\mathbf{x}-\mathbf{x}')^2 + (\mathbf{x}^2+(\mathbf{x}')^2)(\cosh t - 1)}{2\ \sinh t}\right]. \quad (49)$$

We first insert (49) into (48) and then use Eq. (26). Performing the Gaussian integration over $\mathbf{y}$ we obtain

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int_C \frac{dt}{2\pi i t} e^{\mu t} \frac{1}{\left(\cosh\frac{t}{2}\right)^d} e^{-(\mathbf{x}^2+\mathbf{p}^2)\tanh\frac{t}{2}}, \quad (50)$$

an exact formula. Note that the N dependence of $W_N(\mathbf{x},\mathbf{p})$ is only through μ. As in the $d = 1$ case we now analyze this formula both in the bulk as well as in the edge regime.

Bulk behavior. In the bulk one can show that the values of t which dominate the integral are $O(1/\mu)$. Hence, we need to expand the factor $\tanh(t/2)$ inside the exponential only to $O(t)$. This leads to

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int_C \frac{dt}{2\pi i t} e^{(\mu - \frac{\mathbf{p}^2+\mathbf{x}^2}{2})t}. \quad (51)$$

The integral over t just gives a Heaviside θ function, hence establishing the result for the bulk in Eq. (5) in the case of the harmonic potential.

Edge behavior. To analyze the edge behavior in the large μ limit, we need to expand the exponential up to order t^3, as in [25] for the study of the edge in real space. We obtain [discarding terms of $O(t^4)$ in the exponential]

$$W_N(\mathbf{x},\mathbf{p}) = \int_C \frac{dt}{(2\pi)^{d+1} i t} e^{\left(\mu - \frac{\mathbf{x}^2+\mathbf{p}^2}{2}\right)t - \frac{dt^2}{8} + (\mathbf{x}^2+\mathbf{p}^2)\frac{t^3}{24}}. \quad (52)$$

Keeping only the leading $O(t)$ term in the exponential in (52) immediately leads to the result in the bulk (5). Precisely at the edge, the coefficient of the $O(t)$ term vanishes. Hence, to study the vicinity of the edge one must keep terms up to $O(t^3)$. To this aim, we parametrize the distance to the edge [following Eq. (17)] as $\sqrt{\mathbf{x}^2+\mathbf{p}^2} - \sqrt{2\mu} = w_N\, a$, where $w_N = \frac{1}{\sqrt{2}}\mu^{-1/6}$ and $a = O(1)$. We now expand the argument of the exponential (53), denoted by S,

$$S = -\sqrt{2\mu}\, a w_N t - d\frac{t^2}{8} + 2\mu\frac{t^3}{24} + O(t^4, w_N t^3\sqrt{\mu}). \quad (53)$$

Let us now define $t = t_N\tau$, with $t_N = 2^{2/3}\mu^{-1/3}$. Discarding the term $\propto t^2$ which is subdominant, this leads to

$$S = -a2^{2/3}\tau + \frac{\tau^3}{3} + O(\mu^{-2/3}). \quad (54)$$

One thus obtains, upon restoring the physical units, the scaling form (10) for the Wigner function at large μ as

$$W_N(\mathbf{x},\mathbf{p}) \simeq \frac{\mathcal{W}(a)}{(2\pi)^d}, \quad \mathcal{W}(a) = \int_C \frac{d\tau}{2\pi i\tau} e^{-a2^{2/3}\tau + \frac{\tau^3}{3}}, \quad (55)$$

which is precisely the integral representation given in Eq. (44) with the substitution $2^{2/3}\,\tau = t$. Thus, we obtain the remarkable result that the edge scaling form $\mathcal{W}(a)$ of the Wigner function for the harmonic oscillator is completely independent of the space dimension d. The same result can also be obtained (see Appendix B), directly from the known scaling behavior of the kernel [25]. A natural question is whether this scaling form is also universal with respect to the details of the shape of the confining potential, as we will discuss below.

In $d = 1$, we have seen in the previous subsection that there is an additional anomalous regime when $x^2 + p^2 = O(1/N)$. In $d > 1$, similar anomalous regimes are likely to exist, though we have not investigated them in detail.

C. Wigner function for other smooth confining potentials: Beyond the harmonic oscillator

The case of the harmonic oscillator, treated in the previous section, is special because $\mathbf{x}$ and $\mathbf{p}$ [appropriately rescaled in the dimensionless units (27)] play symmetric roles, and the Wigner function depends only on the single variable $\mathbf{x}^2 + \mathbf{p}^2$.

For a general potential $V(\mathbf{x})$ this is no longer the case and a different treatment is needed, as we now show.

Consider a more general smooth potential $V(\mathbf{x})$ in d dimensions. In this case we show, using the above propagator method, that in the bulk the Wigner function is given by (5). Putting together formulas (48) and (26) one can express the Wigner function directly in terms of the Euclidean propagator

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^d}\int_C \frac{dt}{2\pi i t} e^{\frac{\mu t}{\hbar}} \int_{-\infty}^{+\infty} d\mathbf{y}\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\, G\left(\mathbf{x}+\frac{\mathbf{y}}{2},\mathbf{x}-\frac{\mathbf{y}}{2},t\right). \tag{56}$$

Let us recall that the Euclidean quantum propagator satisfies the Feynman-Kac equation $G(\mathbf{x},\mathbf{x}',t)$

$$\partial_t G = -\hat{H}G = \left(\frac{\hbar^2}{2m}\nabla_x^2 - V(\mathbf{x})\right)G, \tag{57}$$

with $G(\mathbf{x},\mathbf{x}',0) = \delta^d(\mathbf{x}-\mathbf{x}')$. As was shown in [25], and further detailed in Appendix C, the large N limit can be obtained from the small time t expansion of the quantum propagator, which up to $O(t^3)$ reads

$$G(\mathbf{x},\mathbf{x}',t) = \left(\frac{m}{2\pi\hbar t}\right)^{d/2} \exp\left[-\frac{m}{2\hbar t}(\mathbf{x}-\mathbf{x}')^2\right] \times \exp\left[-\frac{t}{\hbar}S_1(\mathbf{x},\mathbf{x}') - \frac{t^2}{2m}S_2(\mathbf{x},\mathbf{x}') + \frac{t^3}{2m\hbar}S_3(\mathbf{x},\mathbf{x}')\right], \tag{58}$$

where S_1,S_2,S_3 for an arbitrary potential $V(\mathbf{x})$ are given explicitly in Eqs. (243)–(245) of [25].

