

Asymptotic quantum algorithm for the Toeplitz systems

Lin-Chun Wan, Chao-Hua Yu, Shi-Jie Pan, Fei Gao,* Qiao-Yan Wen, and Su-Juan Qin

State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China



(Received 31 January 2018; published 15 June 2018)

Solving the Toeplitz systems, which involves finding the vector x such that $T_n x = b$ given an $n \times n$ Toeplitz matrix T_n and a vector b , has a variety of applications in mathematics and engineering. In this paper, we present a quantum algorithm for solving the linear equations of Toeplitz matrices, in which the Toeplitz matrices are generated by discretizing a continuous function. It is shown that our algorithm's complexity is nearly $O(\kappa \text{poly}(\log n))$, where κ and n are the condition number and the dimension of T_n , respectively. This implies our algorithm is exponentially faster than its classical counterpart if $\kappa = O(\text{poly}(\log n))$. Since no assumption on the sparseness of T_n is demanded in our algorithm, it can serve as an example of quantum algorithms for solving nonsparse linear systems.

DOI: [10.1103/PhysRevA.97.062322](https://doi.org/10.1103/PhysRevA.97.062322)

I. INTRODUCTION

Quantum information processing has been shown to be enormously advantageous in preserving security and privacy in communication and information retrieval [1,2], as well as in computing for solving certain problems [3], such as association rules mining [4]. In recent years, since the first example—a quantum algorithm for linear systems of equations (HHL algorithm)—was presented by Harrow *et al.* [5], a number of quantum algorithms for other problems in numerical computation, such as linear regression [6,7], Poisson's equation [8], and the finite element method [9], have been subsequently proposed with significant speedup over their classical counterparts. These works motivate us to design fast quantum algorithms for solving more problems in numerical computation.

One of the most important problems in numerical computation is solving linear equations with a Toeplitz matrix. Solving this linear system has a variety of applications in many areas of science and engineering, such as signal processing [10], time series analysis [11], image restoration problems [12], queueing problems [13], minimum realization problems in control theory [14], and numerical integration problems [15]. In general, the Toeplitz systems are obtained by discretization of continuous problems and the dimension n is related to the grid parameter of the discretization. More specifically, the given Toeplitz matrices T_n are generated by a generating function, f , i.e., the elements of every diagonal of T_n are given by the Fourier coefficients of f . Therefore, the linear systems usually are of very large dimensions so that more efficient algorithms for solving these systems deserve to be explored [16].

In the past decades, many scholars have paid their attention to developing the methods for speeding up the solving of Toeplitz systems. A number of advanced methods have been presented, such as fast direct methods [17], iterative methods

[18], and circulant approximation methods [19]. There is a wonderful treasury of classical algorithms for solving Toeplitz systems. Nevertheless, since the time complexity of these methods is $\Omega(n)$, it is still hard work to tackle Toeplitz systems with very large n on a classical computer.

As of now, some work regarding Toeplitz matrices has been done in the quantum setting. In 2016, Mahalinghe and Wang presented an efficient quantum algorithm for implementing sparse or Fourier-sparse Toeplitz matrices [20]. Whereafter, the algorithm presented in Ref. [21] provided a better way to implement Toeplitz matrices, requiring few resources and without the sparsity assumption on T_n . These two algorithms have some significant applications in physics, mathematics, and engineering-related fields. It should be noted that, although Ref. [21] has shown how to implement $T_n|b\rangle$ by embedding T_n in a larger circulant matrix (a special kind of Toeplitz matrix), as well as how to invert a circulant matrix efficiently, it is easy to see that inverting T_n cannot be realized by this trick. However, as mentioned above, many problems can be transformed into solving Toeplitz systems. That is to say, in these practical applications, what is really relevant is the inverse of such matrices rather than the matrices themselves. This is exactly what we focus on in this paper.

Specifically, we present a quantum algorithm to solve the Toeplitz systems, i.e., finding a quantum state $|x\rangle = \sum_i x_i |i\rangle / \|\sum_i x_i |i\rangle\|$ satisfying the linear equation $T_n x = b$, in which the vector b is given by $|b\rangle = \sum_i b_i |i\rangle / \|\sum_i b_i |i\rangle\|$ and T_n is generated by a continuous function and not limited to sparse. The basic idea of our algorithm is that, due to the fact that the Toeplitz matrices can be well approximated by some easier-to-tackle circulant matrices, our quantum algorithm solves the linear equations of Toeplitz matrices by resorting to solve those of the circulant matrices. In addition, we make full use of the relationship between circulant matrices and the generating function, so that we can directly acquire the eigenvalues of the circulant matrices by computing the corresponding value of the generating function. Thus the performing of the phase estimation which requires higher complexity to reveal corresponding eigenvalues can be avoided. It is shown

*gaof@bupt.edu.cn

that our algorithm is exponentially faster than the classical methods when the generating functions f are strictly positive continuous real-valued functions and the Toeplitz matrices T_n are well-conditioned. We call Toeplitz matrices well-conditioned when $\kappa = O(\text{poly}(\log n))$, where κ is the ratio between the largest eigenvalue and the smallest eigenvalue. Moreover, since the solution is encoded in the final quantum state of our algorithm, it is advantageous to extract some interesting features of the solution and it can also be used as an ingredient in other quantum algorithms.

The rest of this paper is organized as follows. In Sec. II, we review some basic concepts and properties of Toeplitz matrices. And we describe the details of our quantum algorithm in Sec. III. The error and runtime analysis of this algorithm are given in Sec. IV. We then study some special cases and give some discussions in Secs. V and VI, respectively. In the last section, we draw conclusions.

II. A REVIEW ABOUT TOEPLITZ MATRICES

A Toeplitz matrix, T_n , is a matrix of size $n \times n$ whose coefficients along each diagonal are constant. More precisely, a Toeplitz matrix has the form

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \dots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \ddots & \vdots \\ t_2 & t_1 & t_0 & \ddots & t_{-2} \\ \vdots & \ddots & \ddots & \ddots & t_{-1} \\ t_{(n-1)} & \dots & t_2 & t_1 & t_0 \end{pmatrix}, \quad (1)$$

where $t_{k,j} = t_{k-j}$; thus, it can be determined by the sequence $\{t_k\}_{k=-n+1}^{n-1}$ with only $2n - 1$ entries.

