Trade-off relation for coherence and disturbance

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The measurement process necessarily disturbs the state of a quantum system, unless the state is an eigenstate of the observable being measured. Once we perform a complete measurement in a given basis, the system undergoes decoherence and loses its coherence. If there is no disturbance, the state retains all of its coherence. It is therefore natural to ask if there is a trade-off between disturbance caused to a state and its coherence. We present coherence disturbance trade-off relations using the relative entropy of coherence for measurement channel as well as for general channels (completely positive trace-preserving maps). For bipartite states we prove a trade-off relation between the quantum coherence, entanglement, and disturbance. Similar relation also holds for quantum coherence, quantum discord, and disturbance for any bipartite state. We illustrate the trade-off between the coherence and the disturbance for single-qubit and -qutrit states subject to various quantum channels.

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I. INTRODUCTION

Measurement is an integral part of quantum theory. In order to extract information about a physical system we need to measure some observable. Unlike classical systems, in quantum mechanics the measurement process necessarily disturbs the state of the system unless the state is prepared in one of the eigenstate of the observable being measured. Intuitively, we know that if we want to extract information, the state of the system is necessarily disturbed, however, for most information processing tasks one would like to keep the disturbance to be minimum. There are several papers aimed at proving this statement quantitatively by deriving information vs disturbance trade-off relations, using different definitions of information and disturbance in varied scenarios [1–13]. It is well known that a state of a single quantum system cannot be determined if we demand no disturbance and possess no prior information about the quantum state [14,15]. However, the disturbance of the quantum system can be made arbitrarily small by using weak measurements. The weak measurement also has potential applications in quantum information processing [16–20].

The measurement process not only disturbs the state, but also leads to loss of coherence. Note that when we say the loss of coherence, we mean that the coherence is defined in a particular basis as it is intrinsically a basis-dependent quantity. If we change the basis we may see a different amount of coherence [21]. This results in the decoherence of a system which is seen as diminishing of the off-diagonal elements of the density matrix that indicates the loss of superposition. Both direct and indirect (here one uses the ancilla) measurement processes can cause decoherence of a system [22–24] and result in transfer of information from the system to the apparatus and environment.

Coherence is a property of the physical system in the quantum world that can be used to drive various nonclassical phenomena. Hence, coherence can be viewed as a resource, which enables us to perform useful quantum information processing tasks. Much before the resource theory of coherence was developed [25–28], coherence was viewed as a resource similar to entanglement. In fact, similar to entanglement swapping, coherence swapping has been proposed that can create coherent superposition from two incoherent states [29]. After the development of the resource theory of coherence, this was shown to be complementary to the path distinguishability in an interferometer [30]. Similarly, a complementarity relation between quantum coherence and entanglement was proved in Ref. [31]. Also, coherence in two incompatible bases were shown to be complementary to each other by proving that they indeed satisfy an uncertainty relation [32]. Complementarity of coherence with mixedness and asymmetry was also investigated in Refs. [33–35].

This motivates us to explore the idea that the initial coherence should respect a trade-off relation with the disturbance caused to the system state, whenever some information is extracted from the system or a measurement is performed on a quantum system. In this work, we present a trade-off relation between the initial coherence of the system state and the disturbance caused by a completely positive trace-preserving (CPTP) map. The trade-off relation is tight and the equality is satisfied for amplitude damping and depolarizing channels in the case of a single-qubit state, while the bound is lowered as the measurement strength is reduced. To prove the trade-off relation, we use a measure of disturbance given in Ref. [36] and the relative entropy quantum coherence given in Ref. [25]. Moving on to bipartite systems, we know that these systems can contain other nonclassical features such as the entanglement and quantum discord in addition to quantum coherence. Therefore, it is natural to seek a generalization of the trade-off relations between the coherence, entanglement, quantum discord, and disturbance caused to the system under CPTP maps. For a bipartite state we prove a trade-off relation for quantum coherence, entanglement, and the disturbance induced by quantum operation. In addition, we also prove a similar relation for quantum coherence, quantum discord, and disturbance.

