Entanglement and quantum combinatorial designs

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We introduce several classes of quantum combinatorial designs, namely quantum Latin squares, cubes, hypercubes, and a notion of orthogonality between them. A further introduced notion, quantum orthogonal arrays, generalizes all previous classes of designs. We show that mutually orthogonal quantum Latin arrangements can be entangled in the same way in which quantum states are entangled. Furthermore, we show that such designs naturally define a remarkable class of genuinely multipartite highly entangled states called k-uniform, i.e., multipartite pure states such that every reduction to k parties is maximally mixed. We derive infinitely many classes of mutually orthogonal quantum Latin arrangements and quantum orthogonal arrays having an arbitrary large number of columns. The corresponding multipartite k-uniform states exhibit a high persistency of entanglement, which makes them ideal candidates to develop multipartite quantum information protocols.

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I. INTRODUCTION

One of the key problems in the theory of quantum information is to identify multipartite quantum states with the strongest possible quantum correlations. Contrary to the classical behavior, information stored in multipartite quantum systems is not equivalent to information provided by the parties. The extremal situation occurs when information stored in an N-partite pure quantum state is not present *at all* in any subset of k collaborating parties, for some integer $k \leqslant N/2$. Such pure states are called *k-uniform* [1–3], meaning that every reduction to k parties is given by the maximally mixed state. When $k = \lfloor N/2 \rfloor$, where $k = \lfloor . \rfloor$ denotes the floor function, the state is called *absolutely maximally entangled* (AME). Sometimes, these states are also called maximally multipartite entangled states [4], or MMES for short.

For instance, the generalized Bell states of two subsystems with d levels each and the tripartite GHZ-like states belong to the AME class. These highly entangled states find applications in quantum secret sharing [5], quantum error correction codes [1], and holographic codes [6]. They can be constructed from graph states [7,8], orthogonal arrays [3], multiunitary matrices [9], and perfect tensors [1,6]. Furthermore, from gluing AME states further multipartite classes of such states can be constructed in higher dimensions [10]. However, to determine the existence of AME(N,d) for any number of parties N and internal levels d is a difficult problem, specially if d is not a power of a prime number [11]. Many approaches were tried in order to give an answer to this question, including recasting of the problem in the language of statistical mechanics [12–14].

In this work, we introduce certain classes of combinatorial designs by extending classical symbols to pure quantum states. Our starting point is the notion of *quantum Latin squares* (QLS) [15], which we generalize to quantum Latin cubes (QLC) and hypercubes (QLH). We also introduce a notion of orthogonality between them and identify a crucial ingredient missing in the previous approach [16]: two orthogonal QLS could be entangled in such a way that they cannot be expressed as two separated arrangements. These entangled designs are intrinsically associated with a larger class of quantum designs that includes all previous quantum Latin arrangements: *quantum orthogonal arrays*. After setting up the quantum combinatorial tools we apply our method to the problem to construct k-uniform states and absolutely maximally entangled states in particular, for multipartite systems having an arbitrary large number of parties.

The paper is organized as follows: In Sec. II, we recall the standard concepts of (classical) Latin squares, Latin cubes, Latin hypercubes, and orthogonal arrays and review their basic properties. In Sec. III we define quantum Latin squares, cubes, and hypercubes and introduce a notion of orthogonality between them. Simple examples in low dimensions are provided. In Sec. IV, we introduce the concept of quantum orthogonal arrays. We show that quantum Latin arrangements arise from quantum orthogonal arrays in the same way that Latin arrangements arise from orthogonal arrays in combinatorics. In Sec. V we show a connection existing between quantum orthogonal arrays and multiunitary matrices, the last ones introduced in Ref. [9]. In Sec. VI we derive simple constructions of k-uniform and AME states from quantum orthogonal arrays. A summary of results and concluding remarks are presented in Sec. VII.

II. LATIN ARRANGEMENTS AND ORTHOGONAL ARRAYS

In this section, we review some basic combinatorial concepts used in this work. A Latin square LS(d) is a square arrangement of size d such that every entry, taken from the set $\{0,\dots,d-1\}$, occurs once in each row and each column. For instance, arrangements

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}, \quad \begin{array}{ccc} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{array}, \quad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array} \tag{1}$$

are Latin squares of size d equal to two, three, and four, respectively.

An *orthogonal array*, denoted as OA(r,N,d,k), is an arrangement composed by r rows, N columns, and entries taken from the set $\{0,\dots,d-1\}$, such that every subset of k columns contains all possible combinations of symbols, occurring the same number (λ) of times along the rows. Here, parameters k and λ are called *strength* and *index* of the OA, respectively [17]. An OA is called *irredundant* if every subset of $(N-k)$ columns contains no repeated rows [3]. Two OA are called equivalent if one array can be transformed into the other one by applying permutations or relabeling of symbols in rows or columns.

It is simple to show that any LS(d) is equivalent to an OA(d^2,3,d,2); see Chapter 8 in Ref. [17]. For example, the array OA(4,3,2,2) produces a LS(2), as shown below:

$$\mathrm{OA} = \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ \text{--} & \text{--} & \text{--} \\ i & j & \mathrm{LS} \end{array} \quad \Rightarrow \quad \mathrm{LS} = \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}. \tag{2}$$

Here, the first two columns of the OA identify coordinates (i,j) of symbols for the LS, whose values are determined by the third column LS of the OA.

Two Latin squares LS^A and LS^B of size d are *orthogonal* if the set of ordered pairs $[(\mathrm{LS}^A)_{ij},(\mathrm{LS}^B)_{ij}]$ is composed by all possible d^2 combinations symbols, where $i,j\in\{0,\dots,d-1\}$. A collection of m LS of order d is called mutually orthogonal (MOLS) if they are pairwise orthogonal. For instance, any OA(d^2,2 + m,d,2) defines a set of m MOLS of size d [17]. In particular, an OA(9,4,3,2) implies two classical OLS of size 3. As before, the first two columns (i,j) of the OA address entries of OLS, while the two latter yield the values of the squares A and B,

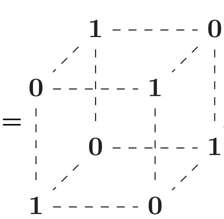
$$\mathrm{OA}(9,4,3,2) = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ \text{--} & \text{--} & \text{--} & \text{--} \\ i & j & A & B \end{array} \quad \Rightarrow \quad \begin{array}{l} \mathrm{LS}^A = \begin{array}{ccc} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{array} \\ \\ \mathrm{LS}^B = \begin{array}{ccc} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{array}. \end{array} \tag{3}$$

Entries of two OLS are typically denoted as ordered pairs in a single array. For instance, the two OLS of Eq. (3) are denoted as

$$\mathrm{OLS} = \begin{array}{ccc} 00 & 21 & 12 \\ 22 & 10 & 01 \\ 11 & 02 & 20 \end{array}. \tag{4}$$

Furthermore, orthogonal arrays can be associated with Latin cubes. An OA(d^3,4,d,3) defines a Latin cube LC(d), which consists of a cubic arrangement composed by d rows, d columns, and d files, such that every entry taken from the set $\{0,\dots,d-1\}$ occurs once in each row, each column, and each file. For instance, OA(8,4,2,3) defines a LC of size 2, where now the first three bits (i,j,k) determine the position of a given element of the cube LC, while the last bit determines its value,

$$\mathrm{OA}(8,4,2,3) = \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ \text{--} & \text{--} & \text{--} & \text{--} \\ i & j & k & \mathrm{LC} \end{array}, \quad \mathrm{LC} =$$



(5)

In general, an OA(d^k,$k+m$,d,k) defines m mutually orthogonal Latin hypercubes (LH) of size d in dimension k, denoted MOLH(d). Figure 1 summarizes existing relations between OA and Latin arrangements.

To emphasize the difference between the above described standard combinatorial designs and their quantum generalizations discussed in subsequent sections we will refer to OA, LS, MOLS, and MOLC as the *classical* arrangements. An OA having r rows, N columns, and d symbols can be associated with a pure quantum state of an N-qudit system having r terms [3]. Each row of the array corresponds to a single term of the state, so the left-hand side of the arrangement (3) yields the unnormalized state of four parties

$$\begin{aligned} |\phi_{4,3}\rangle = {} & |0000\rangle + |0121\rangle + |1022\rangle \\ & + |1110\rangle + |1201\rangle + |2102\rangle \\ & + |2220\rangle + |2011\rangle + |0212\rangle. \end{aligned} \tag{6}$$

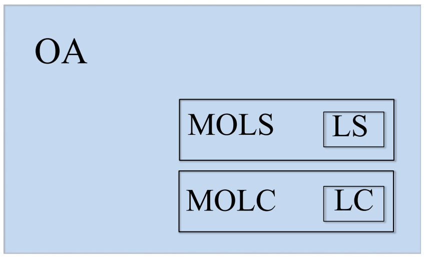


FIG. 1. Orthogonal arrays generalize some classes of combinatorial arrangements: Latin squares (LS), Latin cubes (LC), and mutually orthogonal LS and LC (MOLS and MOLC, respectively). These arrangements can be generalized to Latin hypercubes (LH) and mutually orthogonal LH (MOLH), respectively. In this work, we develop a theory of quantum combinatorial designs and show that quantum Latin arrangements arise from QOA in the same way that classical Latin arrangements arise from OA.