To analyze the bulk behavior of the Wigner function we only need the leading $O(t)$ term S_1, which then reads

$$S_1\left(\mathbf{x}-\frac{\mathbf{y}}{2},\mathbf{x}+\frac{\mathbf{y}}{2}\right) = \int_0^1 du\, V\left[\mathbf{x}+\left(u-\frac{1}{2}\right)\mathbf{y}\right]. \tag{59}$$

We substitute this expression of S_1 in (58) and keep only up to $O(t)$ terms. Next we substitute this propagator in (56)

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^d}\int_C \frac{dt}{2\pi i t}\left(\frac{m}{2\pi\hbar t}\right)^{d/2}\int_{-\infty}^{+\infty} d\mathbf{y}\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \times \exp\left[-\frac{m}{2\hbar t}\mathbf{y}^2 + \frac{t}{\hbar}\left(\mu - \int_0^1 du\, V\left[\mathbf{x}+\left(u-\frac{1}{2}\right)\mathbf{y}\right]\right)\right]. \tag{60}$$

One can show that the values of t which dominate the integral are $O(1/\mu)$. Hence, by rescaling $\mathbf{y}$ as shown in Appendix C, one can neglect the term $(u-1/2)\mathbf{y}$ in the argument of V, leading, after integration over $\mathbf{y}$, to

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^d}\int_C \frac{dt}{2\pi i t} e^{(\mu - \frac{p^2}{2m} - V(\mathbf{x}))t}. \tag{61}$$

The integral over t just gives a Heaviside θ function, hence establishing the result for the bulk in Eq. (5).

From Eq. (5) it is clear that the Wigner function vanishes beyond the boundary of a bounded support, which defines an edge in phase space, i.e., a surface parameterized by $(\mathbf{x}_e,\mathbf{p}_e)$, which satisfies the equation

$$\frac{\mathbf{p}_e^2}{2m} + V(\mathbf{x}_e) = \mu. \tag{62}$$

As for the case of the harmonic oscillator, for large but finite N the jump of the Wigner function described by (5) is smoothed over a scale w_N which now explicitly depends on the potential. However, the appropriately centered and scaled Wigner function at any point of the edge surface is again universal and is given by $\mathcal{W}(a)$ in (11). More precisely the Wigner function takes the following edge scaling form for $\frac{p^2}{2m} + V(x) - \mu \sim \mu^{1/3} \ll \mu$:

$$W_N(\mathbf{x},\mathbf{p}) \simeq \frac{\mathcal{W}(a)}{(2\pi\hbar)^d}, \tag{63}$$

where the scaled variable $a = O(1)$ is now naturally expressed as the ratio of two energies

$$a = \frac{1}{e_N}\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) - \mu\right), \tag{64}$$

where the energy scale e_N is given:

$$e_N = \frac{(\hbar)^{2/3}}{(2m)^{1/3}}\left(\frac{1}{m}(\mathbf{p}_e\cdot\nabla)^2 V(\mathbf{x}_e) + |\nabla V(\mathbf{x}_e)|^2\right)^{1/3}. \tag{65}$$

For the derivation of these results, see Appendix C. For the harmonic oscillator $V(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$, one finds

$$e_N = \hbar\omega\left(\frac{\mu}{\hbar\omega}\right)^{1/3} = m\omega^2 r_e w_N = \frac{\hbar\omega}{2\alpha^2 w_N^2}. \tag{66}$$

We thus see that Eq. (64) is consistent with Eq. (8) and Eq. (65) is consistent with (13), as discussed earlier. For a general potential the above results are valid on any point $(\mathbf{x}_e,\mathbf{p}_e)$ on the edge surface. In particular for $\mathbf{p}_e = 0$ we have

$$a = \frac{1}{e_N}(V(\mathbf{x}) - \mu) \simeq \frac{1}{e_N}\nabla V(\mathbf{x}_e)\cdot(\mathbf{x}-\mathbf{x}_e), \tag{67}$$

$$e_N = \frac{(\hbar)^{2/3}}{(2m)^{1/3}}|\nabla V(\mathbf{x}_e)|^{2/3}. \tag{68}$$

Furthermore, to make contact with the 1D result derived in Ref. [9], let us consider a point in the $(\mathbf{x},\mathbf{p})$ plane near the edge $(\mathbf{x}_e,\mathbf{p}_e)$, where we set $\mathbf{x} = \mathbf{x}_e$ and $\mathbf{p} = \mathbf{p}_e + \tilde{\mathbf{p}}$ with $|\tilde{\mathbf{p}}| \ll 1$ (see Fig. 1). In particular, if we focus on $d = 1$, the formulas (64) and (65) simplify a lot. In this case the formula (65) becomes

$$e_N = \frac{(\hbar)^{2/3}}{(2m)^{1/3}}\left(\frac{p_e^2}{m}V''(x_e) + (V'(x_e))^2\right), \tag{69}$$

and Eq. (64) reduces, to leading order in $\tilde{p}$,

$$a = \frac{p_e(x_e)}{e_N}\,\tilde{p}. \tag{70}$$

Here, $p_e(x_e)$ is the point on the edge surface parameterized by the function

$$p_e^2(x) = 2m(\mu - V(x)). \tag{71}$$

In Ref. [9], this $p_e(x)$ was called the "Fermi surf." By taking twice the derivative of (71), the expression for e_N in Eq. (69)

simplifies to

$$e_N = \left(\frac{\hbar^2}{2}\right)^{1/3} |p_e''(x_e)|^{1/3} \frac{p_e(x_e)}{m}. \tag{72}$$

Consequently, from Eq. (70), one gets

$$a = \kappa\, \tilde{p}, \quad \kappa = \left(\frac{\hbar^2}{2} p_e''(x_e)\right)^{-1/3}. \tag{73}$$

Thus, the argument a of the scaling function $\mathcal{W}(a)$ in Eq. (63) reduces, in $d = 1$, to precisely the argument derived in Ref. [9].

III. WIGNER FUNCTION AT FINITE TEMPERATURE

We now extend our analysis to finite temperature. The Wigner function at temperature $T = 1/\beta$ in the canonical ensemble can be defined from the many-body density matrix $\hat{\mathcal{D}}_N = e^{-\beta\hat{\mathcal{H}}_N}/Z_N(\beta)$ as

$$\begin{aligned} W_{N,T}(\mathbf{x},\mathbf{p}) &= \frac{N}{(2\pi\hbar)^d}\int_{-\infty}^{+\infty} d\mathbf{y}\, d\mathbf{x}_2\ldots d\mathbf{x}_N, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\ &\times\left\langle \mathbf{x}+\frac{\mathbf{y}}{2},\mathbf{x}_2,\ldots,\mathbf{x}_N\middle|\hat{\mathcal{D}}_N\middle|\mathbf{x}-\frac{\mathbf{y}}{2},\mathbf{x}_2,\ldots,\mathbf{x}_N\right\rangle. \end{aligned} \tag{74}$$

By decomposing on the basis of eigenvectors $|E\rangle$ of $\hat{\mathcal{H}}_N$ one can equivalently write it as

$$W_{N,T}(\mathbf{x},\mathbf{p}) = \frac{1}{Z_N(\beta)}\sum_E e^{-\beta E} W_{N,E}(\mathbf{x},\mathbf{p}), \tag{75}$$

where $Z_N(\beta) = \sum_E e^{-\beta E}$ is the canonical partition sum and

$$W_{N,E}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^d}\int_{-\infty}^{+\infty} d\mathbf{y}\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\, K\left(\mathbf{x}+\frac{\mathbf{y}}{2},\mathbf{x}-\frac{\mathbf{y}}{2};\{n_\mathbf{k}\}\right), \tag{76}$$

where the N-body eigenstate $|E\rangle$ is labeled by a set of occupation numbers $n_\mathbf{k} = 0,1$ of the single-particle eigenstates, such that $E = \sum_\mathbf{k} n_\mathbf{k}\epsilon_\mathbf{k}$ and $N = \sum_\mathbf{k} n_\mathbf{k}$. The kernel $K(\mathbf{x},\mathbf{x}';\{n_\mathbf{k}\})$ has the expression

$$K(\mathbf{x},\mathbf{x}';\{n_\mathbf{k}\}) = \sum_\mathbf{k} n_\mathbf{k}\psi_\mathbf{k}^*(\mathbf{x})\psi_\mathbf{k}(\mathbf{x}'), \tag{77}$$

where $\psi_\mathbf{k}(\mathbf{x})$ is the single-particle eigenstate labeled by $\mathbf{k}$. Note that in the $T = 0$ limit (77) reduces to (28) and (76) reduces to (26).