The rest of the paper is organized as follows. In Sec. II, we briefly review the basics of quantum measurement and quantification of coherence and disturbance. Next, in Sec. III, we derive the trade-off relation between the coherence and the disturbance, which is our main result. For a bipartite state we prove a trade-off relation for the relative entropy of coherence, the relative entropy of entanglement, and the disturbance induced by quantum operation. We also prove that a similar relation holds for quantum coherence, quantum discord, and disturbance. In Sec. IV, we give a few examples to illustrate the trade-off between quantum coherence and disturbance for various quantum channels. In Sec. V, we present the conclusions and physical meaning of our results.

II. BASIC DEFINITIONS AND PRELIMINARIES

A. Quantum measurement

Quantum measurement is a distinct type of evolution compared to the Schrödinger evolution of a quantum system. The measurement process is nonunitary in nature and gives us classical information about the system. In quantum mechanics one can measure the observables represented by Hermitian operators. After the measurement process, the state collapses to one of the eigenstates of the observable being measured with probability given by the Born rule. This kind of measurement is described using the projection operators Π_j with $\sum_j \Pi_j = 1$. However, the most general measurement can be described using a set of measurement operators M_j such that $\sum_j M_j^\dagger M_j = 1$. For an initial state ρ, the probability of obtaining an outcome j is $p_j = \mathrm{Tr}(E_j\rho)$, where E_j is the apparatus positive-operator-valued measure. Whenever there is loss of information in measurement outcome, the final state of the system is given by a noisy map $\mathcal{N}(\rho) = \Sigma_j M_j \rho M_j^\dagger$. In order that the final state is a proper density-matrix operator, the map $\mathcal{N}(\rho)$ should be a CPTP map.

B. Quantifying coherence and disturbance

In this section we briefly review the definitions of coherence of a quantum system and the disturbance caused to the quantum system due to a quantum operation.

1. Quantum coherence

Quantum coherence arises from the superposition principle and thus marks a departure from classical physics. It is a basis-dependent quantity and hence it is necessary to fix the reference basis in which we define a quantitative measure of coherence. An axiomatic approach to quantify quantum coherence was developed by Baumgratz *et al.* in Ref. [25] by characterizing incoherent states $\mathcal{I}$ and incoherent operations Λ. For a given reference basis $|i\rangle$ $(i = 0,1,\dots,d-1)$, all incoherent states are of the form $\rho = \sum_i p_i |i\rangle\langle i|$ such that $\sum_i p_i = 1$. All incoherent operators are defined as CPTP maps, which map the set of incoherent states onto itself. A genuine measure of quantum coherence should fulfill the following requirements: (i) non-negativity, where $C(\rho) \geqslant 0$ in general and the equality is satisfied if and only if ρ is an incoherent state; (ii) monotonicity, where $C(\rho)$ does not increase under the action of incoherent operations, i.e., $C(\Lambda(\rho)) \leqslant C(\rho)$, where Λ is an incoherent operation; (iii) strong monotonicity, where $C(\rho)$ does not increase on average under selective incoherent operations, i.e., $\sum_i q_i C(\sigma_i) \leqslant C(\rho)$, where $q_i = \mathrm{Tr}[K_i \rho K^\dagger{}_i]$ are the probabilities, $\sigma_i = K_i \rho K^\dagger{}_i / q_i$ are postmeasurement states, and K_i are the incoherent Kraus operators; and (iv) convexity, where $C(\rho)$ is a convex function of the state, i.e., $\sum_i p_i C(\rho_i) \geqslant C(\sum_i p_i \rho_i)$. It can be noted that conditions (iii) and (iv) put together imply condition (ii).

The measures that fulfill the above requirements are the l_1-norm of coherence and the relative entropy of coherence. In the present work we will use the relative entropy of coherence, which is given by

$$C_r(\rho) = S(\rho^D) - S(\rho), \tag{1}$$

where $S(\rho) = -\mathrm{Tr}[\rho \log_2(\rho)]$ is the von Neumann entropy of the density matrix ρ and ρ^D denotes the state obtained by deleting the off-diagonal elements of ρ in a given basis $\{|i\rangle\}$. For a d-dimensional state, $0 \leqslant C_r(\rho) \leqslant \log_2(d)$. Hence, using the above definition, we can define the maximally coherent state with $C_r(\rho) = \log_2(d)$, which is the case when $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle$. Another definition of coherence based on the matrix norm is the l_1-norm of coherence, which is given by $C_{l_1}(\rho) = \sum_{i\neq j}|\rho_{ij}|$, where $\rho_{ij} = \langle i|\rho|j\rangle$. Also a geometric measure of coherence was defined in Ref. [37], as $C_g(\rho) = 1 - \max_{\sigma\epsilon I} F(\rho,\sigma)$, where I is the set of all incoherent states and the fidelity $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$.