This state is maximally entangled with respect to the $\binom{4}{2} = 6$ possible balanced bipartitions and it is called an *absolutely maximally entangled state*, denoted AME(4,3). In this work we consider unnormalized pure states, for the sake of simplicity.

III. QUANTUM LATIN ARRANGEMENTS

Recently, quantum Latin squares (QLS) [15] and *weakly orthogonal* QLS [16] have been introduced, where classical symbols appearing in entries of arrangements were extended to quantum states. These concepts were used to define unitary error bases [15] and mutually unbiased bases [16]. In this section, we extend those results by introducing some classes of quantum Latin arrangements, where QLS are a particular case.

The following notion of quantum Latin squares was introduced by Musto and Vicary [15].

Definition 1. A quantum Latin square of size d is a square arrangement

$$\mathrm{QLS}(d) = \begin{matrix} |\psi_{0,0}\rangle & \dots & |\psi_{0,d-1}\rangle \\ \vdots & & \vdots \\ |\psi_{d-1,0}\rangle & \dots & |\psi_{d-1,d-1}\rangle \end{matrix} \tag{7}$$

composed of d^2 single-particle quantum states $|\psi_{ij}\rangle \in \mathcal{H}_d$, $i,j \in \{0,\dots,d-1\}$, such that each row and each column determine an orthonormal basis for a qudit system.

For instance, the following example of a quantum Latin square was given in Ref. [15],

$$\begin{matrix} |0\rangle & |1\rangle & |2\rangle & |3\rangle \\ |3\rangle & |2\rangle & |1\rangle & |0\rangle \\ |\chi_-\rangle & |\xi_-\rangle & |\xi_+\rangle & |\chi_+\rangle \\ |\chi_+\rangle & |\xi_+\rangle & |\xi_-\rangle & |\chi_-\rangle \end{matrix}, \tag{8}$$

where the two lower rows contain entangled states, $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |0\rangle)$, $|\xi_+\rangle = \frac{1}{\sqrt{5}}(i|0\rangle + 2|3\rangle)$, and $|\xi_-\rangle = \frac{1}{\sqrt{5}}(2|0\rangle + i|3\rangle)$. As a first observation, we realize that any QLS is naturally related to a tripartite pure state having maximally mixed single-particle reductions.

Proposition 1. A set of d^2 vectors $|\psi_{ij}\rangle \in \mathcal{H}_d$ forms a QLS(d) if and only if every single-particle reduction of the three-qudit state

$$|\Phi\rangle = \sum_{i,j=0}^{d-1} |i\rangle|j\rangle|\psi_{ij}\rangle \tag{9}$$

is maximally mixed.

Proof. Let $|\psi_{ij}\rangle \in \mathcal{H}_d$ be the d^2 entries of a QLS(d) and let us define the state $|\Phi\rangle = \sum_{i,j=0}^{d-1} |i\rangle|j\rangle|\psi_{ij}\rangle$. Therefore

$$\begin{aligned} \rho_A &= \mathrm{Tr}_{BC}|\Phi\rangle\langle\Phi| \\ &= \mathrm{Tr}_{BC}\left(\sum_{i,j,i',j'=0}^{d-1} |ij\rangle_{AB}\langle i'j'| \otimes |\psi_{ij}\rangle_C\langle\psi_{i'j'}|\right) \\ &= \sum_{i,j,i'=0}^{d-1} \langle\psi_{ij}|\psi_{i'j}\rangle_{BC}|i\rangle_A\langle i'| = \sum_{i,j,i'=0}^{d-1} |i\rangle_A\langle i| = \mathbb{I}_d, \end{aligned}$$

where we used the fact that $|\psi_{ij}\rangle \in \mathcal{H}_d$ defines a QLS(d) and denoted A,B,C for the first, second, and third party, respectively. Analogously, $\rho_B = \mathbb{I}_d$, as we work with non-normalized states. Furthermore, we have

$$\begin{aligned} \rho_C &= \mathrm{Tr}_{AB}\left(\sum_{i,j,i',j'=0}^{d-1} |ij\rangle\langle i'j'| \otimes |\psi_{ij}\rangle\langle\psi_{i'j'}|\right) \\ &= \sum_{i,j=0}^{d-1} |\psi_{ij}\rangle\langle\psi_{ij}| = \mathbb{I}_d, \end{aligned}$$

and, therefore, state (9) has every single-particle reduction maximally mixed. The reciprocal implication works in the reverse way. ■

Let us exemplify Proposition 1 by considering the 1-uniform state of a three-qudit system,

$$|\phi\rangle = F_d \otimes F_d \otimes \mathbb{I}_d|\mathrm{GHZ}_d\rangle = \sum_{l,m=0}^{d-1} |lm\rangle|\psi_{l,m}\rangle. \tag{10}$$

Here $|\mathrm{GHZ}_d\rangle = \sum_{n=0}^{d-1}|nnn\rangle$ denotes a generalized GHZ state of three subsystems with d levels each, $F_d = \sum_{l,m=0}^{d-1} \omega^{lm}|l\rangle\langle m|$ is the discrete Fourier transform of size d containing a unimodular number $\omega = e^{2\pi i/d}$, and the state reads

$$|\psi_{l,m}\rangle = \sum_{n=0}^{d-1} \omega^{n(l+m)}|n\rangle. \tag{11}$$

This construction works for any $d \geqslant 2$. The d^2 states from Eq. (11) determine a QLS of size d, which is equivalent to the classical $[\mathrm{LS}(d)]_{lm} = l + m|$ mod d with $l,m = 0,\dots,d-1$, as the classical arrangement can be obtained by applying the same suitable local unitary operation to every column of the QLS. The state (10) is 1-uniform and it is equivalent to the three-qudit GHZ state, in agreement with Proposition 1. Let us generalize this fact in the following observation.

Observation 1. A QLS(d) is equivalent to a classical LS(d) if and only if one arrangement can be transformed into the other by applying the same local unitary operation to every column.

Furthermore, note that a unitary operation U applied to a single column of a LS implies a controlled U operation acting on the third party of the corresponding three-partite 1-uniform state (see Proposition 1). As consequence, the entanglement of the state is changed and the Latin arrangement is spoiled by a single-column unitary operation.

The notion of a *weakly orthogonal* QLS has been recently introduced [16]:

Definition 2. A pair of QLS of size d having entries $\{\varphi_{ij}\}$ and $\{\varphi'_{ij}\}$ are weakly orthogonal when for every $i,j \in \{0,\ldots,d-1\}$, there exists a unique $t_{ij} \in \{0,\ldots,d-1\}$ such that

$$\sum_{l=0}^{d-1} \langle \varphi_{li}|\varphi'_{lj}\rangle |l\rangle = |t_{ij}\rangle. \tag{12}$$

This definition reflects some desired aspects in orthogonal QLS. Indeed, it is reduced to the standard definition of the LS if the states φ_{ij} belong to the computational basis. However, other fundamental ingredients seem to be missing here—for instance, the astonishing property that a pair of orthogonal QLS are *not* necessarily equivalent to two QLS satisfying an orthogonality criteria, as we will see below. Those sets of orthogonal QLS that cannot be separated will be called *essentially quantum Latin squares*. This new concept of nonseparability of combinatorial designs is analogous to the nonseparability of quantum states.

Let us now introduce the notion of orthogonality for QLS, which is *not* equivalent to orthogonality for two separated quantum arrangements.

Definition 3. A set of d^2 pure quantum states $|\psi_{i,j}\rangle \in \mathcal{H}_d^{\otimes 2}$ arranged as

$$\begin{array}{ccc} |\psi_{0,0}\rangle & \ldots & |\psi_{0,d-1}\rangle \\ \vdots & & \vdots \\ |\psi_{d-1,0}\rangle & \ldots & |\psi_{d-1,d-1}\rangle \end{array} \tag{13}$$

forms a pair of orthogonal quantum Latin squares (OQLS) if the following properties hold:

(1) The set of d^2 states $\{|\psi_{i,j}\rangle\}$ are orthogonal and form a basis in $\mathcal{H}_d \otimes \mathcal{H}_d$.

(2) The sum of every row in the array (13), i.e., $\sum_{j=0}^{d-1} |\psi_{i,j}\rangle$, is a 1-uniform state.

(3) The sum of every column in the array (13), i.e., $\sum_{i=0}^{d-1} |\psi_{i,j}\rangle$, is a 1-uniform state.