As we mentioned above, in many applications, the Toeplitz matrices are obtained by discretization of continuous problems. More explicitly, let $C_{2\pi}^+$ be the set of all 2π -periodic strictly positive continuous real-valued functions defined on $[0, 2\pi]$. For all $f \in C_{2\pi}^+$,

$$t_k = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) e^{-ik\lambda} d\lambda, \quad k = 0, \pm 1, \pm 2, \dots \quad (2)$$

are the Fourier coefficients of f . Then, let $T_n (1 \leq n < \infty)$ be the sequence of Toeplitz matrices whose entries along the k th diagonal are t_k . The function f is called the generating function of the sequence of Toeplitz matrices T_n (see Ref. [22]), and the sequence of matrices T_n is often denoted as $T_n(f)$. We are interested in solving the Toeplitz systems $T_n(f)x = b$.

The properties of Toeplitz matrices are well known and easily derived. We describe a simple version here; for details, see Ref. [23]. The Toeplitz matrices $T_n(f)$ will be Hermitian if f is real-valued function. In fact,

$$\begin{aligned} t_k^* &= \frac{1}{2\pi} \int_0^{2\pi} f^*(\lambda) e^{ik\lambda} d\lambda \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) e^{ik\lambda} d\lambda \\ &= t_{-k}. \end{aligned}$$

Another useful property is that when $T_n(f)$ is Hermitian, let λ_k be the eigenvalues of the Toeplitz matrix $T_n(f)$, then

$$f_{\min} \leq \lambda_k \leq f_{\max},$$

where f_{\min} and f_{\max} represent the smallest value and the largest value of f , respectively. In particular,

$$\begin{aligned} \lim_{n \rightarrow \infty} \max_k \lambda_k &= f_{\max}, \\ \lim_{n \rightarrow \infty} \min_k \lambda_k &= f_{\min}. \end{aligned}$$

Note that the generating function f is strictly positive then the Toeplitz matrix $T_n(f)$ is nonsingular. Another thing worth emphasizing is that in practical applications, we are often given the generating functions f instead of the Toeplitz matrices $T_n(f)$. Typical examples of generating functions can be found in Refs. [10,12,18,22,24].

There is a common special case of Toeplitz matrix where every row of the matrix is a right cyclic shift of the row above it. In this case, the structure becomes

$$C_n = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{(n-1)} \\ c_{(n-1)} & c_0 & c_1 & \dots & c_{(n-2)} \\ c_{(n-2)} & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & c_1 \\ c_1 & \dots & & c_{(n-1)} & c_0 \end{pmatrix}. \quad (3)$$

A matrix of this form is called a *circulant* matrix. The following theorem summarizes the properties regarding eigenvalues and eigenvectors of circulant matrices, and more details can be found in Ref. [23].

Theorem 1 [23]. Every circulant matrix C_n can be diagonalized by the Fourier matrix F_n . That means it has the form $C_n = F_n^\dagger \Lambda_n F_n$, where the entries of F_n are given by

$$[F_n]_{j,k} = \frac{1}{\sqrt{n}} e^{-2\pi ijk/n}, \quad 0 \leq j, k \leq n - 1,$$

and Λ_n is a diagonal matrix whose corresponding eigenvalues are given by

$$\psi_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}, \quad m = 0, 1, \dots, n - 1. \quad (4)$$

Apparently, C_n is normal.

Corollary 1 [23]. Let C and B be $n \times n$ circulant matrices with eigenvalues

$$\psi_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}, \quad \beta_m = \sum_{k=0}^{n-1} b_k e^{-2\pi imk/n},$$

respectively. Then we have the following.

(i) C and B commute and

$$CB = BC = F_n^\dagger \gamma F_n, \quad (5)$$

where $\gamma = \text{diag}(\psi_m \beta_m)$ and CB is also a circulant matrix.

(ii) $C + B$ is a circulant matrix and

$$C + B = F_n^\dagger \Omega F_n, \quad (6)$$

where $\Omega = \text{diag}\{(\psi_m + \beta_m)\}$.

(iii) If $\psi_m \neq 0, m = 0, 1, \dots, n - 1$, then C is nonsingular and

$$C^{-1} = F_n^\dagger \Lambda_n^{-1} F_n. \tag{7}$$

One technique to solve the problem involving Toeplitz matrices is to construct a sequence of circulant matrices which are asymptotically equivalent to the Toeplitz matrices. Obviously the choice of constructing a sequence of circulant matrices to approximate the sequence of Toeplitz matrices is not unique; therefore we need to choose a construction which has the most desirable properties. It will prove useful to adopt the circulant matrices defined in Refs. [22,23]. In particular, define $C_n(f)$ to be the circulant matrix with the top row $(c_0, c_1, \dots, c_{n-1})$, where

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} f(2\pi j/n) e^{2\pi ijk/n}. \tag{8}$$

According to Theorem 1, the eigenvalues of $C_n(f)$ are simply $f(2\pi m/n)$:

$$\begin{aligned} \psi_m &= \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n} \\ &= \sum_{k=0}^{n-1} \left(\frac{1}{n} \sum_{j=0}^{n-1} f(2\pi j/n) e^{2\pi ijk} \right) e^{-2\pi imk} \\ &= \sum_{j=0}^{n-1} f(2\pi j/n) \left\{ \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i(j-m)k} \right\} \\ &= f(2\pi m/n), \quad m = 0, 1, \dots, n - 1, \end{aligned} \tag{9}$$

by using the orthogonality of the complex exponentials [23]. $C_n(f)$ are called associated circulant matrices of $T_n(f)$ [25]. To put it simply, the associated circulant matrices have the form

$$C_n(f) = F_n^\dagger \Lambda_n F_n,$$

where

$$\Lambda_n = \text{diag}\{f(0), f(2\pi/n), \dots, f[2\pi(n-1)/n]\}.$$

It has been observed that in many applications the substitution of $T_n(f)$ with $C_n(f)$ often leads to very useful and dramatic simplification [25]. Our quantum algorithm also follows this idea to achieve exponential speedup over the classical algorithm.

III. QUANTUM ALGORITHM

Now, we design the quantum algorithm to solve the linear system of associated circulant matrices. In particular, from Eqs. (7) and (9), we know $C_n^{-1}(f) = F_n^\dagger \Lambda_n^{-1} F_n$, where Λ_n^{-1} is a diagonal matrix with the corresponding eigenvalue $1/f(2\pi j/n)$. Thus, the essential problem is how to implement Λ_n^{-1} . It is a natural candidate for applying the HHL algorithm [5]. Surprisingly, because of the distinctive structure of the associated circulant matrices, we can bypass the phase estimation.