2. Disturbance

In quantum mechanics, disturbance caused by a measurement process can be defined with respect to both the observable and the state. The disturbance for an observable due to measurement of another observable was defined in Refs. [38,39] and for states in connection with the error-disturbance relations [40–42]. However, here we consider the disturbance caused to a state when the system is subjected to a measurement process and CPTP maps and do not aim to formulate error-disturbance relations. We say that a system is disturbed when the initial and final states do not coincide. We define disturbance as an irreversible change in the state of the system, caused by CPTP evolution. It is thus required that the quantity D that measures disturbance should satisfy the following conditions: (i) D should be a function of the initial state ρ and the CPTP map $\mathcal{E}$ only, i.e., $D = D(\rho,\mathcal{E})$; (ii) $D(\rho,\mathcal{E})$ should be null if and only if the CPTP map is invertible on the initial state ρ, because for invertible maps the change in state can be reversed and hence the system is not disturbed by our definition; (iii) $D(\rho,\mathcal{E})$ should be monotonically nondecreasing under successive application of CPTP maps, which ensures that the disturbance cannot be reversed by subsequent measurements; and (iv) $D(\rho,\mathcal{E})$ should be continuous for maps and initial states which do not differ too much.

Several definitions of disturbance have been proposed using the fidelity and the Bures distance between the initial and final states [1,3,43], but they fail to satisfy the irreversibility condition. Moreover, the fidelity-based definition is nonzero for unitary transformations and disturbs a system in a nonclassical way [44], as we know that change in the quantum state due to unitary operations is reversible and hence does not cause any disturbance as per our definition. Also, these definitions can be null for noninvertible maps and they are not monotonically nondecreasing under successive application of CPTP maps, therefore they fail to satisfy conditions (ii) and (iii). These are the main reasons why we have adopted the measure of disturbance given in Ref. [36]. For the sake of completeness, in the Appendix we discuss the trade-off relation for the geometric measure of coherence and the fidelity-based measure of disturbance. With the physically motivated conditions given in (i)–(iv), it was shown by Maccone that all the above conditions are met by the definition of disturbance [36]

$$D(\rho,\mathcal{E}) \equiv S(\rho) - I_c(\rho,\mathcal{E}) \\ = S(\rho) - S(\mathcal{E}(\rho)) + S((\mathcal{E}\otimes I)(|\Psi\rangle_{SR}\langle\Psi|)), \quad (2)$$

where $I_c = S(\mathcal{E}(\rho)) - S((\mathcal{E}\otimes I)(|\Psi\rangle_{SR}\langle\Psi|))$ is the coherent information [45,46] of the system passing through a noisy channel and $|\Psi\rangle_{SR}\langle\Psi|$ is a purification of ρ such that $\rho = \rho_S = \mathrm{Tr}_R(|\Psi\rangle_{SR}\langle\Psi|)$. Since $I_c(\rho,\mathcal{E}) \leqslant S(\rho)$, we have $D(\rho,\mathcal{E}) \geqslant 0$. We know that a CPTP map is invertible if and only if the coherent information is equal to the von Neumann entropy $S(\rho)$ of the state [45] and hence disturbance will always be null for all the invertible maps. The map $\mathcal{E}\otimes I$ acts on $|\Psi\rangle_{SR}$ with $\mathcal{E}$ acting on the system Hilbert space and I acting on the ancilla Hilbert space. The quantity I_c is nonincreasing under successive application of CPTP maps, which makes the disturbance measure monotonically nondecreasing under CPTP maps. It is clear from the definition of $D(\rho,\mathcal{E})$ that for a d-dimensional density matrix ρ it satisfies, $0 \leqslant D \leqslant 2\log_2(d)$. With these basic definitions for the quantum coherence and disturbance, now we present our main results.

III. TRADE-OFF RELATIONS: COHERENCE, ENTANGLEMENT, QUANTUM CORRELATIONS, AND DISTURBANCE

In quantum information processing the role of coherence, entanglement, and quantum correlations cannot be overlooked. However, when we send a quantum state through a noisy channel the system tends to lose these delicate quantum features. In practical scenarios, the action of noise and measurement cannot be evaded. In this section, we will investigate how the initial coherence of the density matrix respects a trade-off relation with the disturbance caused by a quantum operation. Similarly, for a bipartite state we will explore how the quantum features such as coherence, entanglement, and quantum discord respect a trade-off relation with the disturbance caused by a CPTP map.