Observation 2. Two OQLS composed of separable states, $|\psi_{i,j}^{AB}\rangle = |\eta_{ij}^{A}\rangle \otimes |\eta_{ij}^{B}\rangle$ for every $i,j \in \{0,\ldots,d-1\}$, imply that both arrangements $\{|\eta_{ij}^{A}\rangle\}$ and $\{|\eta_{ij}^{B}\rangle\}$ determine QLS, according to Definition 1.

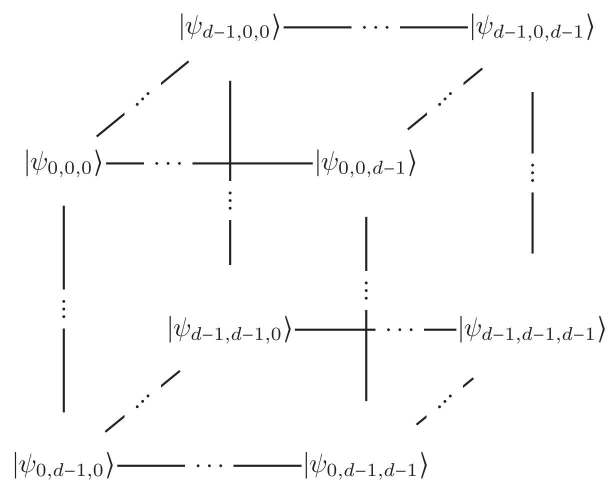
Indeed, single-party reductions to A and B of the states defined in items 2 and 3 above are proportional to the maximally mixed state, so that every row and every column of arrangements $\{|\eta_{ij}^{A}\rangle\}$ and $\{|\eta_{ij}^{B}\rangle\}$ form an orthonormal basis. Moreover, if entries of each QLS are given by elements of the computational basis then Definitions 1 and 3 reduce to the classical definition of LS and OLS, respectively (see Sec. II).

As we will show in Sec. IV, OQLS are closely related to 2-uniform states. In order to achieve higher classes of multipartite entanglement, i.e., k-uniformity for $k > 2$, one has to generalize quantum combinatorial arrangements to higher dimensions. To this end, let us go a step forward and introduce quantum Latin cubes.

Definition 4. A quantum Latin cube (QLC) of size d is a cubic arrangement composed of d^3 single-particle quantum pure states $|\psi_{x,y,z}\rangle \in \mathcal{H}_d$, $x,y,z \in \{0,\ldots,d-1\}$, such that every row, every column and every file form a set of orthogonal states.

For instance, in the case of a cubic arrangement composed by qubit quantum states, i.e., $d = 2$, we have the cube (5). Let us introduce a notion of orthogonality between cubic arrangements.

Definition 5. A set of d^3 tripartite pure states $|\psi_{x,y,z}\rangle$ belonging to a composed Hilbert space $\mathcal{H}_d^3$, arranged as



forms a triple of mutually orthogonal quantum Latin cubes (MOQLC) if the following properties hold:

(1) The set of d^3 states $\{|\psi_{x,y,z}\rangle\}$ are orthogonal.

(2) The sum of every row in this array, i.e., $\sum_{i=0}^{d-1} |\psi_{x,y,z}\rangle$, is a 1-uniform state.

(3) The sum of every column in this array, i.e., $\sum_{j=0}^{d-1} |\psi_{x,y,z}\rangle$, is a 1-uniform state.

(4) The sum of every file in this array, i.e., $\sum_{k=0}^{d-1} |\psi_{x,y,z}\rangle$, is a 1-uniform state.

Analogously to Definition 3, if the d^3 states forming a set of MOQLC are fully separable, i.e., $|\psi_{x,y,z}^{ABC}\rangle = |\eta_{x,y,z}^{A}\rangle \otimes |\eta_{x,y,z}^{B}\rangle \otimes |\eta_{x,y,z}^{C}\rangle$, then each set of states $\{|\eta_{x,y,z}^{A}\rangle\}$, $\{|\eta_{x,y,z}^{B}\rangle\}$, and $\{|\eta_{x,y,z}^{C}\rangle\}$ forms a QLS according to Definition 1. Furthermore, in such a case a fully separable MOQLC is equivalent to a classical MOLC, in the sense that one can be connected to the other by applying local unitary operations acting in columns of the arrangements. This is so because any single-party orthonormal basis can be transformed into the computational basis by applying a suitable local unitary transformation. Also, if the states forming the cube given in Definition 5 are biseparable with respect to a given partition, e.g., $|\psi_{x,y,z}^{ABC}\rangle = |\eta_{x,y,z}^{A}\rangle \otimes |\eta_{x,y,z}^{BC}\rangle$ for every $x,y,z \in \{0,\ldots,d-1\}$, then the single-party arrangement $\{|\eta_{x,y,z}^{A}\rangle\}$ defines a QLC according to Definition 5. It is important here to note that the bipartite arrangement $\{|\eta_{x,y,z}^{BC}\rangle\}$ *not necessarily* forms a pair of OQLC. This surprising fact is closely related to the lack of some classes of multipartite absolutely maximal entanglement; e.g., AME(N,2) states exist only if the number of qubits is given by $N = 2,3,5,6$ [1,18,19].

As the concepts of OQLS and OQLC are settled, let us define an arbitrary-dimensional kind of quantum combinatorial arrangement, called *quantum Latin hypercubes*. These quantum arrangements can be connected to k-uniform states for N-qudit systems having d levels each for any k, N, and d, as we will show in Sec. IV.

Definition 6. A quantum Latin hypercube of size d and dimension k, denoted QLH(d,k), is an arrangement composed of d^k single-particle quantum states $|\psi_{i_1,\ldots,i_k}\rangle \in \mathcal{H}_d^{\otimes k}$, $i_1,\ldots,i_k \in \{0,\ldots,d-1\}$, such that all states belonging to an edge of the hypercube are orthogonal.

In particular, for $k=2$ quantum hypercube QLH$(d,2)$ reduces to the square QLS(d), while for $k=3$ they form a cube, QLH$(d,3)$ = QLC(d). For instance, the state AME(8,7) with minimal support determines an $m=4$ hypercubes MOQLH of size $d=7$ in dimension $k=4$, equivalent to 4 classical MOLH. These four separable hypercubes can be easily found from the orthogonal array OA$(7^4,8,7,4)$ with index equal to unity, associated with the AME(8,7) state; see Theorem 3 and Proposition 2 (i) in [3]. This state is also related to a maximum distance separable (MDS) code [13,20]. Furthermore, the AME(8,5) state, which has nonminimal support, defines $m=4$—essentially quantum—orthogonal Latin hypercubes in dimension $k=4$, with entangled entries. This state can be constructed from *ququint codes* [21].

We can extend the sets of OQLS and OQLC to sets of m mutually orthogonal quantum Latin hypercubes (MOQLH) of size d and dimension $k \leqslant m$. The following definition contains all previously defined combinatorial designs.

Definition 7. A set of m mutually orthogonal quantum Latin hypercubes of size d in dimension k, denoted m MOQLH(d), is a k-dimensional arrangement composed of m-qudit states $|\psi_{i_1,\ldots,i_k}\rangle \in \mathcal{H}_d^{\otimes m}$, $i_1,\ldots,i_m \in \{0,\ldots,d-1\}$, such that the following properties hold:

(1) The set of d^k states $\{|\psi_{i_1,\ldots,i_k}\rangle\}$ are orthogonal.

(2) The sum of d states belonging to the same edge of the hypercube, i.e., $\sum_{i_s=0}^{d-1}|\psi_{i_1,\ldots,i_s,\ldots,i_m}\rangle$ for every $1\leqslant s\leqslant m$, forms a 1-uniform state.

In particular, a set of m MOLS are also MOQLS; e.g., the classical arrangements (3) agree with Definition 7. In Sec. IV, we introduce a suitable tool to generate quantum Latin arrangements, called *quantum orthogonal arrays*, and also establish its connection with quantum Latin arrangements.

Bounds for MOQLH

Let us now study upper bounds for the maximal number of classical and quantum Latin arrangements. The theory of orthogonal arrays provides a bound [22] for the maximal number of columns of an OA$(d^k,2+m_C,d,k)$, that has index unity. Therefore, it is easy to derive an upper bound for the maximal allowed number m_C of classical MOLH of size d and dimension k:

$$m_C \leqslant \begin{cases} k-1, & \text{if } d\leqslant k,\\ d+k-4, & \text{if } d>k\geqslant 3,\\ d+k-3, & \text{in all other cases.}\end{cases} \tag{14}$$

For example, in dimension $k=2$ we have that m MOLS of size d can only exist for $m_C\leqslant d-1$, for any $d\geqslant 2$. The upper bound $m=d-1$ can be saturated for d being a prime power number. These results, well known in standard combinatorics, motivate us to derive similar results for quantum Latin arrangements. However, derivation of such a generalized bound requires solving a complicated optimization problem formalized by Scott—see Eqs. (39)–(41) in Ref. [1]. Given the set of parameters N,d,k (n,D,d in the original notation) these equations can be solved by considering linear programming techniques. The particular case $k=\lfloor N/2\rfloor$, for which the arrangements are associated with AME states, can be analytically solved. Therefore, we are able to provide an analytic bound for the maximal number m_Q of MOQLH in the case of maximal possible dimension $k=\lfloor N/2\rfloor$ as follows:

$$m_Q \leqslant \begin{cases} 2(d^2-1), & \text{if } N \text{ is even},\\ 2d(d+1)-1, & \text{if } N \text{ is odd}.\end{cases} \tag{15}$$

For instance, for $N=4$ and $k=2$ we have that $m_Q\leqslant 2(d^2-1)$ MOQLS exist for any size d, which is $2(d+1)$ times larger than the classical bound $m_C\leqslant d-1$. It is important to note that bounds (15) are not tight, as the bounds provided by Scott [1] are not tight; see also [19].