For the Toeplitz system, b_i can usually be regarded as some function value of a continuous function. There are some

efficiently implementable schemes that have been presented to provide $|b\rangle$ when $\sum_{i=i_1}^{i_2} |b_i|^2$ are efficiently computable [26,27], or one can use the quantum random access memory (qRAM) [28] to load b_i and then perform the conditional rotation and postselect to provide $|b\rangle$ when b_i are close to uniform. In general, preparing an arbitrary initial quantum state is really challenging and deserves specialized study. For simplicity in the subsequent comparison, we follow the assumption of most of the previous quantum algorithms that the preparation of the initial state $|b\rangle$ is efficient; i.e., we assume that there exists a unitary U_b which can produce the state $|b\rangle$ in time $O(\text{poly}(\log n))$.

To extract the eigenvalues, we assume there is an oracle that accesses the values of the generating function f . Specifically it allows us to perform the following map:

$$\sum_{j=0}^{n-1} b_j |j\rangle \xrightarrow{\text{oracle}} \sum_{j=0}^{n-1} b_j |j\rangle |f(2\pi j/n)\rangle.$$

We don't consider the computation complexity of the oracle since the generating function f is always efficiently computable. In fact, it can be implemented efficiently by using the quantum circuit model. Section 3.2.5 of Ref. [29] shows that given a classical circuit for computing f there is a quantum circuit of comparable efficiency which computes the transformation U_f on a quantum computer. Some quantum algorithms and circuits of fundamental numerical functions proposed by Bhaskar *et al.* [30] may be helpful to implement U_f . For our purposes, we can regard it as a black box.

The specific process of our quantum algorithm is summarized by the following five steps.

(i) Apply Fourier transform on $|b\rangle$ and denote the outputted state by $|b'\rangle$.

(ii) Decompose $|b'\rangle$ in the computational basis, i.e., $|b'\rangle = \sum_{j=0}^{n-1} b_j |j\rangle$, and use the oracle to obtain

$$\sum_{j=0}^{n-1} b_j |j\rangle |f(2\pi j/n)\rangle.$$

The query complexity to prepare this state is $O(1)$.

(iii) Add a qubit and perform a controlled-rotation on $|f(2\pi j/n)\rangle$, yielding

$$\sum_{j=0}^{n-1} b_j |j\rangle |f(2\pi j/n)\rangle \left(\sqrt{1 - \frac{m^2}{f^2(2\pi j/n)}} |0\rangle + \frac{m}{f(2\pi j/n)} |1\rangle \right),$$

where m is an appropriate constant and $m \leq \min_j |\psi_j|$, and ψ_j are the eigenvalues of $C_n(f)$.

(iv) Uncompute the second qubit and use the amplitude amplification [31] on the last register to obtain $|1\rangle$; hence we will have the following state with higher probability:

$$\sqrt{\frac{1}{\sum_{j=0}^{n-1} m^2 |b_j|^2 / |f(2\pi j/n)|^2}} \sum_{j=0}^{n-1} b_j \frac{m}{f(2\pi j/n)} |j\rangle,$$

which is proportional to $\Lambda^{-1} |b'\rangle = \sum_{j=0}^{n-1} b_j / f(2\pi j/n) |j\rangle$ up to normalization and denoted by $|b^*\rangle$.

(v) Perform the inverse Fourier transform and get

$$|x^*\rangle = F_n^\dagger |b^*\rangle.$$

Here $|x^*\rangle$ is the quantum state that we desired, and in the following section we prove it is close to the normalized solution $|x\rangle = \frac{T_n^{-1}(f)|b\rangle}{\|T_n^{-1}(f)|b\rangle\|}$ of the Toeplitz system.

IV. ERROR ANALYSIS AND RUNTIME

In this section we show that the error of the final state of our algorithm to the ideal state can always be small enough. Furthermore, we demonstrate two significative corollaries to ascertain the magnitude of error ϵ for certain cases. We also perform analysis on the time complexity of our quantum algorithm and show its runtime advantage over the classical algorithm.

As our main idea is substituting $T_n(f)$ with $C_n(f)$, in order to make such a substitution meaningful, the sequence of Toeplitz matrices needs to converge to their associated circulant matrices. We define convergence of the matrices as the following theorem.

Theorem 2. Let $T_n(f)$ be a sequence of Toeplitz matrices generated by a strictly positive continuous real-valued function, $C_n(f)$ is the associated circulant matrices defined in Eq. (8), then

$$\lim_{n \rightarrow \infty} \frac{\|T_n(f) - C_n(f)\|_F}{\|T_n(f)\|_F} = 0, \quad (10)$$

where $\|A\|_F$ denotes the Frobenius norm of a matrix A .

Proof. See Appendix A.

Let

$$x^* = C_n^{-1}(f)b, \quad x = T_n^{-1}(f)b,$$

$$|x^*\rangle = \frac{C_n^{-1}(f)|b\rangle}{\|C_n^{-1}(f)|b\rangle\|}, \quad |x\rangle = \frac{T_n^{-1}(f)|b\rangle}{\|T_n^{-1}(f)|b\rangle\|},$$

where $\|x\| = \sqrt{x^H x}$ denotes the 2-norm of a vector x .

Recalling Theorem 2, for a given ϵ , we can always choose an n large enough, so that

$$\frac{\|T_n(f) - C_n(f)\|_F}{\|T_n(f)\|_F} \leq \epsilon.$$

Then, from the relevant conclusion of Ref. [32, Sec. 5.8], we know that

$$\frac{\|x^* - x\|}{\|x\|} = \frac{\|C_n^{-1}(f)b - T_n^{-1}(f)b\|}{\|T_n^{-1}(f)b\|} \leq \frac{\epsilon\kappa}{1 - \epsilon\kappa}, \quad (11)$$

where κ is the condition number of $T_n(f)$, and

$$\begin{aligned} & \| |x^*\rangle - |x\rangle \| \\ &= \left\| \frac{C^{-1}|b\rangle}{\|C^{-1}|b\rangle\|} - \frac{T^{-1}|b\rangle}{\|T^{-1}|b\rangle\|} \right\| \\ &= \left\| \frac{(\|T^{-1}|b\rangle\| - \|C^{-1}|b\rangle\|)C^{-1}|b\rangle}{\|C^{-1}|b\rangle\| \|T^{-1}|b\rangle\|} - \frac{T^{-1}|b\rangle - C^{-1}|b\rangle}{\|T^{-1}|b\rangle\|} \right\| \\ &\leq \frac{|\|T^{-1}|b\rangle\| - \|C^{-1}|b\rangle\||}{\|T^{-1}|b\rangle\|} + \frac{\|T^{-1}|b\rangle - C^{-1}|b\rangle\|}{\|T^{-1}|b\rangle\|} \\ &\leq 2 \frac{\|T^{-1}|b\rangle - C^{-1}|b\rangle\|}{\|T^{-1}|b\rangle\|} \\ &= 2 \frac{\|T^{-1}b - C^{-1}b\|}{\|T^{-1}b\|} \leq \frac{2\epsilon\kappa}{1 - \epsilon\kappa}. \end{aligned} \quad (12)$$

For simplicity, we write $C_n^{-1}(f)$ and $T_n^{-1}(f)$ as C^{-1} and T^{-1} , respectively, in this inequality. Apparently, the output of our algorithm can approximate the normalized solution of a well-conditioned Toeplitz system with desired precision as long as we choose an n large enough.