A. Coherence-disturbance trade-off relation

Here we prove that there exists a trade-off relation between the amount of coherence contained in a quantum state and the disturbance caused to a system by a CPTP map. Consider a d-dimensional system with a density matrix ρ, where initially the system and ancilla ρ_R share a pure bipartite state $|\Psi\rangle_{SR}$, with $\rho = \mathrm{Tr}_R(|\Psi\rangle_{SR}\langle\Psi|)$. When the system undergoes a quantum operation $\mathcal{E}$ the evolution is represented as $\rho \to \mathcal{E}(\rho) = \sum_i K_i\rho K_i{}^\dagger$, where K_i are the Kraus operator elements with $\sum_i K_i{}^\dagger K_i = I$. During the action of the CPTP map, the system undergoes a disturbance as given in Eq. (2), i.e., $D(\rho,\mathcal{E}) = S(\rho) - I_c(\rho,\mathcal{E})$. For such a noisy evolution, we will prove that the trade-off relation between the coherence and the disturbance is given by

$$2C_r(\rho) + D(\rho,\mathcal{E}) \leqslant 2\log_2(d). \quad (3)$$

The proof is

$$\begin{aligned} &2C_{\text{rel ent}}(\rho) + D(\rho,\mathcal{E}) \\ &\quad = 2S(\rho^D) - S(\rho) - S(\mathcal{E}(\rho)) + S(\mathcal{E}\otimes I(|\Psi\rangle_{SR}\langle\Psi|)) \\ &\quad \leqslant 2S(\rho^D) - S(\rho) - S(\mathcal{E}(\rho)) + S(\mathcal{E}(\rho)) + S(\rho_R') \\ &\quad = 2S(\rho^D) - S(\rho) + S(\rho_R) \\ &\quad = 2S(\rho^D) \leqslant 2\log_2(d), \end{aligned}$$

where ρ_R' is the final state of ancilla. The first inequality is obtained by using the subadditivity of quantum entropy. The next inequality is obtained using the fact that there is no change in entropy of the ancilla and the next equality follows using the fact that initial bipartite state is a pure state and thus $S(\rho) = S(\rho_R)$. The final inequality comes from the maximum value of entropy of a state. Thus, for a given value of nonzero disturbance, the quantum coherence cannot reach its maximum value. There is a trade-off between these two quantum features. Also, note that nowhere in the proof do we use the coherence measure in a particular basis. Therefore, the relation holds true in any basis we want to define the coherence and for all CPTP maps.

B. Coherence-disturbance trade-off for the measurement channel

While the trade-off relation holds true for all quantum channels, the bound is tighter in the case of measurement channels. The quantum operation for the measurement channel is given by

$$\rho \to \mathcal{E}(\rho) = \sum_k \Pi_k\rho\Pi_k = \rho^D = \sum_k \rho_{kk}\,|k\rangle\langle k|\,,$$

where Π_k are the projection operators. This is also known as a dephasing channel. Now if we consider an environment state $|0\rangle_E$ so that $|\Psi\rangle_{SR}\otimes|0\rangle_E$ is also a pure state, then the evolution $(\mathcal{E}\otimes I)(|\Psi\rangle_{SR}\langle\Psi|)$ is equivalent to unitary evolution of the tripartite state (U acts on $\mathcal{H}_S\otimes\mathcal{H}_E$)

$$U\otimes\mathcal{I}(|\Psi\rangle_{SR}\otimes|0\rangle_E) \to |\Psi'\rangle_{\text{SRE}}\,.$$

Since $|\Psi'\rangle_{\text{SRE}}$ is also a pure state, we have $S(\rho_{SR}') = S(\rho_E')$, where $\rho_{SR}' = (\mathcal{E}\otimes I)(|\Psi\rangle_{SR}\langle\Psi|) = \mathrm{Tr}[U(|\Psi\rangle_{SR}\langle\Psi| \otimes |0\rangle_E\langle 0|)U^\dagger]$. Then, using subadditivity of entropy, one can obtain the trade-off relation

$$C(\rho) + D(\rho,\mathcal{E}) \leqslant \log_2 d_E, \quad (4)$$

where $d_E = \dim(\mathcal{H}_E)$ is the dimension of the Hilbert space of the environment. In Eq. (4) both the quantities $C(\rho)$ and $D(\rho,\mathcal{E})$ are basis dependent. In the case of $\dim(\mathcal{H}_S) = \dim(\mathcal{H}_E) = d$, the trade-off relation given in Eq. (4) is tighter than the one given in Eq. (3).