Inequalities (14) and (15) can be useful to detect genuine quantumness in MOQLH. In general, given a set of m MOQLH it is hard to detect inequivalence to a classical set of MOLS. Typically, such kind of comparison would require considering a full set of entanglement invariants. However, for those cases where $m>m_C$ it is ensured that a MOQLH is essentially quantum. For instance, a single LS of size two exists and there are no two QOLS of size two. Surprisingly, there exist *three* entangled MOQLS of size two, as we will show in Sec. IV.

IV. QUANTUM ORTHOGONAL ARRAYS

In this section, we introduce quantum orthogonal arrays. This concept allows us to derive a simple rule to generate infinitely many classes of k-uniform states and absolutely maximally entangled states, in particular.

Definition 8. A quantum orthogonal array QOA(r,N,d,k) is an arrangement consisting of r rows composed by N-partite normalized pure quantum states $|\varphi_j\rangle \in \mathcal{H}_d^{\otimes N}$, having d internal levels each, such that

$$k\sum_{j=0}^{r-1}\mathrm{Tr}_{i_1,\ldots,i_{N-k}}(|\varphi_j\rangle\langle\varphi_j|) = r\,\mathbb{I}_k, \tag{16}$$

for every subset of $N-k$ parties $\{i_1,\ldots,i_{N-k}\}$.

In words, a QOA is an arrangement having N columns, possibly entangled, such that every reduction to k columns defines a *positive operator valued measure* (POVM). We recall that a POVM is a set of positive semidefinite operators such that they sum up to identity, determining a generalized quantum measurement [23].

We can also provide a connection to error correction codes that suggest considering generalized measurements instead of projective measurements in QOA. Note that any AME state (or k-uniform state) can be related to a certain quantum error correction code [1]. In particular, an AME state of N parties with local dimension d corresponds to a quantum code—which can be considered as an injective mapping from the space of $K=1$ messages to a subset C of the set of code words with length N—often denoted by $(\!(N,K=1,D=\lfloor N/2\rfloor+1)\!)_d$. In this notation, the parameter D is the distance of the code, i.e., the minimal number of local operations performed on a single qudit that are needed to create a nonzero overlap between two code words [24]. The Knill-Laflamme theorem [25] implies that a subspace C of the Hilbert space $\mathcal{H} = \mathbb{C}_d^{\otimes N}$ generates an error-correcting quantum code, if there exist recovery operators $R_1, R_2, \ldots$ such that for any state ρ with support in C and any collection of error operators $A_1, A_2, \ldots$ with $\sum_e E_e^\dagger E_e = \mathbb{1}$, we have $\sum_{r,e} R_r E_e \rho E_e^\dagger R_r^\dagger = \rho \otimes \mathbb{1}$. In this case $R_1, R_2, \ldots$ are a finite sequence of operators in $\mathcal{H}$ satisfying the relation $\sum_r R_r^\dagger R_r = \mathbb{1}$. This theorem combined with the fact that an AME state yields an error correction code allows us to define quantum orthogonal arrays in a way that every reduction produces a POVM.

Definition 8 forms a natural extension of the classical concept of orthogonal arrays to quantum theory: the classical digits from $(0, \ldots, d-1)$ are generalized to quantum states from $\mathcal{H}_d$, while the classical concept of subsets of columns is replaced by partial trace.

From now on, we assume that columns of quantum arrangements are connected by the Kronecker product. Also, QOA having the minimal possible number of rows, i.e., $r = d^k$, are called *index unity*, as occurs in the classical case.

Let us introduce equivalent classes of QOA as a natural generalization of its classical counterpart, defined in Sec. II. Two QOA are *equivalent* if one can transform one arrangement into the other one by applying suitable local unitary operations to columns and permutation of rows or columns. Note that permutation of columns in quantum states produces states inequivalent under local operations and classical communication (LOCC), in general. Nevertheless, as interchange of particles does not change the amount of entanglement in quantum states, from now on we will restrict our attention to QOA inequivalent under swap operations. Note that the only allowed local unitary operations in classical OA are permutation matrices, equivalent to relabeling of symbols. In contrast to quantum Latin arrangements, in QOA we are allowed to apply any local unitary operation to any column without spoiling the orthogonal array. To illustrate these ideas let us consider the following example:

$$(\mathbb{I} \otimes \sigma_x) \begin{array}{cc} |0\rangle & |0\rangle \\ |1\rangle & |1\rangle \end{array} = \begin{array}{cc} |0\rangle & |1\rangle \\ |1\rangle & |0\rangle \end{array},$$

where $\sigma_x = \{\{0,1\},\{1,0\}\}$ is the Pauli shift operator. In this way, we obtain two equivalent classical OA. Instead, by applying the Hadamard gate $H = \{\{1,1\},\{1,-1\}\}$ to the second column, i.e.,

$$(\mathbb{I} \otimes H) \begin{array}{cc} |0\rangle & |0\rangle \\ |1\rangle & |1\rangle \end{array} = \begin{array}{cc} |0\rangle & |+\rangle \\ |1\rangle & |-\rangle \end{array}, \tag{17}$$

with $|\pm\rangle = |0\rangle \pm |1\rangle$, we obtain a QOA which is equivalent under local unitary operations to a classical OA. The simplest essentially quantum orthogonal array consists of five columns,

$$\mathrm{QOA}(4,5,2,2) = \begin{array}{cccc} |0\rangle & |0\rangle & |0\rangle & |\Phi^+\rangle \\ |0\rangle & |1\rangle & |1\rangle & |\Psi^+\rangle \\ |1\rangle & |0\rangle & |1\rangle & |\Psi^-\rangle \\ |1\rangle & |1\rangle & |0\rangle & |\Phi^-\rangle \end{array}, \tag{18}$$

where $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ denote the Bell basis. To emphasize that some of these columns are separable (classical) and some of them are entangled (quantum), we shall also write QOA$(4,3_C+2_Q,2,2)$, as the second argument denotes three classical and two quantum columns. Note that the number of classical and quantum columns, i.e., N_C and N_Q such that $N = N_C + N_Q$, are invariant under local unitary operations acting on columns of the QOA. Moreover, a QOA is equivalent to a classical OA if and only if $N_Q = 0$, thus also implying a classical set of MOLS and a classical error correction code [17]. Roughly speaking, parameter N_Q quantifies how much quantum is a given QOA and its related MOQLS and error correction code. As a further comment, note that every reduction to two columns of the arrangement (18) forms a POVM, where partial trace should be considered for entangled columns. The fact that QOA (18) is not equivalent to a classical OA is in correspondence with the fact that the AME(5,2) state cannot be written as a convex combination of elements of the 5-qubit computational basis.

As we have seen in Sec. II, OLS arise from OA. The first two columns of the OA provide address to entries of the first and second LS, whose values are determined by the third and fourth columns of the OA; see Eq. (3). In the same way, from QOA(4,5,2,2) of Eq. (18) we derive three MOQLS of size 2, which are essentially quantum. A triple of mutually orthogonal quantum Latin squares reads

$$\mathrm{MOQLS}(2) = \begin{array}{cc} |0\rangle|\Phi^+\rangle & |1\rangle|\Psi^+\rangle \\ |1\rangle|\Psi^-\rangle & |0\rangle|\Phi^-\rangle \end{array}. \tag{19}$$

The first two columns of QOA (18) address entries of the three MOQLS (19). Note that these three MOQLS are entangled, which is a direct consequence of the fact that QOA (18) is not equivalent to a classical one. Indeed, QOA (18) contains entangled columns. According to the results shown in Sec. III, a single-party arrangement belonging to a set of MOQLS determines a QLS, which can be seen from Eq. (19) after tracing out second and third parties. However, the bipartite arrangement obtained from taking a partial trace over the first subsystem of the QOA (19), i.e.,

$$\begin{array}{cc} |\Phi^+\rangle & |\Psi^+\rangle \\ |\Psi^-\rangle & |\Phi^-\rangle \end{array}, \tag{20}$$

is *not* a pair of orthogonal QLS. This is simple to observe if we take into account Definition 3. Indeed, the sum of every column of the arrangement (20) determines a 1-uniform state but the sum of every row gives a separable state. It is possible to prove that such QOA$(r,4,2,2)$ does not exist for any $r \in \mathbb{N}$, which is related to the fact that an AME(4,2) state does not exist [26].