The above process proves the correctness of the algorithm in a certain sense. Naturally, we also need to seek ways to provide error-bound estimates. Depending on different properties of the associated sequence $\{t_k\}$, we get different error estimates as shown in the following corollaries.

Corollary 2. Let $T_n(f)$ be a sequence of well-conditioned Toeplitz matrices generated by a strictly positive continuous real-valued function. If the associated sequence $\{t_k\}$ satisfies $|t_k| \leq M/k$, where $k = 1, 2, \dots$ and M is a constant, then we can acquire a normalized solution of the corresponding Toeplitz system with error $\epsilon = O(\frac{\text{poly}(\log n)}{\sqrt{n}})$.

Proof. See Appendix B.

The error of solving many Toeplitz systems can be estimated conveniently from this corollary, such as Toeplitz systems generated by primitive functions, since it is well known that the magnitude of Fourier coefficients of most of primitive functions is $o(\frac{1}{k})$, where o denotes the higher-order infinitely small quantity, i.e.,

$$\lim_{k \rightarrow \infty} \frac{|t_k|}{1/k} = 0.$$

This implies that there must be an \tilde{M} such that

$$\frac{|t_k|}{1/k} \leq \tilde{M} \Rightarrow |t_k| \leq \tilde{M}/k.$$

And the magnitude of Fourier coefficients of a more general generating function can be estimated by combining the Fourier expansion of primitive functions. Thus the final error can be estimated from this corollary or by performing a similar analysis. Moreover, some useful judgment theorems about the magnitude of Fourier coefficients of functions are listed in Ref. [33]. These theorems show that the magnitude of Fourier coefficients of many functions (such as absolutely continuous functions and bounded variation functions) satisfy the condition as stated in Corollary 2; thus the final error will also be bounded by $O(\frac{\text{poly}(\log n)}{\sqrt{n}})$.

Corollary 3. Let $T_n(f)$ be a sequence of Toeplitz matrices generated by a strictly positive continuous real-valued function. If the associated sequence $\{t_k\}$ satisfies $\sum_{k=-\infty}^{\infty} |kt_k| < \infty$ and the spectrum norm $\|T_n\| \leq 1$, for a vector with the form

$$b = (0, \dots, 0, b_{-L}, \dots, b_0, \dots, b_L, 0, \dots, 0)$$

we can acquire a normalized solution of the corresponding Toeplitz system with error $\epsilon = O(1/\sqrt{n})$.

Proof. See Appendix C.

This corollary is based on the results of literature [25]. It extends the convergence theorem to a form in which b has only a finite number of nonzero terms. Moreover, a large class of communication receiver design problems involving a similar form can be solved efficiently.

Based on these corollaries, we can conveniently estimate the scale of ϵ for a specific Toeplitz system. This seems to be different from the normal way we design and analyze algorithms of which the accuracy ϵ and the dimension n are given beforehand. We would like to emphasize that the Toeplitz

systems are often given by a continuous function, and we are required to design an algorithm to solve the problem with the accuracy ϵ . In the actual operation, we first discretize the continuous function and then solve this discrete system. Thus, the dimension n is not fixed but operational, and it depends on the desired accuracy and can be deduced by the above corollaries or by performing a similar analysis.

Reviewing our algorithm, the state $|b\rangle$ can be produced in time $O(\text{poly}(\log n))$, the (inverse) quantum Fourier transform takes time $O(\log^2 n)$, and the cost of invoking the oracle is $O(1)$ while the computation complexity of the oracle can be ignored. Then, we consider the success probability of the postselection in the process of implementing Λ_n^{-1} . Since $m \leq \min_j |\psi_j|$, for choosing $m = f_{\min}$, the success probability is $\Omega(1/\mu^2)$, where $\mu = f_{\max}/f_{\min}$, and $O(\mu^2)$ measurements are required. Using amplitude amplification, we need only repeat $O(\mu)$ times. Putting these all together, our quantum algorithm takes time $O(\mu \text{poly}(\log n))$. Noting that the condition number κ approximates f_{\max}/f_{\min} when n is getting large, the time complexity of our algorithm is also nearly $O(\kappa \text{poly}(\log n))$.

Replacing Toeplitz matrices with their associated circulant matrices has been widely used to solve the problems involving the inverses of Toeplitz matrices. It takes two steps to complete this process by classical algorithms: (i) compute the top row of the circulant matrix, and (ii) solve the linear system of circulant matrices. Both of these steps can be performed using the classical fast algorithm. In particular, the top row can be computed in $O(n)$ [23]. And according to Theorem 1, the linear systems of circulant matrices can be solved in time $O(n \log n)$ using the fast Fourier transform. Therefore, our quantum algorithm is exponentially faster than the corresponding classical algorithm when the condition numbers of the Toeplitz matrices are $O(\text{poly}(\log n))$. In addition, there are also some exact classical algorithms (n is fixed, such as in the iterative method) for solving Toeplitz systems. Since the exact algorithm and the asymptotic algorithm have different measurements of the complexity, a direct comparison of exact classical algorithms and the asymptotic quantum algorithm is not appropriate. We are looking forward to more research on the exact quantum algorithm for solving Toeplitz systems.

V. SPECIAL CASE

Sometimes, there might be some problems in practical situations where we only know the sequence of Toeplitz matrices T_n but not the generating function f . Fortunately, the problems will often begin with Toeplitz matrices in the Wiener class [23]. A sequence of Toeplitz matrices for which the sequence $\{t_k\}$ is absolutely summable is said to be in the Wiener class, i.e., the infinite sequence $\{t_k; k = \dots, -2, -1, 0, 1, 2, \dots\}$ which defines the matrices T_n satisfies

$$\sum_{k=-\infty}^{\infty} |t_k| < \infty. \tag{13}$$

As mentioned in Ref. [23], one natural idea for estimating the eigenvalues is to approximate the generating function by

$$\hat{f}_n(\lambda) = \sum_{k=-(n-1)}^{n-1} t_k e^{ik\lambda}, \quad \lambda \in [0, 2\pi]. \tag{14}$$

On the one hand, the function $\hat{f}_n(\lambda)$ defines a circulant matrix sequence $C_n(\hat{f}_n)$ by Eq. (8). Note that, since the sequences $\{t_k\}$ are absolutely summable, they are also square summable:

$$\sum_{k=-\infty}^{\infty} |t_k|^2 \leq \left\{ \sum_{k=-\infty}^{\infty} |t_k| \right\}^2 < \infty.$$

Thus, according to the proof in Appendix A, it can be seen that the circulant matrices $C_n(\hat{f}_n)$ converge to $T_n(f)$ in the form (10). Based on this convergence theory, our quantum algorithm is still feasible.