C. Trade-off between coherence, entanglement, and disturbance

In the preceding section, we proved the trade-off of coherence and disturbance for a single system. However, when we deal with a composite system it can have quantum coherence, entanglement, and quantum correlation beyond entanglement such as the discord. Then a natural question to ask here is if there exists any trade-off relation between the coherence, entanglement, and disturbance caused by CPTP maps. Already, we know that for pure bipartite states there is a trade-off relation between the relative entropy of coherence and the bipartite entanglement, i.e., $C(\rho_A) + E(|\Psi\rangle_{AB}) \leqslant \log_2 d$, where d is the dimension of the subsystem Hilbert space of A [31]. Even for mixed bipartite states ρ_{AB} one can prove a trade-off relation between coherence of one subsystem and entanglement of formation $E_f(\rho_{AB})$ [47]. This is given by

$$C_r(\rho_A) + E_f(\rho_{AB}) \leqslant \log_2 d. \qquad (5)$$

The proof follows from the Carlen-Lieb inequality [48]. This inequality says that $E_f(\rho_{AB}) \leqslant \min\{S(\rho_A), S(\rho_B)\}$. Assuming that $S(\rho_A)$ is the minimum one, we have $E_f(\rho_{AB}) \leqslant S(\rho_A)$, which is equivalent to Eq. (5). Below we prove that there is indeed a trade-off relation for the coherence, relative entropy of entanglement, and disturbance caused by measurement or a CPTP map on the bipartite state. Suppose we have the bipartite state ρ_{AB} with purification $|\Psi\rangle_{ABR}$ such that $\rho_{AB} = \mathrm{Tr}_R(|\Psi\rangle_{ABR}\langle\Psi|)$. The relative entropy of entanglement was defined in Refs. [49,50] as

$$E_R(\rho_{AB}) = \min_{\sigma_{AB}} S(\rho_{AB}||\sigma_{AB}),$$

where σ_{AB} belongs to the set of all separable states. Note that a mixed state is called separable if it can be written in the form $\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k$, where ρ_A^k and ρ_B^k are states of the subsystems with probability p_k. Now suppose that $\rho_{AB} \to \mathcal{E}(\rho_{AB})$; then the disturbance for the bipartite state under a quantum channel is defined as

$$\begin{aligned} D(\rho_{AB},\mathcal{E}) &= S(\rho_{AB}) - I_c((\rho_{AB})) \\ &= S(\rho_{AB}) - S(\mathcal{E}(\rho_{AB})) + S(\mathcal{E}\otimes\mathcal{I}(|\Psi\rangle_{ABR}\langle\Psi|)). \end{aligned} \qquad (6)$$

Here the relative entropy of quantum coherence for the bipartite state can be defined as

$$C(\rho_{AB}) = S(\rho_{AB}^D) - S(\rho_{AB}), \qquad (7)$$

where ρ_{AB}^D is the diagonal part of ρ_{AB} in the basis $\{|i\rangle \otimes |\mu\rangle\} \in \mathcal{H}_{AB}$. Using the above definitions of coherence, entanglement, and disturbance for the bipartite state ρ_{AB}, we can get a trade-off relation of the form

$$C(\rho_{AB}) + E_R(\rho_{AB}) + D(\rho_{AB},\mathcal{E}) \leqslant 2\log_2(d_{AB}). \qquad (8)$$