As a further example, we consider the following array consisting of three classical and three quantum columns,

$$\mathrm{QOA}(8,3_C+3_Q,2,3) = \begin{array}{cccc} |0\rangle & |0\rangle & |0\rangle & |\mathrm{GHZ}_{000}\rangle \\ |0\rangle & |0\rangle & |1\rangle & |\mathrm{GHZ}_{001}\rangle \\ |0\rangle & |1\rangle & |0\rangle & |\mathrm{GHZ}_{010}\rangle \\ |0\rangle & |1\rangle & |1\rangle & |\mathrm{GHZ}_{011}\rangle \\ |1\rangle & |0\rangle & |0\rangle & |\mathrm{GHZ}_{100}\rangle \\ |1\rangle & |0\rangle & |1\rangle & |\mathrm{GHZ}_{101}\rangle \\ |1\rangle & |1\rangle & |0\rangle & |\mathrm{GHZ}_{110}\rangle \\ |1\rangle & |1\rangle & |1\rangle & |\mathrm{GHZ}_{111}\rangle \end{array}, \tag{21}$$

which produces three MOQLC of size 2:

$$\mathrm{MOQLC}(2) = \begin{array}{cccc} & |\mathrm{GHZ}_{100}\rangle & & |\mathrm{GHZ}_{101}\rangle \\ |\mathrm{GHZ}_{000}\rangle & & |\mathrm{GHZ}_{001}\rangle & \\ & |\mathrm{GHZ}_{110}\rangle & & |\mathrm{GHZ}_{111}\rangle \\ |\mathrm{GHZ}_{010}\rangle & & |\mathrm{GHZ}_{011}\rangle & \end{array} \tag{22}$$

Here, the tripartite orthonormal basis is composed by eight states locally equivalent to the 3-qubit GHZ state, $|\mathrm{GHZ}\rangle = |000\rangle + |111\rangle$. These states form an orthonormal basis in $\mathcal{H}_8 = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$,

$$|\mathrm{GHZ}_{ijk}\rangle = (-1)^{\alpha_{ijk}} \sigma_i \otimes \sigma_j \otimes \sigma_k |\mathrm{GHZ}\rangle, \tag{23}$$

where $i,j,k = \{0,1\}$ and σ_0 and σ_1 represent the Pauli matrices σ_x and σ_z, respectively. Global phases given by $\alpha_{ijk} = 1$ if $i = j = k$ and $\alpha_{ijk} = 0$ otherwise are added to states (23) forming the GHZ basis, in such a way that the construction (22) forms a quantum Latin cube.

Let us show that a QOA(r,N,d,k) determines a k-uniform state of N qudits, in the same way as an irredundant OA(r,N,d,k) implies a k-uniform state of N subsystems with d levels each [3].

Proposition 2. The sum of rows of a QOA(r,N,d,k) produces a k-uniform state of a quantum system composed of N parties with d levels each.

Proof. Every reduction to k columns of a QOA(r,N,d,k) defines a POVM, and thus the sum of its elements produces the identity operator. ■

For instance, QOA(4,5,2,2) of Eq. (18), related to the squares (19), produces the 2-uniform five-qubit state [27]

$$\mathrm{AME}(5,2) = |000\rangle|\Phi^+\rangle + |011\rangle|\Psi^+\rangle + |101\rangle|\Psi^-\rangle + |110\rangle|\Phi^-\rangle. \tag{24}$$

Furthermore, the array QOA(8,6,2,3) presented in Eq. (21), and related to the cube (22), produces the AME state for six-qubit systems [28],

$$\mathrm{AME}(6,2) = \sum_{x,y,z=0}^{1} |x,y,z\rangle|\mathrm{GHZ}_{xyz}\rangle. \tag{25}$$

Proposition 2 reveals that QOA generalizes the notion of irredundant OA and *not* the entire set of OA. For instance, the non-irredundant classical array,

$$\mathrm{OA}(4,3,2,1) = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}, \tag{26}$$

is not equivalent to a QOA$(r,3,2,1)$ for any r. This is so because OA (26) does not produce a 1-uniform state and, by definition,

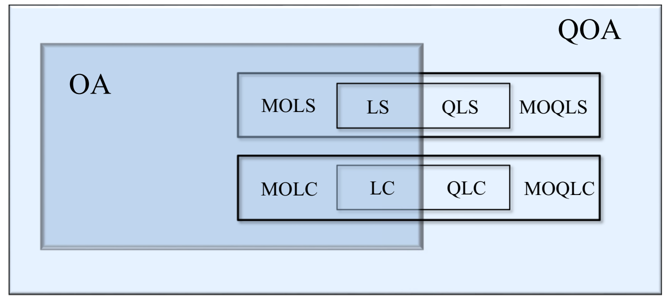


FIG. 2. Generalization of orthogonal arrays (OA) to quantum orthogonal arrays (QOA). This extension allows us to naturally generalize some classical arrangements to quantum mechanics: quantum Latin squares (QLS), quantum Latin cubes (QLC), and mutually orthogonal quantum arrangements (MOQLS and MOQLC).

any QOA produces at least a 1-uniform state. The key difference existing between classical and quantum OA relies on the fact that the action of removing columns in classical OA is *not* equivalent to taking the partial trace in the quantum case. Precisely, these operations are equivalent only if the orthogonal array considered is irredundant. Furthermore, the juxtaposition of two OA is still an OA, whereas the same statement does not hold for QOA. This is connected to the fact that the sum of two k-uniform states is not necessarily k-uniform (see Sec. VI). Nonetheless, all classical OA$(d^k,2+m,d,2)$, associated with m mutually orthogonal hypercubes size d, are irredundant [3]. Thus, any set of m mutually orthogonal Latin hypercubes, in particular any set of m MOQLS, is linked to a QOA; see Fig. 2. As a natural generalization of this result, we have the following proposition.

Proposition 3. A QOA$(d^k,k+m,d,k)$ generates m MOQLH of size d in dimension k.

We generate MOQLH from QOA in the same way in which MOLH arise from classical OA. That is, first k classical columns of a QOA address the location of entries and the remaining m columns determine the values of every entry of the quantum Latin arrangement.

Let us discuss some important open issues. The lowest-dimensional open case for MOQLS occurs for $k = m = 2$ and $d = 6$, that is, two OQLS of size six. It is well known that the classical problem of *36 officers of Euler* has no solution [29], as there are no orthogonal Latin squares of order six. After an exhaustive numerical exploration we are tempted to advance the following conjecture.

Conjecture 1. Two orthogonal QLS of size 6 do not exist.

This conjecture is equivalent to saying that the famous problem of Euler has no solution also in the generalized quantum setup, as 36 officers are now allowed to be described by entangled quantum states.

It also would imply a negative answer to the existence of AME state for a system composed of four systems with 6 levels each; compare related studies in Refs. [3,11]. The existence of the AME(4,6) state currently represents the lowest-dimensional open case, and the only open case in the family of states AME$(4,d)$. We recall that AME$(4,d)$ exist for any $d \neq 2,6$. Indeed, all of these states have minimal support and can be easily generated from two classical MOLS(d), equivalently from OA$(d^2,4,d,2)$ [3].

Let us now relate quantum Latin arrangements defined through QOA with those established in Definition 7. The special subset of MOQLH satisfying Definition 7 produces highly entangled k-uniform states (e.g., cluster states), robust under the presence of a noisy environment. Indeed, we might interpret the hyperfaces of MOQLS as a protection of multipartite entanglement contained in lower-dimensional faces of the hypercube. For instance, the generalized, N-qudit GHZ state, $\sum_{i=0}^{d-1}|i\rangle^{\otimes N}$, defines the following set of N MOQLH of size d defined in dimension $k=1$:

$$|\underbrace{0,\dots,0}_{N}\rangle -- |\underbrace{1,\dots,1}_{N}\rangle -- |\underbrace{d-1,\dots,d-1}_{N}\rangle. \tag{27}$$

Here, the double line ($--$) denotes edges in the same way as depicted before; cf. Definition 5. Arrangement (27) has a unique 1-dimensional face, evidencing fragility of entanglement of GHZ states with respect to the noisy environment. On the other hand, the square (13) produces a state having a higher robustness, as the square transforms to an edge under the presence of a local measurement on any of its parties. For instance, the 5-qubit state produced by three MOQLS of size $d=2$, see Eq. (19), defines a *perfect code* for quantum error correction [27].

In order to understand robustness of entanglement produced by states coming from Definition 7, we need to recall two quantifiers of robustness [30]:

Maximum connectedness ($\mathcal{C}$). A multipartite quantum state is maximally connected if any two qudits can be projected, with certainty, into a Bell state by implementing local measurements on the complementary subset of parties.