One can also define the Wigner function in the grand-canonical ensemble with chemical potential $\tilde{\mu}$ as

$$W_{\tilde{\mu}}(\mathbf{x},\mathbf{p}) = \frac{1}{Z_{gr}(\beta,\tilde{\mu})}\sum_N Z_N(\beta)W_{N,T}(\mathbf{x},\mathbf{p})e^{\tilde{\mu}\beta N}, \tag{78}$$

where $Z_{gr}(\beta,\tilde{\mu}) = \sum_N Z_N(\beta)e^{\tilde{\mu}\beta N}$ is the grand-canonical partition function. Substituting (75) in (78) and summing Eq. (76) over the eigenstates $|E\rangle$ and N, i.e., over the independent variables $n_\mathbf{k}$'s, leads to

$$W_{\tilde{\mu}}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^d}\int_{-\infty}^{+\infty} d\mathbf{y}\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\, \tilde{K}_{\tilde{\mu}}\left(\mathbf{x}+\frac{\mathbf{y}}{2},\mathbf{x}-\frac{\mathbf{y}}{2}\right), \tag{79}$$

where $\tilde{K}_{\tilde{\mu}}(\mathbf{x},\mathbf{x}')$ is the kernel defined in the grand-canonical ensemble

$$\tilde{K}_{\tilde{\mu}}(\mathbf{x},\mathbf{x}') = \sum_\mathbf{k}\langle n_\mathbf{k}\rangle\psi_\mathbf{k}^*(\mathbf{x})\psi_\mathbf{k}(\mathbf{x}'), \tag{80}$$

where

$$\langle n_\mathbf{k}\rangle = \frac{1}{1+e^{\beta(\epsilon_\mathbf{k}-\tilde{\mu})}} \tag{81}$$

is the mean occupation number of state $\mathbf{k}$, over the Fermi distribution.

In the grand-canonical ensemble one can relate the finite-temperature Wigner function $W_{\tilde{\mu}}(\mathbf{x},\mathbf{p})$ to the zero-temperature Wigner function $W_N(\mathbf{x},\mathbf{p})$. To see this, we consider first the zero-temperature kernel in (24). Taking a derivative with respect to the Fermi energy μ gives

$$\partial_\mu K_\mu(\mathbf{x},\mathbf{x}') = \sum_\mathbf{k}\delta(\mu-\epsilon_\mathbf{k})\psi_\mathbf{k}^*(\mathbf{x})\psi_\mathbf{k}(\mathbf{x}'). \tag{82}$$

Now we start with Eqs. (80) and (81) and rewrite them as

$$\begin{aligned} \tilde{K}_{\tilde{\mu}}(\mathbf{x},\mathbf{x}') &= \int_{-\infty}^{+\infty}\frac{d\mu'}{1+e^{\beta(\mu'-\tilde{\mu})}}\sum_\mathbf{k}\delta(\mu'-\epsilon_\mathbf{k})\psi_\mathbf{k}^*(\mathbf{x})\psi_\mathbf{k}(\mathbf{x}') \\ &= \int_{-\infty}^{+\infty}\frac{d\mu'}{1+e^{\beta(\mu'-\tilde{\mu})}}\partial_{\mu'}K_{\mu'}(\mathbf{x},\mathbf{x}'), \end{aligned} \tag{83}$$

where in the last line we used (82). This equation was previously derived as Eq. (240) in [25]. We can now apply this formula to the Wigner function using (79), leading to

$$W_{\tilde{\mu}}(\mathbf{x},\mathbf{p}) = \int_{-\infty}^{+\infty} d\mu'\frac{1}{1+e^{\beta(\mu'-\tilde{\mu})}}\partial_{\mu'}(W_N(\mathbf{x},\mathbf{p})|_{\mu=\mu'}), \tag{84}$$

which is exact in the grand-canonical ensemble. We have denoted the integration variable by μ' to avoid confusion with the variable μ which denotes the Fermi energy. Note that the zero-temperature Wigner function $W_N(\mathbf{x},\mathbf{p})$ depends implicitly on μ. This relation (84) will now allow us to derive the bulk and edge properties of the Wigner function at finite temperature.

The above results are obtained in the grand-canonical ensemble where $\tilde{\mu}$ is a given parameter. However, our main goal is to describe the Wigner function in the canonical ensemble where N is fixed. Indeed, in the large N limit, adapting the saddle-point method of [25] to the Wigner function, we expect that one can use the grand-canonical results for the canonical ensemble, provided we determine $\tilde{\mu}$ as a function of N by the relation

$$N = \sum_\mathbf{k}\langle n_\mathbf{k}\rangle = \sum_\mathbf{k}\frac{1}{1+e^{\beta(\epsilon_\mathbf{k}-\tilde{\mu})}}. \tag{85}$$

Note that by definition of μ one also has

$$N = \sum_\mathbf{k}\langle n_\mathbf{k}\rangle = \sum_\mathbf{k}\Theta(\mu-\epsilon_\mathbf{k}), \tag{86}$$

which is also the $T = 0$ limit of (85). This implies that $\tilde{\mu}$ is related to μ by equating the two relations (85) and (86).

Bulk behavior. Substituting the result (5) for the bulk zero-temperature Wigner function in (84) we obtain

$$W_{\tilde{\mu}}(\mathbf{x},\mathbf{p}) = \frac{1}{1+e^{\beta(\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-\tilde{\mu})}}. \tag{87}$$

This equation is valid for all $\mathbf{x},\mathbf{p}$ in the phase space where $\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-\tilde{\mu}=O(T)$. In the canonical ensemble $\tilde{\mu}$ is related to N via (85), and using also (86), one sees that the bulk regime corresponds to scaling $T\sim\tilde{\mu}$ and $T\sim\mu$. By integrating (87) over $\mathbf{p}$ (respectively over $\mathbf{x}$) one obtains the finite-temperature bulk density $\rho_N(\mathbf{x})$ [respectively momentum density $\bar{\rho}_N(\mathbf{p})$] [see, e.g., Eqs. (270)– (273) in [25]].