The proof of the relation is

$$\begin{aligned} &C(\rho_{AB}) + E_R(\rho_{AB}) + D(\rho_{AB},\mathcal{E}) \\ &= S(\rho_{AB}^D) + \min_{\sigma_{AB}} S(\rho_{AB}||\sigma_{AB}) - S(\mathcal{E}(\rho_{AB})) \\ &\quad + S(\mathcal{E}\otimes\mathcal{I}(|\Psi\rangle_{ABR}\langle\Psi|)) \\ &\leqslant S(\rho_{AB}^D) + S(\rho_{AB}||\rho_A\otimes\rho_B) + S(\rho_{AB}) \\ &= S(\rho_{AB}^D) + S(\rho_A) + S(\rho_B) \leqslant 2\log_2(d_{AB}), \end{aligned}$$

where d_{AB} is the dimension of the state ρ_{AB}. The first inequality is obtained using subadditivity of $S(\mathcal{E}\otimes\mathcal{I}(|\Psi\rangle_{ABR}\langle\Psi|))$ and the fact that $\min_\sigma S(\rho_{AB}||\sigma) \leqslant S(\rho_{AB}||\rho_A\otimes\rho_B)$. The final inequality follows from the maximum value of the entropy of the states, i.e., $S(\rho_A) \leqslant \log_2(d_A)$, $S(\rho_B) \leqslant \log_2(d_B)$, and $S(\rho_{AB}) \leqslant \log_2(d_{AB})$.

The trade-off relation (8) suggests that the sum of quantumness such as the coherence and entanglement cannot be large if the disturbance is also large. Also, for a fixed coherence $C(\rho)$, there is a trade-off between entanglement and disturbance caused to the quantum system. For separable states, $E_R(\rho_{AB}) = 0$ and we have $C(\rho_{AB}) + D(\rho_{AB},\mathcal{E}) \leqslant 2\log_2(d_{AB})$.

D. Trade-off between coherence, quantum discord, and disturbance

In the preceding section we proved a trade-off relation for coherence, entanglement, and disturbance caused by a CPTP map on a bipartite system. Similarly, one can ask if other quantum correlations such as quantum discord satisfy a similar trade-off relation. It was shown in Ref. [51] that for multipartite states, creation of quantum discord with multipartite incoherent operations is bounded by the amount of quantum coherence consumed in its subsystems during the process. This interplay between coherence and quantum discord suggests that coherence, quantum discord, and disturbance of a bipartite system may satisfy a trade-off relation. We will now prove that they also satisfy a trade-off relation. Quantum discord of a bipartite state is defined in Ref. [52] as

$$Q_D(\rho_{AB}) = \min_{\Pi_i^B}[I(\rho_{AB}) - J(\rho_{AB})_{\Pi_i^B}],$$

where $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ is the mutual information between subsystems A and B and $J(\rho_{AB})_{\Pi_i^B} = S(\rho_A) - S(A|\Pi_i^B)$ represents the amount of information gained about subsystem A by measuring subsystem B. Here Π_i^B are the measurement operators corresponding to the von Neumann measurement on subsystem B and $S(A|\Pi_i^B)$ is the conditional entropy after the measurement has been performed on subsystem B. Using the definitions of disturbance and coherence given in Eqs. (7) and (6), respectively, for a bipartite state, we get a trade-off relation of the form

$$C(\rho_{AB}) + Q_D(\rho_{AB}) + D(\rho_{AB},\mathcal{E}) \leqslant 2\log_2(d_{AB}). \qquad (9)$$

The proof of the relation is

$$\begin{aligned} &C(\rho_{AB}) + Q_D(\rho_{AB}) + D(\rho_{AB},\mathcal{E}) \\ &= S(\rho_{AB}^D) + \min_{\Pi_i^B}[I(\rho_{AB}) - J(\rho_{AB})_{\Pi_i^B}] - S(\mathcal{E}(\rho_{AB})) \\ &\quad + S(\mathcal{E}\otimes\mathcal{I}(|\Psi\rangle_{ABR}\langle\Psi|)) \\ &\leqslant S(\rho_{AB}^D) + I(\rho_{AB}) + S(\rho_{AB}) \\ &= S(\rho_{AB}^D) + S(\rho_A) + S(\rho_B) \leqslant 2\log_2(d_{AB}), \end{aligned}$$

where the proof is similar to the proof of the trade-off relation of entanglement, coherence, and disturbance of bipartite state. Thus, for separable states the coherence and discord cannot be arbitrarily large for a given disturbance $D(\rho_{AB},\mathcal{E})$ caused to the quantum system. For classical-classical states such as $\rho_{AB} = \sum p_k |k\rangle\langle k| \otimes |k\rangle\langle k|$, one has $Q_D(\rho_{AB}) = 0$ and in that case one has $C(\rho_{AB}) + D(\rho_{AB},\mathcal{E}) \leqslant 2\log_2(d_{AB})$.