Persistency of entanglement ($\mathcal{P}$). The minimal number of local measurements to be implemented such that, for all measurement outcomes, the state is completely disentangled.

Now we are in position to establish the following result:

Proposition 4. A set of m MOQLH $\{|\varphi_{i_1,\dots,i_m}\rangle\}$ of size d defined in dimension k, composed by d^k states of m-qudit systems having d levels each, defines a k-uniform state for $N=k+m$ qudit systems, given by

$$|\phi\rangle = \sum_{i_1,\dots,i_k=0}^{d-1} |i_1,\dots,i_k\rangle|\varphi_{i_1,\dots,i_m}\rangle. \tag{28}$$

Even more, if $k'\leqslant k$ subsystems belonging to the first k qudits are measured then the remaining entangled state is $(k-k')$-uniform. In particular, if a state $|\phi\rangle$ can be written in the form (28) for its $\binom{N}{k}$ possible bipartitions of k parties out of N then it has maximum connectedness $\mathcal{C}=k-1$ and persistency of entanglement $\mathcal{P}\geqslant k$.

Proof. The state $|\phi\rangle$ defined for $N=k+m$ subsystems with d levels each is k-uniform, since the following two facts hold: (i) the set of m MOQLH defined in dimension k defines a QOA(d^k,N,d,k) and (ii) Proposition 2 applies. The fact that maximum connectedness is at least $\mathcal{C}=k-1$ comes straight from Property 2 in Definition 7. By the same reason, we have $\mathcal{P}\geqslant k$, as an additional measurement may possibly destroy the 1-uniformity of the remaining m-partite entangled states $|\phi'\rangle=\sum_{i_1,\dots,i_k=0}^{d-1}|\varphi_{i_1,\dots,i_m}\rangle$. ■

For instance, the state AME(5,2) defined in (24), constructed through MOQLS (19), satisfies $\mathcal{C}=1$, and defines a 1-dimensional subspace protected under decoherence [27].

Let us summarize some important connections existing between classical and quantum arrangements and k-uniform states derived along this section. First, we start considering previously known connections. The following standard ("classical") notions are equivalent:

(1) QOA with fully separable columns ($\equiv$OA) [e.g., QOA$(9,4_C+0_Q,3,2)\equiv$ OA(9,4,3,2) in Eq. (3)].

(2) Sets of m separable MOQLH(d) in dimension k ($\equiv$ MOLH) [e.g., classical LS$_A$ and LS$_B$ in Eq. (3)].

(3) N-qudit k-uniform states with minimal support [e.g., AME(4,3) state in Eq. (6)].

Here, the symbol $\equiv$ denotes equivalence under local unitary operations applied to columns of an array. Connection 1-2 is well known in mathematics since the early times of orthogonal arrays theory; see Chapter 8 in Ref. [17]. Connections 1-3 and 2-3 have been recently established; see Refs. [3] and [9], respectively. Furthermore, in the case of $N=2k$ there exists a link between AME states and multiunitary permutation matrices [9].

In a similar manner, the following generalized ("quantum") notions are equivalent:

(a) QOA with entangled columns ($\not\equiv$OA) [e.g., Eqs. (18) and (21)].

(b) Entangled MOQLH ($\not\equiv$ fully separable MOQLH) [e.g., Eqs. (19)].

(c) N-qudit k-uniform states with nonminimal support ($\not\equiv$ to minimal support states) [e.g., Eqs. (24) and (25)].

The above relations a-b, a-c, and b-c form a novel contribution of the present work. A further connection to general multiunitary matrices occurs when $N=2k$ [9].

Note that a QOA having at least one pair of entangled columns necessarily implies the existence of entangled OQLS that cannot be separated, in the same way as entangled states cannot be represented as the tensor product of two single-party pure states.

V. QOA AND MULTIUNITARY MATRICES

Let us consider a quantum system consisting of $N=2k$ parties having d level systems each, where $k\geqslant 1$ and the system is prepared in the pure state

$$|\phi\rangle = \sum_{\substack{n_1,\dots,n_k\\ \nu_1,\dots,\nu_k}} a_{\substack{n_1,\dots,n_k\\ \nu_1,\dots,\nu_k}} |n_1,\dots,n_k,\nu_1,\dots,\nu_k\rangle, \tag{29}$$

where every sum goes from 0 to $d-1$. The matrix

$$(A)_{\substack{n_1,\dots,n_k\\ \nu_1,\dots,\nu_k}} = \langle n_1,\dots,n_k|A|\nu_1,\dots,\nu_k\rangle = a_{\substack{n_1,\dots,n_k\\ \nu_1,\dots,\nu_k}}$$

is called k-unitary if it is unitary for all possible $\binom{2k}{k}$ reordering of its indices, corresponding to all possible choices of k indices out of $2k$. Matrices k-unitary for $k>1$ are called *multiunitary* [9]. Furthermore, multiunitary matrices are one-to-one connected with *perfect tensors* [6], which play an important role in construction of holographic codes.

For instance, a matrix A is 2-unitary if A, A^{T_2}, and A^R are unitary, where T_2 and R stand for partial transposition and reshuffling operations, respectively; see Appendix 2 in Ref. [9].

As a remarkable property, a matrix A is k-unitary if and only if the state (29) is AME($2k,d$).

A multiunitary matrix A of size d^k allows us to write a multipartite pure state as the action of a nonlocal gate acting on k parties over a generalized Bell-like state, that is,

$$|\phi\rangle = \sum_{n_1,\dots,n_k} (\mathbb{I}_{d^k} \otimes A)\,|n_1,\dots,n_k\rangle|n_1,\dots,n_k\rangle$$
$$= (\mathbb{I}_{d^k} \otimes A) \sum_{n_1,\dots,n_k} |n_1,\dots,n_k\rangle|n_1,\dots,n_k\rangle. \quad (30)$$

For any AME state $|\phi\rangle$, the operator A is a nonlocal k-unitary gate acting on k parties. Furthermore, if A is a 2-unitary matrix of size d^2, then the quantum arrangement

$$\begin{array}{ccc} A|0,0\rangle & \dots & A|0,d-1\rangle \\ \vdots & & \vdots \\ A|d-1,0\rangle & \dots & A|d-1,d-1\rangle \end{array} \quad (31)$$

forms a pair of QLS. In particular, A is a 2-unitary permutation matrix if and only if the arrangement (31) is a classical MOLS(d). This implies that a matrix A being 2-unitary but not permutation defines a quantum QOA. Even more, if the QOA is not equivalent under LOCC to a QOA having associated a permutation matrix A then the QOA is essentially quantum. This is the case of the essentially quantum array QOA($4,3_C+2_Q,2,2$), presented in Eq. (18). In general, A is a k-unitary permutation matrix if and only if the state $|\phi\rangle$ defined by (30) is an AME($2k,d$) state with a minimal support.

In the same way, a 3-unitary matrix of size d^3 defines a set of 3 MOQLC. As an interesting observation, from Eq. (30) we realize that any AME($2k,d$) state can be associated with a QOA having at least $N_C = k$ classical columns and the minimal possible number of rows $r = d^k$; i.e., it always has index unity.

Let us generalize these finding in the following proposition.

Proposition 5. A k-unitary matrix A of order d^k defines $m = 2k$ MOQLH of size d in dimension k. Even more, if A is a permutation matrix then the MOQLH are equivalent to a classical set of MOLH.

Proof. A k-unitary matrix A of size d^k defines an AME state composed by $N = 2k$ subsystems with d levels each. Due to Proposition 2 such a state defines a QOA($d^k,2k,d,k$) and according to Proposition 3 implies a MOQLH of size d in dimension k. The last implication was already proven in Ref. [9]. ■

In terms of bipartite quantum gates [31], the fact that the classical problem of 36 officers has no solution implies that there is no multiunitary permutation matrix of size $6^2 = 36$. That is, there is no permutation matrix P_{36} of order 36 such that its partial transpose $P_{36}^{T_2}$ and its reshuffling P_{36}^{R} are both unitary (for an explicit definition of T_2 and R see Appendix B in Ref. [9]). As a generalization to quantum mechanics, there exists a solution of *36 quantum officers of Euler* if and only if a multiunitary matrix of size 36 exists. Multiunitary matrices are relevant in quantum information theory as they saturate the upper bound of the entangling power [9,32,33]. We remark that on one hand Conjecture 1 is consistent with earlier observations by Clarisse *et al.* [31] and by recent numerical investigations [34–36]. On the other hand, the existence of the AME(4,6) state cannot be excluded by applying the currently known bounds for AME states [1,19,37], so this interesting problem remains still open.

VI. AME STATES FROM QUANTUM ORTHOGONAL ARRAYS

As we have seen in Proposition 2, quantum arrays QOA(r,N,d,k) imply the existence of k-uniform states for N-qudit systems having d levels each. In this section, we derive k-uniform states with maximal possible value $k = \lfloor N/2 \rfloor$ for $N = 5$ and arbitrary $d \geqslant 2$ from QOA. Those states determine AME states for 5-qudit systems.