Edge behavior. For simplicity let us first focus on the harmonic oscillator. There it turns out that, in the edge regime, one needs to scale the temperature as $T\sim\mu^{1/3}$ [25] in the limit of large μ. Consequently, one defines a reduced inverse temperature

$$b=\beta\mu^{1/3}, \tag{88}$$

with $b=O(1)$ in this regime. In addition, in this regime $\tilde{\mu}\simeq\mu$ [25]; hence we set $\tilde{\mu}=\mu$ in the following. Our starting point is the integral in (84). For $T\sim\mu^{1/3}$, this integral is dominated by the regime where μ' is close to μ. In fact, by setting

$$\beta(\mu'-\mu)=-bu, \tag{89}$$

we see that the Fermi factor in (84) takes the dimensionless form $1/(1+e^{-bu})$. This suggests that the integral will be controlled by values of u which are of order unity; hence

$$\mu'-\mu=O(\mu^{-1/3}). \tag{90}$$

Therefore, at any point $\mathbf{x},\mathbf{p}$ in phase space close to the edge defined in (62), we can use the scaling form (10) and (11) of $W_N(\mathbf{x},\mathbf{p})$ inside the integral in (84). We note that the only dependence on μ' of $\mathcal{W}(a)$ is through the scaling variable

$$a=a_{\mu'}:=\sqrt{2}\,(\mu')^{1/6}\left(\sqrt{\mathbf{p}^2+\mathbf{x}^2}-\sqrt{2\mu'}\right). \tag{91}$$

Using the scaling form we have

$$\partial_{\mu'}W_N(\mathbf{x},\mathbf{p})\simeq\mathcal{W}'(a_{\mu'})\partial_{\mu'}a_{\mu'}. \tag{92}$$

We can now expand $a_{\mu'}$ in (91) around $\mu'=\mu$ as

$$a_{\mu'}=a_\mu-\frac{1}{\mu^{1/3}}(\mu'-\mu). \tag{93}$$

Using (89) and $b=\beta\mu^{1/3}$ we obtain

$$a_{\mu'}=a_\mu+u+O(\mu^{-2/3}). \tag{94}$$

Inside the integral in (84) the factor

$$\partial_{\mu'}a_{\mu'}=\frac{du}{d\mu'}\partial_u a_{\mu'}=\frac{du}{d\mu'}(1+O(\mu^{-2/3})). \tag{95}$$

We now rewrite the integral in (84) in terms of the u variable using (94) in the argument of $\mathcal{W}'$ in the right-hand side (RHS) of (92). This gives us the finite-temperature Wigner function, which near the edge takes the scaling form

$$W_N(\mathbf{x},\mathbf{p})\simeq\frac{\mathcal{W}_b(a)}{(2\pi\hbar)^d},\quad a=\frac{1}{w_N}\left(\sqrt{\frac{\mathbf{p}^2}{m^2\omega^2}+\mathbf{x}^2}-r_e\right), \tag{96}$$

where w_N is given in (14). The finite-temperature scaling function, parametrized by $b=\beta\mu^{1/3}$, is

$$\mathcal{W}_b(a)=\int_{-\infty}^{+\infty}\frac{2^{2/3}du}{1+e^{-bu}}\,\mathrm{Ai}(2^{2/3}(u+a)). \tag{97}$$

It reduces to the $T=0$ scaling form in the limit $b\to\infty$, where the Fermi factor becomes a Heaviside θ function. The asymptotics of $\mathcal{W}_b(a)$ can be computed easily. After making the change of variable $2^{2/3}(u+a)=y$, we obtain

$$\mathcal{W}_b(a)=\int_{-\infty}^{\infty}dy\,\frac{\mathrm{Ai}(y)}{1+e^{a\,b}e^{-b\,y2^{-2/3}}}. \tag{98}$$

For $a\to-\infty$, this gives

$$\mathcal{W}_b(a)\to\int_{-\infty}^{\infty}dy\,\mathrm{Ai}(y)=1. \tag{99}$$

In contrast, when $a\to+\infty$, we get to leading order

$$\mathcal{W}_b(a)\sim C\,e^{-a\,b}, \tag{100}$$

where the prefactor

$$C=\int_{-\infty}^{\infty}dy\,\mathrm{Ai}(y)\,e^{b\,y\,2^{-2/3}}=e^{b^3/12}. \tag{101}$$

As in the zero-temperature case discussed in Sec. II C, we can extend the above finite-temperature results to the case of arbitrary smooth potentials. As discussed in Sec. II C the zero-temperature scaling form $\mathcal{W}(a)$ of the Wigner function near the edge is identical to that of the harmonic oscillator. In contrast the scaling variable takes the form

$$a=a_{\mu'}=\frac{1}{e_N(\mu')}\left(\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-\mu'\right), \tag{102}$$

where the energy scale $e_N(\mu')$ is an implicit function of μ' which can be obtained from formula (65). It thus depends nonuniversally on the shape of the potential $V(\mathbf{x})$ and on the precise location of $\mathbf{x}_e,\mathbf{p}_e$ on the edge surface in the phase space. Following the same steps as for the harmonic-oscillator case, the reduced inverse temperature variable is now

$$b=\beta\,e_N(\mu). \tag{103}$$

Defining u as in (89) we can again expand $a_{\mu'}$ in (102) around $\mu'=\mu$ as

$$a_{\mu'}\simeq a_\mu-\left[\frac{1}{e_N(\mu)}+a_\mu\partial_\mu\ln e_N(\mu)\right](\mu'-\mu), \tag{104}$$

which leads exactly to Eq. (94) to leading order in μ. The rest of the argument simply goes through, and we find that, for arbitrary smooth potentials, the finite-temperature Wigner function near the edge takes the form

$$W_N(\mathbf{x},\mathbf{p})\simeq\frac{\mathcal{W}_b(a)}{(2\pi\hbar)^d},\quad a=\frac{1}{e_N}\left(\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-\mu\right), \tag{105}$$

with exactly the same scaling function $\mathcal{W}_b(a)$ given in (96).

IV. CONCLUSION

In this paper we have studied the Wigner function $W_N(\mathbf{x},\mathbf{p})$ for N noninteracting fermions in a confining trap in d dimensions. At zero temperature and large N, we have shown that there are two main regimes for $W_N(\mathbf{x},\mathbf{p})$ in the phase space. A bulk regime where the Wigner function is flat over a finite support and vanishes outside. The edge of this support in the phase space is given by $\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})=\mu$, where μ is the Fermi energy. Around this edge $W_N(x,p)$, appropriately centered and scaled, is described by a scaling function $\mathcal{W}$. We have shown that this scaling function is universal, i.e., the same for a large class of confining smooth potentials, and, strikingly, is independent of the space dimension d. We then extended these results to finite temperature and found a one-parameter edge scaling function $\mathcal{W}_b$ where b is the scaled temperature. The finite-temperature scaling function is also universal and independent of the space dimension d. In both the bulk and the edge regimes, the scaling functions are non-negative everywhere.

In addition to these universal features (bulk and edge) there appears to be anomalous small scale regimes around special points in the phase space where the Wigner function is rapidly varying. For the harmonic oscillator in $d=1$ we have analyzed such an anomalous regime [4] near $x=p=0$ in detail. Indeed, it is well known, already at the single-particle level, that the Wigner function is not guaranteed to be non-negative for arbitrary potentials $V(x)$. However, for the scaling regimes at large N (bulk and edge), our results show that the Wigner function remains positive, for a large class of smooth potentials.