IV. EXAMPLES

To gain some physical insight, in this section, we analyze the coherence disturbance trade-off relation for different quantum channels for a single qubit and later for the single-qutrit density matrix. The trade-off relations can be neatly presented for a few channels. Let us consider a two-qubit pure composite state of the system and ancilla in a $\{|+\rangle, |-\rangle\}$ basis

$$|\Psi\rangle_{SR} = \sqrt{\lambda_0}\,|+\rangle_S\,|+\rangle_R + \sqrt{\lambda_1}\,|-\rangle_S\,|-\rangle_R\,.$$

For this composite state, the density matrix of the system in a computational basis is given by

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & \lambda_0-\lambda_1 \\ \lambda_0-\lambda_1 & 1 \end{bmatrix}.$$

In this basis, it has nonzero coherence, which is given by

$$\begin{aligned} C_r(\rho) &= -\mathrm{Tr}[\rho^D \log_2 \rho^D] + \mathrm{Tr}[\rho \log_2 \rho] \\ &= 1 + \lambda_0 \log_2 \lambda_0 + \lambda_1 \log_2 \lambda_1. \end{aligned} \tag{10}$$

The disturbance of a state depends on both the state density matrix and the quantum channel. We give Kraus operators and the corresponding expressions of the disturbance and present the trade-off relations for a few channels as examples.

A. Weak measurement channel

The theory of weak measurement channels using the measurement operator formalism can be found in Refs. [53,54]. This approach provides a tool to handle strong as well weak measurements. The Kraus operators for the weak measurement channel are given by $K(x) = \sqrt{\frac{1-x}{2}}\Pi_0 + \sqrt{\frac{1+x}{2}}\Pi_1$ and $K(-x) = \sqrt{\frac{1+x}{2}}\Pi_0 + \sqrt{\frac{1-x}{2}}\Pi_1$, where Π_0 and Π_1 are the two projection operators in the computational basis. The weak measurement Kraus operators satisfy $K(x)^\dagger K(x) + K(-x)^\dagger K(-x) = \mathcal{I}$. The parameter $x \in [0,1]$ denotes the measurement strength, where the measurement strength increases as x goes from 0 to 1. The operators satisfy the following properties: (i) For $x=0$, we have no measurement, i.e., $K(x) = K(-x) = \frac{\mathcal{I}}{\sqrt{2}}$, resulting in no state change; (ii) for $x=1$, in the strong measurement limit we have the projective measurements, i.e., $K(x) = \Pi_1$ and $K(-x) = \Pi_0$; and (iii) $[K(x), K(-x)] = 0$. Under the weak measurement channel the state changes as

$$\rho \rightarrow \mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{(1-x^2)}(\lambda_0-\lambda_1) \\ \sqrt{(1-x^2)}(\lambda_0-\lambda_1) & 1 \end{bmatrix}.$$

The disturbance for the weak measurement channel is given by

$$\begin{aligned} D(\rho,\mathcal{E}) =& -\mathrm{Tr}[\rho \log_2 \rho] + \mathrm{Tr}[\mathcal{E}(\rho) \log_2 \mathcal{E}(\rho)] \\ & - \mathrm{Tr}[\mathcal{E}\otimes I(|\Psi\rangle\langle\Psi|) \log_2 \mathcal{E}\otimes I(|\Psi\rangle\langle\Psi|] \\ =& -\lambda_0 \log_2 \lambda_0 - \lambda_1 \log_2 \lambda_1 - \frac{1-\sqrt{1-4\lambda_0\lambda_1 x^2}}{2} \\ & \times \log_2 \frac{1-\sqrt{1-4\lambda_0\lambda_1 x^2}}{2} - \frac{1+\sqrt{1-4\lambda_0\lambda_1 x^2}}{2} \\ & \times \log_2 \frac{1+\sqrt{1-4\lambda_0\lambda_1 x^2}}{2} \\ & + \left(\frac{1}{2} - \frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right) \\ & \times \log_2 \left(\frac{1}{2} - \frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right) \\ & + \left(\frac{1}{2} + \frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right) \\ & \times \log_2 \left(\frac{1}{2} + \frac{\lambda_0-\lambda_1}{2}\sqrt{1-x^2}\right). \end{aligned} \tag{11}$$