Let us present a simple construction for AME($5,d$) states for every $d \geqslant 2$ derived from QOA. These states are known to exist [38] but their explicit closed form has not been presented before, as far as we know. We first define the state

$$\mathrm{AME}(3,d) = \sum_{i=0}^{d-1} |i,j,i+j\rangle, \quad (32)$$

which has associated a classical array IrOA($d^2,3,d,1$). Here and from now on, sums inside kets are understood to be modulo d. By considering this state and the generalized Bell basis for 2-qudit systems, we are going to construct a QOA composed of 5 columns and d^2 rows that defines an AME($5,d$) state for every integer d. The first three classical columns of the quantum arrangement are induced by the state (32), whereas the remaining two essentially quantum columns are given by elements of the Bell basis

$$|\phi_{i,j}\rangle = \sum_{l=0}^{d-1} \omega^{il} |l+j,l\rangle, \quad (33)$$

where $\omega = e^{2\pi i/d}$. We are now in a position to establish the following result.

Proposition 6. The following three existing quantum objects, determined by a collection of d^2 states $|\phi_{i,j}\rangle \in \mathcal{H}_d^{\otimes 2}$, are equivalent:

(A) QOA($d^2,3_C+2_Q,d,2$)

$$\begin{array}{cccc} |0\rangle & |0\rangle & |0\rangle & |\phi_{0,0}\rangle \\ |0\rangle & |1\rangle & |1\rangle & |\phi_{0,1}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |d-1\rangle & |d-1\rangle & |d-2\rangle & |\phi_{d-1,d-1}\rangle \end{array}. \quad (34)$$

(B) Triple of MOQLS of size d

$$\begin{array}{ccc} |0\rangle|\phi_{0,0}\rangle & \dots & |d-1\rangle|\phi_{0,d-1}\rangle \\ \vdots & \ddots & \vdots \\ |d-1\rangle|\phi_{d-1,0}\rangle & \dots & |d-2\rangle|\phi_{d-1,d-1}\rangle \end{array}. \quad (35)$$

(C) Quantum state

$$\mathrm{AME}(5,d) = \sum_{i,j=0}^{d-1} |i,\,j,\,i+j\rangle|\phi_{i,j}\rangle, \quad (36)$$

for any integer $d \geqslant 2$.

Proof. Proof of (A) follows from two facts: (i) every subset of two columns produces an orthonormal basis, and (ii) every reduction to three columns contains orthogonal rows. These conditions ensure that every reduction to two columns produces a POVM. These two properties are an extension of the so-called *uniformity* and *irredundancy*, considered to construct k-uniform states from classical OA (see Sec. IV in Ref. [3]). Equivalence between (A) and (C) follows directly from Proposition 2, while the last relation between (A) and (B) can be obtained by Proposition 3. ■

For instance, in the case of $d = 2$, this construction reduces to QOA (18), MOQLS (19), and the AME(5,2) state (24). Note that the QOA (34) has its last two columns entangled, implying that MOQLS (35) are necessarily entangled and AME state (36) does not have minimal support. This is consistent with the summary of results presented at the end of Sec. IV.

Observation 3. QOA allow us to add a classical column to arrangement (34) in order to define the following 2-uniform states of 6 qudits, i.e.,

$$|\psi(6,d)\rangle = \sum_{i,j=0}^{d-1} |i,j,i+j,i+2j\rangle|\phi_{i,j}\rangle, \tag{37}$$

where d is an odd prime number and both sums in kets are taken modulo d. When d is a prime power number, it is convenient to use a polynomial representation based on irreducible polynomials. In such cases, the 2-uniform states of 6 qubits can be written as

$$|\psi(6,d)\rangle = \sum_{i,j=0}^{d-1} |i,j,i+j,i+a_1 j\rangle|\phi_{i,j}\rangle,$$

where a_1 is the first element of the finite set using the polynomial representation for which $a_1 \neq 0,1$.

Here, note that the classical and quantum parts of the underlying QOA are composed of four and two columns, respectively. It is simple to check that this arrangement is a QOA(d^2,$4_C + 2_Q$,d,2).

In the constructions presented above, the key point was to produce a QOA from combining a classical OA and an orthonormal basis composed of generalized Bell states. It is simple to realize that multiplication of quantum columns produces another QOA having a larger number of columns. For example, the QOA (18) can be extended by considering m copies of the quantum part in the following way:

$$\begin{array}{cccl} 1 & 1 & 1 & |\Phi^+\rangle \;\ldots\; |\Phi^+\rangle \\ 0 & 0 & 1 & |\Phi^-\rangle \;\ldots\; |\Phi^-\rangle \\ 0 & 1 & 0 & |\Psi^+\rangle \;\ldots\; |\Psi^+\rangle, \\ 1 & 0 & 0 & \underbrace{|\Psi^-\rangle \;\ldots\; |\Psi^-\rangle}_{m} \end{array} \tag{38}$$

which produces a 2-uniform state of $3 + 2m$ qubit systems. Furthermore, constructions (36) and (39) can be generalized in the same way. That is, we construct 2-uniform states for an odd number of $N = 5 + 2m$ qudits

$$|\psi(5+m,d)\rangle = \sum_{i,j=0}^{d-1} |i,\, j,\, i+j\rangle \underbrace{|\phi_{i,j}\rangle \cdots |\phi_{i,j}\rangle}_{m},$$

and also 2-uniform states for an even number of $N = 6 + 2m$ qudits

$$|\psi(6+m,d)\rangle = \sum_{i,j=0}^{d-1} |i,j,i+j,i+2j\rangle \underbrace{|\phi_{i,j}\rangle \cdots |\phi_{i,j}\rangle}_{m},$$

where d is a prime number. As we described in Eq. (39), when d is a prime power we should consider the set of polynomial representation of the finite sets. For these constructions it is straightforward to check that every reduction to two parties forms a POVM.

We recently learned that QOA composed by six columns exist for any prime number of levels d. By using qudit graph states [8], the following solution can be found [39] for any prime number of levels d:

$$|\mathrm{AME}(6,d)\rangle = \sum_{i_1,i_2,i_3=0}^{d-1} |i_1,i_2,i_3\rangle|\phi_{i_1,i_2,i_3}\rangle, \tag{39}$$

where

$$|\phi_{i_1,i_2,i_3}\rangle = \sum_{i_4,i_5,i_6=0}^{d-1} \omega^{A_{i_1,\ldots,i_6}} |i_4,i_5,i_6\rangle, \tag{40}$$

with $\omega = e^{2\pi i/d}$ and

$$\begin{aligned} A_{i_1,\ldots,i_6} = {} & i_1 i_2 + i_2 i_3 + i_3 i_4 + i_4 i_5 + i_5 i_6 + i_6 i_1 \\ & + i_1 i_3 + i_4 i_6 + i_2 i_5. \end{aligned} \tag{41}$$

Note that these states determine the d^3 entries of three MOQLC of a prime size d. Furthermore, these states also imply the existence of a 3-unitary complex Hadamard matrix of size d^3 whose entries are given by $M_{\mu,\nu} = \omega^{A_{\mu,\nu}}$, where $\mu = d^2 i_1 + d i_2 + i_3$ and $\nu = d^2 i_4 + d i_5 + i_6$, with $\mu,\nu = 0 \ldots d^3 - 1$.

VII. SUMMARY AND CONCLUSIONS

A generalization of classical combinatorial arrangements to quantum mechanics has been established. We introduced the notion of quantum Latin squares (QLS), quantum Latin cubes (QLC), and quantum Latin hypercubes (QLH) and established a suitable notion of orthogonality between them; see Sec. III. We also introduced the notion of quantum orthogonal arrays (QOA) in Sec. IV, that generalizes all the classical and quantum arrangements studied in Secs. II and III. Moreover, we derived quantum Latin arrangements from QOA in the same way as classical Latin arrangements can be obtained from classical OA; see Proposition 3.