Finally, in this paper, we have focused on the standard Wigner function $W_N(\mathbf{x},\mathbf{p})$ of an N-body system, as defined in Eq. (4). This can be interpreted as the one-point probability density function in the phase space (in the semiclassical sense). Naturally, one can also investigate higher-order correlation functions in the phase space. At $T=0$, this can be naturally done by introducing a generalized Wigner function [1]

$$W_N^{(N)}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_N,\mathbf{p}_N) = \frac{1}{(2\pi\hbar)^{Nd}}\int d\mathbf{y}_1\cdots d\mathbf{y}_N\, e^{\frac{i}{\hbar}\sum_{i=1}^N \mathbf{p}_i\cdot\mathbf{y}_i} \times \Psi_0^*\left(\mathbf{x}_1+\frac{\mathbf{y}_1}{2},\ldots,\mathbf{x}_N+\frac{\mathbf{y}_N}{2}\right) \times \Psi_0\left(\mathbf{x}_1-\frac{\mathbf{y}_1}{2},\ldots,\mathbf{x}_N-\frac{\mathbf{y}_N}{2}\right), \quad (106)$$

where Ψ_0 is the ground-state many-body wave function. From this generalized Wigner function, one can construct successively n-point correlation functions by integrating out $N-n$ phase-space coordinates as follows:

$$C_n^{(N)}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_n,\mathbf{p}_n) = \frac{N!}{(N-n)!}\int d\mathbf{x}_{n+1}d\mathbf{p}_{n+1} \times\cdots d\mathbf{x}_N d\mathbf{p}_N W_N^{(N)}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_N,\mathbf{p}_N). \quad (107)$$

For instance, for $n=1$, Eq. (107) reduces precisely to the standard Wigner function $W_N(\mathbf{x},\mathbf{p})$ defined in Eq. (4), i.e., $C_1^{(N)}(\mathbf{x},\mathbf{p}) = W_N(\mathbf{x},\mathbf{p})$. Investigations of these higher-order correlation functions with $n>1$, both at $T=0$ and $T>0$, would be interesting and will be studied in a future publication [38].

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APPENDIX A: DERIVATION OF EQ. (25)

We start from the expression of the ground state for N noninteracting fermions as a Slater determinant:

$$\Psi_0(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N) = \frac{1}{\sqrt{N!}}\det_{1\leqslant i,j,\leqslant N}\psi_{\mathbf{k}_j}(\mathbf{x}_i). \quad (A1)$$

We now evaluate the integral on the RHS of Eq. (25) as

$$N\int_{-\infty}^{+\infty} d\mathbf{x}_2\ldots d\mathbf{x}_N\,\Psi_0^*(\mathbf{x},\mathbf{x}_2,\ldots,\mathbf{x}_N)\Psi_0(\mathbf{x}',\mathbf{x}_2,\ldots,\mathbf{x}_N) \quad (A2)$$

$$= \frac{1}{(N-1)!}\sum_{\sigma,\tau\in\mathcal{S}_N}(-1)^{\epsilon(\sigma)+\epsilon(\tau)}\psi^*_{\mathbf{k}_{\sigma(1)}}(\mathbf{x})\psi_{\mathbf{k}_{\tau(1)}}(\mathbf{x}') \times\int_{-\infty}^{+\infty} d\mathbf{x}_2\ldots d\mathbf{x}_N\prod_{i=2}^N \psi^*_{\mathbf{k}_{\sigma(i)}}(\mathbf{x}_i)\psi_{\mathbf{k}_{\tau(i)}}(\mathbf{x}_i) \quad (A3)$$

$$= \frac{1}{(N-1)!}\sum_{\sigma\in\mathcal{S}_N}\psi^*_{\mathbf{k}_{\sigma(1)}}(\mathbf{x})\psi_{\mathbf{k}_{\sigma(1)}}(\mathbf{x}') = \sum_{i=1}^N \psi^*_{\mathbf{k}_i}(\mathbf{x})\psi_{\mathbf{k}_i}(\mathbf{x}') = K_\mu(\mathbf{x},\mathbf{x}'), \quad (A4)$$

where $\mathcal{S}_N$ denotes the group of permutations over N elements and $\epsilon(\sigma)$ is the signature of the permutation σ. In the middle line each integral over x_i, $i=2,\ldots,N$ constrains the permutations in the double sum to be the same, i.e., $\sigma(i)=\tau(i)$, for each i between 2 and N, from orthonormality of the single-particle eigenfunctions. However, this also constrains $\sigma(1)=\tau(1)$ and we thus have $\sigma=\tau$. The sum in the last line is obtained by setting $\sigma(1)=i$ and summing over σ. Since we are dealing with the ground state, the $\mathbf{k}_i$'s correspond to the N lowest eigenstates, recovering (24).

APPENDIX B: EDGE KERNEL AND WIGNER FUNCTION

In [25] it was shown that, for a d-dimensional harmonic oscillator at $T=0$, the kernel near a point on the edge in position space $\mathbf{x}_{\rm em}=\mathbf{r}_e$ takes the scaling form

$$K_\mu(\mathbf{x},\mathbf{x}') \simeq \frac{1}{w_N^d}\mathcal{K}_d^{\rm edge}\left(\frac{\mathbf{x}-\mathbf{r}_e}{w_N},\frac{\mathbf{x}'-\mathbf{r}_e}{w_N}\right), \quad (B1)$$

where the scaling function $\mathcal{K}_d^{\rm edge}(\mathbf{a},\mathbf{b})$ is given by

$$\mathcal{K}_d^{\rm edge}(\mathbf{a},\mathbf{b}) = \int\frac{d^dq}{(2\pi)^d}e^{-i\mathbf{q}\cdot(\mathbf{a}-\mathbf{b})}\mathrm{Ai}_1\left[2^{2/3}\left(q^2+\frac{a_n+b_n}{2}\right)\right], \quad (B2)$$

where $a_n=\mathbf{a}\cdot\mathbf{r}_e/r_e$ and $b_n=\mathbf{b}\cdot\mathbf{r}_e/r_e$. In Eq. (B2) we have $\mathrm{Ai}_1(x)=\int_x^\infty du\,\mathrm{Ai}(u)$. Setting $\mathbf{x}\to\mathbf{x}-\frac{\mathbf{y}}{2}$ and $\mathbf{x}'\to\mathbf{x}'+\frac{\mathbf{y}}{2}$, gives, after rescaling $\mathbf{q}\to\mathbf{q}w_N$,

$$K_\mu\left(\mathbf{x}-\frac{\mathbf{y}}{2},\mathbf{x}+\frac{\mathbf{y}}{2}\right) \simeq \int\frac{d^dq}{(2\pi)^d}e^{-i\mathbf{q}\cdot\mathbf{y}}\mathrm{Ai}_1\left[2^{2/3}\left(w_N^2q^2+\frac{(\mathbf{x}-\mathbf{r}_e)\cdot\mathbf{r}_e}{w_N r_e}\right)\right]. \quad (B3)$$