On the other hand, the eigenvalues of circulant matrices $C_n(\hat{f}_n)$ are

$$\begin{aligned} \hat{f}_n(2\pi j/n) &= \sum_{k=-(n-1)}^{n-1} t_k e^{2\pi ijk/n} \\ &= \sum_{k=0}^{n-1} t_k e^{2\pi ijk/n} + \sum_{k=0}^{n-1} t_{-k} e^{-2\pi ijk/n} - t_0, \end{aligned}$$

where $j = 0, 1, \dots, n - 1$. Apparently, they can be seen as the results of performing discrete Fourier transformation on the sequences $\{t_k\}$ and then subtracting an extra t_0 .

In order to solve these Toeplitz systems, we first rescale the matrices T_n by the factor f_{\max} , such that $\lambda_k \in [1/\mu, 1]$. Then we use the algorithm presented in Ref. [34] to perform Fourier transformation and encode the Fourier coefficients in the computational basis. More formally, the quantum Fourier transform in the computational basis (QFTC) can perform the transformation

$$|j\rangle \xrightarrow{\text{QFTC}} |j\rangle|y_j\rangle,$$

where $y_j = \sum_{k=0}^{n-1} t_k e^{2\pi ijk/n}$ in this context.

Since the $|j\rangle$ is only used to control the application of quantum operators acting on other registers, we can do the similar operations in the additional registers and get

$$\sum_{j=0}^{n-1} b_j |j\rangle|0\rangle|0\rangle|t_0\rangle \rightarrow \sum_{j=0}^{n-1} b_j |j\rangle|y_j\rangle|y'_j\rangle|t_0\rangle,$$

where $y'_j = \sum_{k=0}^{n-1} t_{-k} e^{-2\pi ijk/n}$. Then, we calculate $y_j + y'_j - t_0$ by the quantum adder [35] and encode the result in another register:

$$\sum_{j=0}^{n-1} b_j |j\rangle|y_j\rangle|y'_j\rangle|t_0\rangle|\hat{f}_n(2\pi j/n)\rangle.$$

Finally, we uncompute the ancillas to obtain

$$\sum_{j=0}^{n-1} b_j |j\rangle|\text{Anc}_j\rangle|\hat{f}_n(2\pi j/n)\rangle \rightarrow \sum_{j=0}^{n-1} b_j |j\rangle|\hat{f}_n(2\pi j/n)\rangle.$$

The above process actually completes step (ii) of our algorithm, i.e.,

$$\sum_{j=0}^{n-1} b_j |j\rangle \xrightarrow{\text{oracle}} \sum_{j=0}^{n-1} b_j |j\rangle|\hat{f}_n(2\pi j/n)\rangle.$$

Proceeding to the next steps of the algorithm, where $m = O(1/\mu)$, we can get a quantum state approximating the solution of the Toeplitz system.

It is worth noting that the algorithm QFTC requires an oracle O_t , where $O_t|0\rangle = \sum_{k=0}^{n-1} t_k|k\rangle$. The oracle can be efficiently implemented if $\{t_k\}_{k=0}^{n-1}$ is efficiently computable or using the qRAM [28]. And the QFTC can be performed to accuracy ϵ with fidelity $1 - \delta$ using $O[\log^2 n / (\delta\epsilon)]$ one- or two-qubit gates. According to the error analysis in Ref. [5], taking the error as $O(\epsilon_0/\mu)$ in computing $\hat{f}_n(2\pi j/n)$ induces a final error ϵ_0 of $|x^*$. Thus, employing the QFTC as a subroutine to extract the eigenvalues, the complexity of our quantum algorithm is nearly $O[\kappa^2 \text{poly}(\log n) / (\delta\epsilon_0)]$. This suggests that our algorithm is exponentially fast when $1/\epsilon_0, \kappa = O(\text{poly}(\log n))$.

VI. DISCUSSION

We notice that some meaningful results about the asymptotic equivalence of Toeplitz matrices and the associated circulant matrices have been presented recently [36]. These results establish the individual asymptotic convergence of the eigenvalues between T_n and C_n while Theorem 2 of this paper actually characterizes the certain collective asymptotic behaviors of the eigenvalues. The individual asymptotic equivalence seems to be stronger than the collective asymptotic equivalence. We describe these results here to demonstrate that our algorithm might achieve a faster convergence rate if the sequence $\{t_k\}$ satisfies certain conditions.

Theorem 3 [36]. Suppose that the sequence $\{t_k\}$ is absolutely summable and C_n are the associated circulant matrices of the Toeplitz matrices T_n . Then

$$\lim_{n \rightarrow \infty} \max_{l \in n} |\lambda_l(T_n) - \lambda_{\rho(l)}(C_n)| = 0, \quad (15)$$

where $\lambda_l(T_n)$ are the eigenvalues of T_n satisfying $\lambda_0(T_n) \geq \dots \geq \lambda_{n-1}(T_n)$, and $\lambda_{\rho(l)}(C_n)$ are the eigenvalues of C_n satisfying $\lambda_{\rho(0)}(C_n) \geq \dots \geq \lambda_{\rho(n-1)}(C_n)$.

Theorem 4 [36]. Suppose that the $t_k = 0$ for all $|k| > r$, i.e., T_n is a band Toeplitz matrix when $n > r$. Then

$$\max_{l \in n} |\lambda_l(T_n) - \lambda_{\rho(l)}(C_n)| = O\left(\frac{1}{n}\right) \quad (16)$$

as $n \rightarrow \infty$.