Our findings allowed us to realize that a pair of orthogonal quantum Latin arrangements not necessarily implies the existence of two separated arrangements satisfying an orthogonality criteria. Indeed, *orthogonal Latin arrangements can be entangled in the same way as quantum states are entangled*; see for instance Eqs. (19) and (22). This astonishing property is one-to-one related to the fact that columns of QOA can be entangled; see Eqs. (18) and (21). This turned out to be a crucial property in order to reproduce some classes of highly entangled multipartite states, so-called AME states with nonminimal support, for instance the states AME(5,2) and

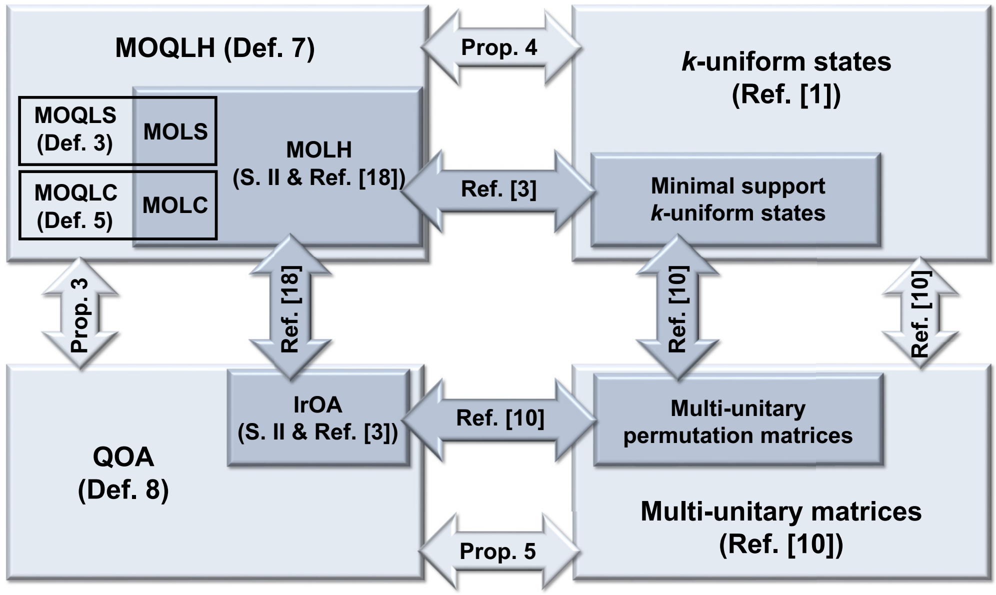


FIG. 3. Connection between the quantum combinatorial arrangements introduced in this work, i.e., mutually orthogonal quantum Latin arrangements (see Sec. III) and quantum orthogonal arrays (see Sec. IV). This also shows the connections existing between our findings, k-uniform states, and multiunitary matrices.

AME(6,2) consisting of five and six qubits; see Eqs. (24) and (25), respectively.

Furthermore, QOA define k-uniform states; see Proposition 2. We demonstrated that k-uniform states constructed from quantum Latin arrangements have high persistency of entanglement, which makes them ideal candidates for quantum information protocols; see Proposition 4. We also established a link between multiunitary matrices and mutually orthogonal Latin arrangements; see Proposition 5.

We constructed three genuinely entangled MOQLS of size d, QOA composed of five columns and an arbitrary number d of internal levels and AME states for five parties with d levels each, for every $d \geqslant 2$; see Proposition 6. This result shows the usefulness of the quantum combinatorial designs introduced in this work.

Figure 3 summarizes the relations existing between the studied concepts and the most relevant results derived in this work. On one hand, we proposed new mathematical tools and described original techniques to construct multipartite quantum states with remarkable properties. On the other hand, we established some further links between problems and objects studied in classical combinatorics and quantum theory. We are tempted to believe that such an approach might be fruitful in future as it can lead to further development of "quantum combinatorics," a branch of mathematics which investigates various arrangements composed of elements of the continuous and connected space of d-dimensional quantum states instead of elements of a discrete set containing d elements.

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[1] A. J. Scott, Multipartite entanglement, quantum-error-correcting codes, and entangling power of quantum evolutions, Phys. Rev. A **69**, 052330 (2004).

[2] L. Arnaud and N. Cerf, Exploring pure quantum states with maximally mixed reductions, Phys. Rev. A **87**, 012319 (2013).

[3] D. Goyeneche and K. Życzkowski, Genuinely multipartite entangled states and orthogonal arrays, Phys. Rev. A **90**, 022316 (2014).

[4] P. Facchi, G. Florio, G. Parisi, and S. Pascazio, Maximally multipartite entangled states, Phys. Rev. A **77**, 060304 (2008).

[5] W. Helwig, W. Cui, J. I. Latorre, A. Riera, and H. K. Lo, Absolute maximal entanglement and quantum secret sharing, Phys. Rev. A **86**, 052335 (2012).

[6] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence, J. High Energy Phys. 06 (2015) 149.

[7] M. Hein, J. Eisert, and H. J. Briegel, Multi-party entanglement in graph states, Phys. Rev. A **69**, 062311 (2004).

[8] W. Helwig, Absolutely maximally entangled qudit graph states, arXiv:1306.2879.

[9] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, Absolutely maximally entangled states, combinatorial designs, and multiunitary matrices, Phys. Rev. A **92**, 032316 (2015).

[10] Z. Raissi and V. Karimipour, Creating maximally entangled states by gluing, Quantum Inf. Process. **16**, 81 (2017).

[11] F. Huber, Existence of absolutely maximally entangled states, Problem 35 in the list of Open Quantum Problems, IQOQI Vienna (May 2017), https://oqp.iqoqi.univie.ac.at/existence-of-absolutely-maximally-entangled-pure-states.

[12] P. Facchi, G. Florio, U. Marzolino, G. Parisi, and S. Pascazio, Classical statistical mechanics approach to multipartite entanglement, J. Phys. A **43**, 225303 (2010).

[13] Z. Raissi, C. Gogolin, A. Riera, and A. Acín, Constructing optimal quantum error correcting codes from absolute maximally entangled states, J. Phys. A **51**, 075301 (2017).

[14] S. Di Martino, P. Facchi, and G. Florio, Feynman graphs and the large dimensional limit of multipartite entanglement, J. Math. Phys. **59**, 012201 (2018).

[15] B. Musto and J. Vicary, Quantum Latin squares and unitary error bases, Quantum Inf. Comput. **16**, 1318 (2016).

[16] B. Musto, Constructing mutually unbiased bases from quantum Latin squares, EPTCS **236**, 108 (2017).

[17] A. S. Hedayat, N. J. A. Sloane, and J. Stufken, *Orthogonal Arrays: Theory and Applications* (Springer-Verlag, New York, 1999).

[18] F. Huber, O. Gühne, and J. Siewert, Absolutely Maximally Entangled States of Seven Qubits Do Not Exist, Phys. Rev. Lett. **118**, 200502 (2017).

[19] F. Huber, C. Eltschka, J. Siewert, and O. Gühne, Bounds on absolutely maximally entangled states from shadow inequalities, and the quantum MacWilliams identity, J. Phys. A **51**, 175301 (2018).

[20] W. Helwig and W. Cui, Absolutely maximally entangled states: Existence and applications, arXiv:1306.2536.

[21] M. Grassl and M. Roetteler, Quantum MDS codes over small fields, in *Proc. IEEE International Symposium on Information Theory (ISIT 2015), Hong Kong, 14-19 June 2015* (IEEE, 2015), pp. 1104-1108.

[22] K. Bush, Orthogonal arrays of index unity, Ann. Math. Statist. **23**, 426 (1952).

[23] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[24] D. Gottesman, Stabilizer codes and quantum error correction, arXiv:quant-ph/9705052.

[25] E. Knill and R. Laflamme, A theory of quantum error correcting codes, Phys. Rev. A **55**, 900 (1997).

[26] A. Higuchi and A. Sudbery, How entangled can two couples get? Phys. Lett. A **273**, 213 (2000).

[27] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Perfect Quantum Error Correction Code, Phys. Rev. Lett. **77**, 198 (1996).

[28] A. Borras, A. R. Plastino, J. Batle, C. Zander, M. Casas, and A. Plastino, J. Phys. A **40**, 13407 (2007).

[29] L. Euler, Recherches sur une nouvelle espece de quarres magiques, in *Verhandelingen uitgegeven door het zeeuwsch Genootschap der Wetenschappen te Vlissingen* (Middelburg, 1782), Vol. 9 pp. 85–239. Also published in Commentationes arithmeticae **2**, 302 (1849). Available online in The Euler Archive, http://eulerarchive.maa.org/pages/E530.html.

[30] H. J. Briegel and R. Raussendorf, Persistent Entanglement in Arrays of Interacting Particles, Phys. Rev. Lett. **86**, 910 (2001).

[31] L. Clarisse, S. Ghosh, S. Severini, and A. Sudbery, Entangling power of permutations, Phys. Rev. A **72**, 012314 (2005).

[32] P. Zanardi, Ch. Zalka, and L. Faoro, Entangling power of quantum evolutions, Phys. Rev. A **62**, 030301 (2000).

[33] P. Zanardi, Entanglement of quantum evolutions, Phys. Rev. A **63**, 040304 (2001).

[34] Z. Puchała (private communication, 2016).

[35] W. Bruzda (private communication, 2017).

[36] A. Lakshminarayan (private communication, 2017).

[37] F. Huber and N. Wyderka, Table of absolutely maximally entangled states, http://www.tp.nt.uni-siegen.de/+fhuber/ame.html.

[38] E. Rains, Nonbinary quantum codes, IEEE Trans. Inf. Theory **45**, 1827 (1999).

[39] F. Huber and N. Wyderka (private communication, 2017).