Fourier transforming with respect to $\mathbf{y}$ we obtain

$$W_N(\mathbf{x},\mathbf{p}) \simeq \frac{1}{(2\pi\hbar)^d}\mathrm{Ai}_1\left[2^{2/3}\left(\frac{w_N^2}{\hbar^2}\mathbf{p}^2 + \frac{(\mathbf{x}-\mathbf{r}_e)\cdot\mathbf{r}_e}{w_N r_e}\right)\right] = \frac{1}{(2\pi\hbar)^d}\mathcal{W}\left(\frac{w_N^2}{\hbar^2}\mathbf{p}^2 + \frac{(\mathbf{x}-\mathbf{r}_e)\cdot\mathbf{r}_e}{w_N r_e}\right) \quad \text{(B4)}$$

from Eq. (11). Furthermore, we identify the argument which appears in the scaling function as $2^{2/3}a$, where a has been expanded around the point $\mathbf{x} = \mathbf{r}_e$, $\mathbf{p} = 0$ of the edge surface in the phase space. Indeed, to lowest order in $(\mathbf{x}-\mathbf{r}_e)$ and $\mathbf{p}^2$ one has

$$a = \frac{1}{w_N}\left(\sqrt{\frac{\mathbf{p}^2}{m^2\omega^2} + \mathbf{x}^2} - r_e\right) \simeq \frac{(\mathbf{x}-\mathbf{r}_e)\cdot\mathbf{r}_e}{w_N r_e} + \frac{w_N^2}{\hbar^2}\mathbf{p}^2 \quad \text{(B5)}$$

using the relation $w_N^3 r_e/\hbar^2 = 1/(2m^2\omega^2)$ valid for the harmonic oscillator. This provides an alternative derivation of $W_N(\mathbf{x},\mathbf{p})$, which is valid near the special point $\mathbf{x}_e = \mathbf{r}_e$, $\mathbf{p}_e = 0$ on the edge surface in the phase space. However, for the harmonic oscillator, since $W_N(\mathbf{x},\mathbf{p})$ is isotropic in the phase space (in dimensionless units) it clearly suffices to establish the result (10) for any point on the edge surface. This result is thus fully consistent with our derivation of the scaling behavior of the Wigner function for the harmonic oscillator.

Furthermore, using the fact that the scaling form (B1), (B2) is universal for a broad class of smooth potentials, one can similarly show that the zero-temperature Wigner scaling function is also universal around the special point $\mathbf{x} = \mathbf{r}_e$, $\mathbf{p}_e = 0$, and matches with the formula for e_N, given in Eq. (9) at this special point only. However, since for a general potential the isotropy of the Wigner function in phase space no longer holds, this method does not allow one to obtain the scaling form at a generic point on the edge surface. However, the method used in the text does not rely on isotropy and is valid anywhere on the edge surface.

APPENDIX C: SHORT-TIME EXPANSION AND UNIVERSALITY FOR A CLASS OF SMOOTH POTENTIALS

In this appendix, for simplicity, we use units such that $m = \hbar = 1$ and restore the units in the text. Consider a generic point in phase space $(\mathbf{x}_e,\mathbf{p}_e)$ on the edge surface defined by

$$\frac{\mathbf{p}_e^2}{2} + V(\mathbf{x}_e) = \mu. \quad \text{(C1)}$$

We demonstrate (the result given in the main text) that for $\mathbf{x},\mathbf{p}$ near such a point, and for the scaling variable $a = O(1)$ with a defined as

$$a = \frac{1}{e_N}\left(\frac{\mathbf{p}^2}{2} + V(\mathbf{x}) - \mu\right),$$
$$e_N = \frac{1}{2^{1/3}}\left((\mathbf{p}_e\cdot\nabla)^2 V(\mathbf{x}_e) + |\nabla V(\mathbf{x}_e)|^2\right)^{1/3}, \quad \text{(C2)}$$

the Wigner function takes the form

$$W_N(\mathbf{x},\mathbf{p}) \simeq \frac{\mathcal{W}(a)}{(2\pi\hbar)^d}. \quad \text{(C3)}$$

This statement is valid in the limit of large μ, which can be studied using the short-time expansion of the Euclidean propagator, extending the calculation performed in Appendix A of [25] (see below). It is useful to anticipate the main idea of the proof. First, at a generic point (C1) one has $|\mathbf{p}_e| \sim \mu^{1/2}$ and $V(\mathbf{x}_e) \sim \mu$. The two terms in the energy scale e_N in (C2) are thus both of the same order, with $e_N \sim |\nabla V(\mathbf{x}_e)|^{2/3}$. The typical time scale $t = t_N$ which will control the final integral over t [see below in Eq. (C4)] is $t_N \sim 1/e_N$, which, in the particular case $\mathbf{p}_e = 0$, also agrees with the result given by Eq. (282) in [25]. For a potential $V(\mathbf{x}) \sim |\mathbf{x}|^p$ at large $|\mathbf{x}|$ the estimate is $e_N \sim (\mu/x_e)^{2/3} \sim \mu^{2(p-1)/(3p)}$ (with $x_e = |\mathbf{x}_e|$) and $t_N \sim (x_e/\mu)^{2/3} \sim \mu^{-2(p-1)/(3p)}$, consistent for $p = 2$ with $t_N \sim \mu^{-1/3}$ obtained for the harmonic oscillator in the text. We will justify these statements below, but it is useful to keep them in mind for estimating the various terms.

Before performing the short-time expansion, let us first derive some useful exact representations for the Wigner function. We use the relation (56) between the Wigner function and the Euclidean propagator

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int_C \frac{dt}{2\pi i t}e^{\mu t}\int_{-\infty}^{+\infty} d\mathbf{y}\, e^{i\mathbf{p}\cdot\mathbf{y}} G\left(\mathbf{x}+\frac{\mathbf{y}}{2},\mathbf{x}-\frac{\mathbf{y}}{2},t\right). \quad \text{(C4)}$$

From Appendix A of [25] we can write, as an exact starting point, the following representation (using the symmetry of the Euclidean propagator):

$$G\left(\mathbf{x}-\frac{\mathbf{y}}{2},\mathbf{x}+\frac{\mathbf{y}}{2},t\right) = \frac{1}{(2\pi t)^{d/2}}\exp\left[-\frac{\mathbf{y}^2}{2t}\right] \times\left\langle \exp\left(-t\int_0^1 du\, V\left[\mathbf{x}+\mathbf{y}\left(u-\frac{1}{2}\right)+\sqrt{t}\mathsf{B}_u\right]\right)\right\rangle_{\mathsf{B}}, \quad \text{(C5)}$$

where $\langle\dots\rangle_{\mathsf{B}}$ denotes an average over the d-dimensional Brownian bridge $\mathsf{B}_u = \{B_{iu}\}_{i=1}^d$ on the interval [0,1], i.e., a Gaussian process with mean zero and correlation function

$$\langle B_{iu}B_{ju'}\rangle_{\mathsf{B}} = \delta_{ij}\, g(u,u'), \quad g(u,u') = \min(u,u') - uu', \quad \text{(C6)}$$

and consequently we have that $\mathsf{B}_0 = \mathsf{B}_1 = 0$. By first performing a cumulant expansion and then expanding in $\sqrt{t}\mathsf{B}_u$, we generate the short-time expansion of Eq. (C5).