In addition, because the kernel idea of our algorithm is adopting associated circulant matrices to substitute the Toeplitz matrices, it must be intractable to solve the ill-conditioned Toeplitz system. Evidently, the error of the solution may not be controlled from Eq. (11) when the condition number of T_n is unbounded. Fortunately, in some cases, when $|b\rangle$ is in the well-conditioned part of T_n (i.e., the subspace spanned by the eigenvectors with large eigenvalues), we can also implement the inversion. The corresponding process has been demonstrated in detail in Ref. [5]. Another way to handle ill-conditioned Toeplitz systems is to precondition the Toeplitz matrices. A number of preconditioners developed for ill-conditioned Toeplitz systems have been presented, including band-Toeplitz preconditioners [37] and circulant preconditioners [38]. Since the product of two circulant matrices is a circulant matrix, and the product of a circulant matrix times a vector is available by our quantum algorithm with a little change, our algorithm can run much faster with a suitable circulant preconditioner even for ill-conditioned Toeplitz systems. Moreover, how to establish the quantum version of these preconditioners is an interesting problem, and that is our next work.

VII. CONCLUSION

Solving the Toeplitz systems plays a pivotal role in many areas of science and engineering. We have addressed this problem in the quantum settings and proposed an efficient quantum algorithm to solve it. Taking advantages of quantum computation and the structure of Toeplitz matrices, our algorithm achieves exponential speedup over classical algorithms for the well-conditioned Toeplitz matrices.

Besides solving linear equations with a Toeplitz matrix, our quantum algorithm can deal with many other problems regarding the approximation of a Toeplitz matrix by its associated circulant matrix. In fact, many different convergence forms, such as the weak convergence form [23] and the finite-term quadratic form [25], of approximation between Toeplitz matrices and their associated circulant matrices have been exploited for different applications. Although the matrix convergence forms are different, similar to our algorithm, all of these methods adopt the idea of substituting the Toeplitz matrices with their associated circulant matrices to solve corresponding problems. Therefore, the common methodology makes it possible for our quantum algorithm to be successfully applied to these applications.

Solving linear systems of equations is a fundamental problem that arises frequently in science and engineering. The quantum algorithms for solving sparse linear systems have been well studied [5,39,40]. More recently, Wossnig *et al.* [41] presented a quantum linear system algorithm for dense matrices based on a new data structure which prepares quantum states corresponding to the rows and the vector of Euclidean norms of the rows of the matrices. The algorithm achieved a polynomial improvement over known quantum linear system algorithms when the spectral norm of the dense matrix was bounded by a constant. Simply applying this algorithm to solve the Toeplitz systems gives only a polynomial improvement. Thus, designing efficient quantum algorithms for solving nonsparse linear systems with certain special structures remains a crucial challenge. Our work is a significant addition to this direction. It is an interesting open question if one can achieve an exponential improvement for solving Toeplitz systems in the model of Ref. [41] or given a black-box access to the matrix elements.

Moreover, Ye and Lim pointed out that every matrix is a product of Toeplitz matrices [42], and their conclusion that every $n \times n$ matrix can be decomposed into $\lfloor n/2 \rfloor + 1$ Toeplitz matrices is exciting. That is, if a large nonsparse matrix A has a known Toeplitz decomposition, one can solve the corresponding linear systems with lower time complexity by a high-efficiency quantum algorithm for Toeplitz systems (though we still do not know how to compute Toeplitz decompositions efficiently). It should be noted that our algorithm is not suitable for this method because of the asymptotic equivalence, but this method opens up a new horizon in solving general linear equations and deserves further investigation.

ACKNOWLEDGMENTS

We would like to thank the anonymous referees for their helpful comments. This work is supported by the

National Natural Science Foundation of China (Grants No. 61672110 and No. 61671082).

APPENDIX A: PROOF OF THEOREM 2

In this appendix, we prove the conclusion in Theorem 2. For simplicity, we write $\|A\|_F$ as $|A|$ in the following processes. We first give some involved theories.

Lemma 1. The Frobenius norm of an $n \times n$ matrix $A = [a_{k,j}]$ is equal to

$$\begin{aligned} \|A\|_F &= \left(\sum_{k=0}^{n-1} \sum_{j=0}^{n-1} |a_{k,j}|^2 \right)^{1/2} \\ &= [\text{tr}(A^*A)]^{1/2} = \left[\sum_{k=0}^{n-1} \lambda_k(A^*A) \right]^{1/2}. \end{aligned} \tag{A1}$$

Lemma 2. Let A be a matrix with eigenvalues λ_k , then

$$\sum_{k=0}^{n-1} \lambda_k(A^*A) \geq \sum_{k=0}^{n-1} \lambda_k^2 \tag{A2}$$

with equality if and only if A is normal.

Theorem 5 (Parseval's identity). Let $f(\lambda)$ be a function that is square integrable on $[0, 2\pi]$; then the sum of the squares of the Fourier coefficients of the function is equal to the integral of the square of the function:

$$\sum_{-\infty}^{\infty} |t_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda)^2 d\lambda.$$

Consider the truncated Fourier series

$$\hat{f}_n(\lambda) = \sum_{k=-(n-1)}^{n-1} t_k e^{ik\lambda}, \quad \lambda \in [0, 2\pi]. \tag{A3}$$

After some simple analysis and calculation, we know that

$$\frac{1}{2\pi} \int_0^{2\pi} f(\lambda) \hat{f}_n(\lambda) d\lambda = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}_n^2(\lambda) d\lambda = \sum_{k=-(n-1)}^{n-1} |t_k|^2.$$

Therefore

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} [f(\lambda) - \hat{f}_n(\lambda)]^2 d\lambda \\ &= \frac{1}{2\pi} \left\{ \int_0^{2\pi} f^2(\lambda) d\lambda - 2 \int_0^{2\pi} f(\lambda) \hat{f}_n(\lambda) d\lambda + \int_0^{2\pi} \hat{f}_n^2(\lambda) d\lambda \right\} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f^2(\lambda) d\lambda - \sum_{k=-(n-1)}^{n-1} |t_k|^2. \end{aligned}$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} [f(\lambda) - \hat{f}_n(\lambda)]^2 d\lambda \geq 0,$$

it follows that

$$\sum_{k=-(n-1)}^{n-1} |t_k|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} f^2(\lambda) d\lambda.$$

This means the sequence $\{t_k\}$ is square summable as $n \rightarrow \infty$. Thus given ϵ , there is a single N , such that

$$\sum_{k=-\infty}^{-n} |t_k|^2 + \sum_{k=n}^{\infty} |t_k|^2 \leq \epsilon \quad \text{if } n \geq N.$$

Because the generating function is a continuous function on $[0, 2\pi]$, it is square integrable. Using Parseval's identity, when $n \geq N$, we have