We now substitute (C5) into (C4). We note that, in the absence of a potential (or if we neglect the $\mathbf{y}$ dependence in the potential term), we have a Gaussian integral over $\mathbf{y}$ with a saddle point at $\mathbf{y} = it\mathbf{p}$. This suggests that it is natural to make the change of integration variable

$$\mathbf{y} = it\mathbf{p} + \tilde{\mathbf{y}}\sqrt{t} \quad \text{(C7)}$$

and rewrite

$$W_N(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int_C \frac{dt}{2\pi i t}e^{(\mu-\frac{\mathbf{p}^2}{2}-V(\mathbf{x}))t+S(\mathbf{x},\mathbf{p},t)}, \quad \text{(C8)}$$

where

$$S(\mathbf{x},\mathbf{p},t) = \ln\left\langle \exp\left(-t\int_0^1 du\left[V\left(\mathbf{x}+(it\mathbf{p}+\tilde{\mathbf{y}}\sqrt{t})\left(u-\frac{1}{2}\right)+\sqrt{t}\mathsf{B}_u\right)-V(\mathbf{x})\right]\right)\right\rangle_{\mathsf{B},\tilde{\mathbf{y}}}, \tag{C9}$$

where the "average" over $\tilde{\mathbf{y}}$ is over a unit Gaussian random variable, uncorrelated with B, i.e.,

$$\langle\dots\rangle_{\mathsf{B},\tilde{\mathbf{y}}} = \int_{-\infty}^{+\infty}\frac{d\tilde{\mathbf{y}}}{(2\pi)^{d/2}}\exp\left[-\frac{\tilde{\mathbf{y}}^2}{2}\right]\langle\dots\rangle_{\mathsf{B}}. \tag{C10}$$

Since the averaging measure is even in $\tilde{\mathbf{y}}$ and even in B_u it is clear that $S(\mathbf{x},\mathbf{p},t)$ starts at $O(t^2)$. Hence the form (C8) is quite convenient to study the short-time expansion. The leading term, $O(t)$ in the exponential, is obtained by setting $S(\mathbf{x},\mathbf{p},t)$ to zero, which recovers the result Eq. (5) of the text for the Wigner function in the bulk.

Since $\tilde{\mathbf{y}}$, B, and $\mathbf{p}$ do not depend on t, the t dependence of $S(\mathbf{x},\mathbf{p},t)$ in (C9) is explicit, and its expansion in powers of t at small time is straightforward, although tedious. It is done by a gradient expansion of the argument of V around $\mathbf{x}\simeq\mathbf{x}_e$. In doing so we also need to check that this gradient expansion is consistent with the expansion at large μ, i.e., that in the large μ limit all terms in the argument of V are small compared to $\mathbf{x}_e$. This is clearly the case for the terms proportional to $\tilde{\mathbf{y}}$ and B_u, for which the gradient expansion is an expansion in the parameter $t_N^{1/2}/x_e \sim \mu^{-1/3}x_e^{-2/3} \ll 1$, using our above anticipated estimate for t_N. For the term $it\mathbf{p}$, the gradient expansion parameter is $t_N p_e/x_e \sim \mu^{-1/6}x_e^{-1/3} \ll 1$.

We now calculate $S(\mathbf{x},\mathbf{p},t)$ up to $O(t^3)$. For this it is easy to see that we need only the first two cumulants in (C9). The first cumulant is

$$S_1(\mathbf{x},\mathbf{p},t) = -t\int_0^1 du\left[\left\langle V\left(\mathbf{x}+(it\mathbf{p}+\tilde{\mathbf{y}}\sqrt{t})\left(u-\frac{1}{2}\right)+\sqrt{t}\mathsf{B}_u\right)\right\rangle_{\mathsf{B},\tilde{\mathbf{y}}}-V(\mathbf{x})\right] \tag{C11}$$

$$= -t^2\left[\left(\frac{1}{24}\langle\tilde{y}_j\tilde{y}_k\rangle_{\tilde{\mathbf{y}}}+\frac{1}{2}\int_0^1 du\langle B_{ju}B_{ku}\rangle_{\mathsf{B}}\right)\nabla_j\nabla_k V(\mathbf{x})\right] - t^3\left[-\frac{1}{8}\int_0^1 du(1-2u)^2(\mathbf{p}\cdot\nabla)^2V(\mathbf{x}) + \left(\frac{1}{1920}\langle\tilde{y}_j\tilde{y}_k\tilde{y}_\ell\tilde{y}_n\rangle_{\tilde{\mathbf{y}}}\right.\right.$$
$$\left.\left.+\frac{1}{16}\langle\tilde{y}_j\tilde{y}_k\rangle_{\tilde{\mathbf{y}}}\int_0^1 du(1-2u)^2\langle B_{\ell u}B_{nu}\rangle_{\mathsf{B}}+\frac{1}{24}\int_0^1 du\langle B_{ju}B_{ku}B_{\ell u}B_{nu}\rangle_{\mathsf{B}}\right)\nabla_j\nabla_k\nabla_\ell\nabla_n V(\mathbf{x})\right]+O(t^4). \tag{C12}$$

Note that all terms proportional to an odd number of gradients vanish due to the symmetry $u\to 1-u$ in the integral over u. Performing all the averages we obtain

$$S_1(\mathbf{x},\mathbf{p},t) = -\,t^2\left(\frac{1}{24}+\frac{1}{12}\right)\nabla^2V(\mathbf{x}) - t^3\left[-\frac{1}{24}(\mathbf{p}\cdot\nabla)^2V(\mathbf{x})+\left(\frac{1}{640}+\frac{1}{480}+\frac{1}{240}\right)\nabla^2\nabla^2V(\mathbf{x})\right], \tag{C13}$$

where we have purposely indicated the contribution of each term, and checked that the last term is identical to the result in Eqs. (246)– (248) in [25]. The second cumulant expanded to $O(t^3)$ reads

$$S_2(\mathbf{x},\mathbf{p},t) = \frac{t^2}{2}\int_0^1 du\int_0^1 du' \tag{C14}$$

$$\times\left\langle\left[V\left(\mathbf{x}+(it\mathbf{p}+\tilde{\mathbf{y}}\sqrt{t})\left(u-\frac{1}{2}\right)+\sqrt{t}\mathsf{B}_u\right)-V(\mathbf{x})\right]\left[V\left(\mathbf{x}+(it\mathbf{p}+\tilde{\mathbf{y}}\sqrt{t})\left(u'-\frac{1}{2}\right)+\sqrt{t}\mathsf{B}_{u'}\right)-V(\mathbf{x})\right]\right\rangle^c_{\mathsf{B},\tilde{\mathbf{y}}} \tag{C15}$$

$$= \frac{t^3}{2}\int_0^1 du\int_0^1 du'\left[\left(u-\frac{1}{2}\right)\left(u'-\frac{1}{2}\right)\tilde{y}_j\tilde{y}_k+B_{ju}B_{ku'}\right]\nabla_jV(\mathbf{x})\nabla_kV(\mathbf{x})+O(t^4) \tag{C16}$$

$$= \frac{t^3}{24}|\nabla V(\mathbf{x})|^2+O(t^4). \tag{C17}$$

The sum $S(\mathbf{x},\mathbf{p},t) = S_1(\mathbf{x},\mathbf{p},t)+S_2(\mathbf{x},\mathbf{p},t)+O(t^4)$ together with (C13) and (C14) provides the exact short-time expansion up to $O(t^3)$ of $S(\mathbf{x},\mathbf{p},t)$, which enters the formula (C8) for the Wigner function [it gives in fact the short-time expansion up to $O(t^3)$ of the Fourier transform of the Euclidean propagator $G(\mathbf{x}-\frac{\mathbf{y}}{2},\mathbf{x}+\frac{\mathbf{y}}{2},t)$].

Now remember that our goal is instead the large μ expansion of the Wigner function. If one can show that (i) all terms in S_1 except the term $\frac{t^3}{2}(\mathbf{p}\cdot\nabla)^2V(\mathbf{x})$ are irrelevant in the edge regime and (ii) all terms $O(t^4)$ or higher are also irrelevant, then we see that

$$W_N(\mathbf{x},\mathbf{p}) \simeq \frac{1}{(2\pi)^d}\int_C\frac{dt}{2\pi it}e^{\left(\mu-\frac{\mathbf{p}^2}{2}-V(\mathbf{x})\right)t+\frac{t^3}{24}(|\nabla V(\mathbf{x}_e)|^2+(\mathbf{p}_e\cdot\nabla)^2V(\mathbf{x}_e))}, \tag{C18}$$

where in the cubic term it is consistent to replace $\mathbf{x}$ by $\mathbf{x}_e$. Performing the change of variable $t = 2^{2/3}\tau/e_N$, with e_N given by (C2), we obtain exactly the integral representation (55) of the function $\mathcal{W}(a)$, hence demonstrating the result (C3) with the scaling variable a defined in (C2). Furthermore, this confirms that $t_N \sim 1/e_N$ is the time scale which dominates the integral in the edge regime as anticipated above.

Estimating the term $O(t^2)$ in (C13) to be of order $t_N^2 V(\mathbf{x}_e)/x_e^2 \sim \mu^{-1/3} x_e^{-2/3} \ll 1$, we see that it is indeed negligible, as was already the case for $\mathbf{p}_e = \mathbf{0}$ in [25]. Inside the t^3 term in (C13) we see that the second term is smaller than the first by a factor $\frac{1}{p_e x_e^2} \sim \mu^{-1/2} x_e^{-2} \ll 1$. The examination of terms $O(t^4)$ and higher is very tedious and can be performed along the lines of Appendix A in [25]. We will not reproduce this analysis here.

In summary, the above shows that for a large class of smooth potentials, at a generic point of the edge surface in phase space, the universal edge form of the Wigner function holds. The analysis is a rather simple extension of the one in [25] (simple in the sense that the characteristic scales are not changed, apart from some prefactors). Note that a necessary condition is that e_N does not vanish, which is true at a generic point, but could fail in some exceptional cases, for instance if $\mathbf{p}_e$ and $\nabla V(\mathbf{x}_e)$ vanish simultaneously.

APPENDIX D: WIGNER FUNCTION CLOSE TO THE CENTER ($x = 0, p = 0$)

In this appendix, we show that for the 1D harmonic oscillator at $T = 0$ the Wigner function near $(x = 0, p = 0)$ in the phase space has an anomalous behavior. Indeed, for $r^2 = x^2 + p^2 = O(1/N)$, we will show that $W_N(x,p)$ has the following behavior:

$$W_N(x,p) \sim \frac{1}{2\pi} - (-1)^N F(\sqrt{N(x^2+p^2)}),$$
$$F(z) = \frac{1}{2\pi} J_0(2\sqrt{2}z), \tag{D1}$$

where $J_\nu(x)$ is the Bessel function of index ν. Our starting point is the exact generating function in Eq. (35).

Formally inverting this generating function using Cauchy's formula, we find

$$W_N(x,p) = \frac{1}{2\pi i}\int_{C_0} \frac{dz}{z^N} \frac{1}{\pi(1-z^2)} e^{-\frac{1-z}{1+z}r^2}, \tag{D2}$$

where C_0 is the contour around the origin in the complex z plane, as shown in Fig. 4. For z such that $\mathrm{Re}(z) > -1$, the integrand in Eq. (D2) has a simple pole at $z = 1$ and an Nth order pole at $z = 0$ and is analytic elsewhere. We can thus replace the contour integral on C_0 by three other contour integrals as follows (see Fig. 4):

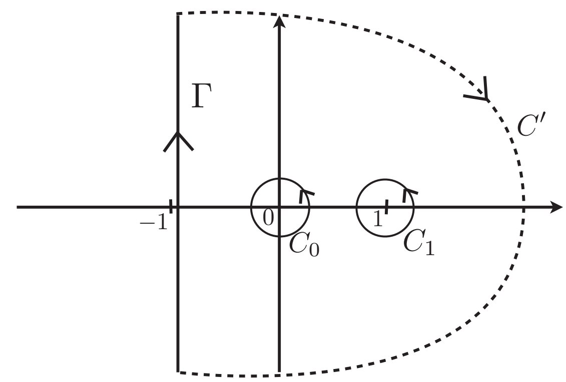

FIG. 4. Contours in the complex z plane used to evaluate the integral in Eq. (D2) using the decomposition in Eq. (D3).

$$\int_{C_0} = -\int_{C_1} - \int_{C'} - \int_{\Gamma}, \tag{D3}$$

where the contours C_1, C', and Γ are shown in Fig. 4. We will eventually deform the contours C' and Γ such that Γ is a straight vertical line passing infinitesimally close to the right of $z = -1$ and C' will be eventually sent to infinity. Evaluating the simple pole around $z = 1$ gives

$$-\int_{C_1} = \frac{1}{2\pi}. \tag{D4}$$

The contribution from the contour integral C' is exponentially small for large N when the contour C' is sent to infinity. It remains to evaluate the contour integral over Γ. It is clear that the leading contribution to this integral over Γ comes from the vicinity of $z = -1$. Hence, it is natural to make the change of variable $z = -1 + z'$ and note that the dominant contribution comes from the regime close to $z' = 0$, i.e., $z' = O(1/N)$ for large N. Therefore, we set $z' = t/N$ and expand the integrand for large N. To leading order, we obtain

$$W_N(x,p) \sim \frac{1}{2\pi} - \frac{(-1)^N}{2\pi}\frac{1}{2\pi i}\int_\Gamma \frac{dt}{t} e^{Nt - \frac{2}{t}r^2}. \tag{D5}$$

Using the integral representation of the Bessel function J_0

$$\frac{1}{2\pi i}\int \frac{dt}{t} e^{Nt - \frac{y}{t}} = J_0(2\sqrt{yN}), \tag{D6}$$

Eq. (D5) immediately gives the result in Eq. (D1).

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