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} [f(\lambda) - \hat{f}_n(\lambda)]^2 d\lambda \\ &= \frac{1}{2\pi} \int_0^{2\pi} f^2(\lambda) d\lambda - \sum_{k=-(n-1)}^{n-1} |t_k|^2 \\ &= \sum_{-\infty}^{\infty} |t_k|^2 - \sum_{k=-(n-1)}^{n-1} |t_k|^2 \\ &= \sum_{k=-\infty}^{-n} |t_k|^2 + \sum_{k=n}^{\infty} |t_k|^2 \\ &\leq \epsilon. \end{aligned}$$

Similar to the proof in Ref. [23], we define a circulant matrix $C_n(\hat{f}_n)$ which has the top row $(\hat{c}_0, \hat{c}_1, \dots, \hat{c}_{n-1})$, where

$$\hat{c}_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_n(2\pi j/n) e^{2\pi ijk/n}. \tag{A4}$$

Since $C_n(f) - C_n(\hat{f}_n)$ is a circulant matrix and it is a normal matrix (Corollary 1), we know from Eqs. (A1) and (A2) that

$$|C_n(f) - C_n(\hat{f}_n)|^2 = \sum_{k=0}^{n-1} |f(2\pi k/n) - \hat{f}_n(2\pi k/n)|^2,$$

and hence for $n \geq N$

$$\begin{aligned} &\frac{|C_n(f) - C_n(\hat{f}_n)|^2}{|T_n(f)|^2} \\ &= \frac{\sum_{k=0}^{n-1} |f(2\pi k/n) - \hat{f}_n(2\pi k/n)|^2}{\sum_{k=-(n-1)}^{n-1} (n - |k|) |t_k|^2} \\ &\leq \frac{\int_0^{2\pi} [f(\lambda) - \hat{f}_n(\lambda)]^2 d\lambda}{\sum_{k=-(n-1)}^{n-1} (n - |k|) |t_k|^2} \\ &\leq \epsilon. \end{aligned} \tag{A5}$$

From the result of Ref. [23],

$$C_n(\hat{f}_n) - T_n(f) = \begin{pmatrix} 0 & t_{n-1} & t_{n-2} & \dots & t_1 \\ t_{-(n-1)} & 0 & t_{n-1} & & \\ t_{-(n-2)} & t_{-(n-1)} & 0 & & \vdots \\ \vdots & & & \ddots & \\ t_{-1} & & \dots & & 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned}
 & \frac{|C_n(\hat{f}_n) - T_n(f)|^2}{|T_n(f)|^2} \\
 &= \frac{\sum_{k=-(n-1)}^{n-1} |k||t_k|^2}{\sum_{k=-(n-1)}^{n-1} (n - |k|)|t_k|^2} \\
 &= \frac{\sum_{k=-N}^N |k||t_k|^2}{\sum_{k=-(n-1)}^{n-1} (n - |k|)|t_k|^2} + \frac{\sum_{k=-(n-1)}^{-(N+1)} |k||t_k|^2 + \sum_{k=N+1}^{n-1} |k||t_k|^2}{\sum_{k=-(n-1)}^{n-1} (n - |k|)|t_k|^2} \\
 &\leq \frac{\sum_{k=-N}^N |k||t_k|^2}{\sum_{k=-(n-1)}^{n-1} (n - |k|)|t_k|^2} + \frac{\sum_{k=-(n-1)}^{-(N+1)} |k||t_k|^2 + \sum_{k=N+1}^{n-1} |k||t_k|^2}{n|t_0|^2} \\
 &\leq \frac{\sum_{k=-N}^N |k||t_k|^2}{\sum_{k=-N}^N (n - |k|)|t_k|^2} + \frac{\sum_{k=-\infty}^{-(N+1)} |t_k|^2 + \sum_{k=N+1}^{\infty} |t_k|^2}{|t_0|^2} \\
 &\leq \frac{N}{n - N} + O(1)\epsilon.
 \end{aligned} \tag{A6}$$

Since ϵ is arbitrary,

$$\begin{aligned}
 & \frac{|C_n(f) - T_n(f)|}{|T_n(f)|} \\
 &\leq \left(\frac{|C_n(f) - C_n(\hat{f}_n)|}{|T_n(f)|} + \frac{|C_n(\hat{f}_n) - T_n(f)|}{|T_n(f)|} \right) \\
 &\leq \sqrt{\frac{N}{n - N}}.
 \end{aligned} \tag{A7}$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{|C_n(f) - T_n(f)|}{|T_n(f)|} = 0. \tag{A8}$$

APPENDIX B: PROOF OF COROLLARY 2

In this appendix, we analyze the magnitude of ϵ , when

$$|t_k| \leq M/k, \quad k = 1, 2, \dots,$$

M is a constant. From Eqs. (A5), (A6), and (A7),

$$\begin{aligned}
 & \frac{\|C_n(f) - T_n(f)\|_F}{\|T_n(f)\|_F} \\
 &\leq \frac{\|C_n(f) - C_n(\hat{f}_n)\|_F + \|C_n(\hat{f}_n) - T_n(f)\|_F}{\|T_n(f)\|_F} \\
 &\leq \frac{\sqrt{\sum_{k=-\infty}^{-n} |t_k|^2} + \sqrt{\sum_{k=n}^{\infty} |t_k|^2} + \sqrt{\sum_{k=-(n-1)}^{n-1} |k||t_k|^2}}{\sqrt{\sum_{k=-(n-1)}^{n-1} (n - |k|)|t_k|^2}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sqrt{2 \sum_{k=n}^{\infty} |t_k|^2} + \sqrt{2 \sum_{k=0}^{n-1} k|t_k|^2}}{\sqrt{nt_0^2 + 2 \sum_{k=1}^{n-1} (n - k)|t_k|^2}} \\
 &\leq \frac{\sqrt{\frac{\pi^2 M^2}{3}} + \sqrt{\frac{2M^2}{1} + \frac{2M^2}{2} + \dots + \frac{2M^2}{n-1}}}{\sqrt{nt_0^2}} \\
 &= O\left(\sqrt{\frac{\ln n}{n}}\right),
 \end{aligned}$$

where the final two steps use the fact

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n\right) = \gamma, \\
 & \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6},
 \end{aligned}$$

and γ is a Euler-Mascheroni constant. In view of Eq. (12), for the well-conditioned Toeplitz matrices, the magnitude of the final error is

$$O\left(\frac{\text{poly}(\log n)}{\sqrt{n}}\right),$$

with the consideration of the equivalency of $\ln n$ and $\log n$ in the notation O .

APPENDIX C: PROOF OF COROLLARY 3

The corollary is proven from following theorem.

Theorem 6 [25]. Let T_n be a family of Toeplitz Hermitian matrices associated with the sequence $\{t_k\}$, and let $F(\lambda)$ be the discrete-time Fourier transform of $\{t_k\}$. If $|F(\lambda)| \neq 0$ for $\lambda \in [0, 2\pi]$ and $\sum_{k=-\infty}^{\infty} |kt_k| < \infty$, for a vector with the form

$$x = (0, \dots, 0, x_{-L}, \dots, x_0, \dots, x_L, 0, \dots, 0),$$

the quadratic form is bounded by

$$\frac{\|(T_n^{-1} - C_n^{-1})x\|}{\|x\|} \leq O(1/\sqrt{n}).$$

It is easy to verify that the conditions of the theorem are met when the generating function is a strictly positive continuous real-valued function. Besides, because of $\lambda_{\max}(T_n) \leq 1$,

$$\frac{\|(T_n^{-1} - C_n^{-1})x\|}{\|T_n^{-1}x\|} \leq \frac{\|(T_n^{-1} - C_n^{-1})x\|}{\|x\|} \leq O(1/\sqrt{n}).$$

Thus, the magnitude of the final error is $O(1/\sqrt{n})$.

- [1] C. H. Bennett and G. Brassard, *Theor. Comput. Sci.* **560**, 7 (2014).
- [2] C.-Y. Wei, X.-Q. Cai, B. Liu, T. Wang, and F. Gao, *IEEE Trans. Comput.* **67**, 2 (2018).
- [3] A. Montanaro, *npj Quantum Inf.* **2**, 15023 (2016).
- [4] C.-H. Yu, F. Gao, Q.-L. Wang, and Q.-Y. Wen, *Phys. Rev. A* **94**, 042311 (2016).
- [5] A. W. Harrow, A. Hassidim, and S. Lloyd, *Phys. Rev. Lett.* **103**, 150502 (2009).
- [6] N. Wiebe, D. Braun, and S. Lloyd, *Phys. Rev. Lett.* **109**, 050505 (2012).
- [7] C.-H. Yu, F. Gao, and Q.-Y. Wen, [arXiv:1707.09524](https://arxiv.org/abs/1707.09524).
- [8] Y. Cao, A. Papageorgiou, I. Petras, J. Traub, and S. Kais, *New J. Phys.* **15**, 013021 (2013).
- [9] A. Montanaro and S. Pallister, *Phys. Rev. A* **93**, 032324 (2016).
- [10] C. Chui and A. Chan, *IEEE Trans. Acoust., Speech, Signal Process.* **30**, 18 (1982).
- [11] M. Ahmadi, M. Azimi-Sadjadi, R. Gorgui-Naguib, R. King, and A. Kwabwe, *Digital Filtering in One and Two Dimensions: Design and Applications* (Springer Science & Business Media, 2013).
- [12] A. K. Jain, *Fundamentals of Digital Image Processing* (Prentice-Hall, Englewood Cliffs, NJ, 1989).
- [13] T. Oda, *IEEE Trans. Commun.* **39**, 737 (1991).
- [14] J. R. Bunch, *SIAM J. Sci. Stat. Comput.* **6**, 349 (1985).
- [15] D. S. Gilliam, C. F. Martin, and J. R. Lund, in *26th IEEE Conference on Decision and Control, 1987*, Vol. 26 (IEEE, New York, 1987), pp. 975–976.
- [16] S. S. Capizzano, *SIAM J. Matrix Anal. Appl.* **20**, 446 (1998).
- [17] G. S. Ammar and W. B. Gragg, *SIAM J. Matrix Anal. Appl.* **9**, 61 (1988).
- [18] R. H. Chan and M. K. Ng, *SIAM Rev.* **38**, 427 (1996).
- [19] P. Sherman, *IEEE Trans. Acoust., Speech, Signal Process.* **33**, 1630 (1985).
- [20] A. Mahasinghe and J. B. Wang, *J. Phys. A: Math. Theor.* **49**, 275301 (2016).
- [21] S. S. Zhou and J. B. Wang, *R. Soc. Open Sci.* **4**, 160906 (2017).
- [22] U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications*, Vol. 321 (University of California, Berkeley 2001).
- [23] R. M. Gray, *Toeplitz and Circulant Matrices: A Review* (now, Boston, 2006).
- [24] I. Gohberg and I. A. Feldman, *Convolution Equations and Projection Methods for Their Solution*, Vol. 41 (American Mathematical Society, Providence, 2005).
- [25] F.-W. Sun, Y. Jiang, and J. S. Baras, *IEEE Trans. Inf. Theory* **49**, 180 (2003).
- [26] L. Grover and T. Rudolph, [arXiv:quant-ph/0208112](https://arxiv.org/abs/quant-ph/0208112).
- [27] A. N. Soklakov and R. Schack, *Phys. Rev. A* **73**, 012307 (2006).
- [28] V. Giovannetti, S. Lloyd, and L. Maccone, *Phys. Rev. Lett.* **100**, 160501 (2008).
- [29] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, Cambridge, England, 2010).
- [30] M. K. Bhaskar, S. Hadfield, A. Papageorgiou, and I. Petras, *Quantum Inf. Comput.* **16**, 197 (2016).
- [31] G. Brassard, P. Høyer, M. Mosca, and A. Tapp, *Contemp. Math.* **305**, 53 (2002).
- [32] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University, Cambridge, England, 2012).
- [33] G. H. Hardy and W. W. Rogosinski, *Fourier Series* (Dover Publications, New York, 2013).
- [34] S. S. Zhou, T. Loke, J. A. Izaac, and J. B. Wang, *Quantum Inf. Process.* **16**, 82 (2016).
- [35] T. G. Draper, [arXiv:quant-ph/0008033](https://arxiv.org/abs/quant-ph/0008033).
- [36] Z. Zhu and M. B. Wakin, *IEEE Trans. Inf. Theory* **63**, 2975 (2017).
- [37] R. H. Chan and P. T. P. Tang, *SIAM J. Sci. Comput.* **15**, 164 (1994).
- [38] D. Potts and G. Steidl, *BIT Numer. Math.* **39**, 513 (1999).
- [39] A. M. Childs, R. Kothari, and R. D. Somma, *SIAM J. Comput.* **46**, 1920 (2017).
- [40] B. D. Clader, B. C. Jacobs, and C. R. Sprouse, *Phys. Rev. Lett.* **110**, 250504 (2013).
- [41] L. Wossnig, Z. Zhao, and A. Prakash, *Phys. Rev. Lett.* **120**, 050502 (2018).
- [42] K. Ye and L.-H. Lim, *Found. Comput. Math.* **16**, 577 (2016).