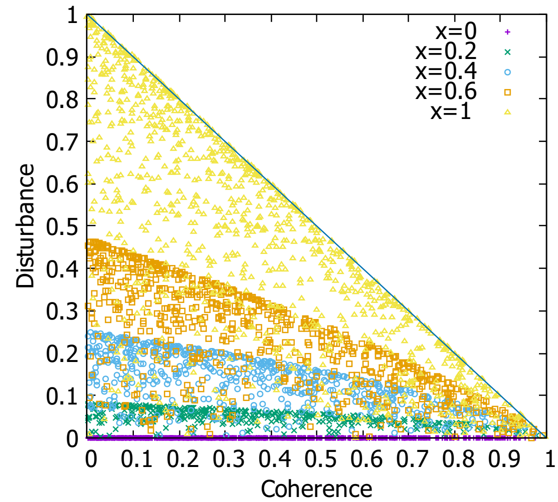

FIG. 1. Trade-off between coherence $C(\rho)$ and disturbance $D(\rho,\mathcal{E})$ for the weak measurement channel. The figure shows coherence along the X axis and disturbance along the Y axis. Random states were generated and coherence and entropy were calculated using the MATLAB package [55].

It can be checked that $D(\rho,\mathcal{E})$ increases monotonically as x is increased from 0 to 1. By using Eqs. (10) and (11) we indeed see that the relation $C(\rho) + D(\rho,\mathcal{E}) \leqslant 1$ holds. This is depicted in Fig. 1. This relation is tighter than our original relation (3). The same trade-off relation is also obtained for the bit-flip, phase-flip, and bit-phase flip channels for a single-qubit system.

B. Depolarizing channel

The Kraus operators for the depolarizing channel are given by $K_1 = \sqrt{1-\frac{3p}{4}}I_2$, $K_2 = \sqrt{\frac{p}{4}}\sigma_x$, $K_3 = \sqrt{\frac{p}{4}}\sigma_y$, and $K_4 = \sqrt{\frac{p}{4}}\sigma_z$, where σ_x, σ_y, and σ_z are the Pauli matrices. Under the depolarizing channel the state changes as

$$\rho \rightarrow \mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} 1 & (1-p)(\lambda_0-\lambda_1) \\ (1-p)(\lambda_0-\lambda_1) & 1 \end{bmatrix}.$$

The disturbance for the depolarizing channel is given by

$$
\begin{aligned}
D(\rho,\mathcal{E}) = & -\lambda_0 \log_2 \lambda_0 - \lambda_1 \log_2 \lambda_1 + \left(\frac{1}{2} - \frac{\lambda_0-\lambda_1}{2}(1-p)\right)\log_2\left(\frac{1}{2} - \frac{\lambda_0-\lambda_1}{2}(1-p)\right) \\
& + \left(\frac{1}{2} + \frac{\lambda_0-\lambda_1}{2}(1-p)\right)\log_2\left(\frac{1}{2} + \frac{\lambda_0-\lambda_1}{2}(1-p)\right) - \frac{p\lambda_0}{2}\log_2\frac{p\lambda_0}{2} - \frac{p\lambda_1}{2}\log_2\frac{p\lambda_1}{2} \\
& - \left(\frac{\left(1-\frac{p}{2}\right) + \sqrt{\left(1-\frac{p}{2}\right)^2 - 4\lambda_0\lambda_1\left(p-\frac{3p^2}{4}\right)}}{2}\right)\log_2\left(\frac{\left(1-\frac{p}{2}\right) + \sqrt{\left(1-\frac{p}{2}\right)^2 - 4\lambda_0\lambda_1\left(p-\frac{3p^2}{4}\right)}}{2}\right) \\
& - \left(\frac{\left(1-\frac{p}{2}\right) - \sqrt{\left(1-\frac{p}{2}\right)^2 - 4\lambda_0\lambda_1\left(p-\frac{3p^2}{4}\right)}}{2}\right)\log_2\left(\frac{\left(1-\frac{p}{2}\right) - \sqrt{\left(1-\frac{p}{2}\right)^2 - 4\lambda_0\lambda_1\left(p-\frac{3p^2}{4}\right)}}{2}\right). \qquad (12)
\end{aligned}
$$

Again it is easy to check that $D(\rho,\mathcal{E})$ increases monotonically with p. Moreover, using Eqs. (10) and (12), we get $2C(\rho) + D(\rho,\mathcal{E}) \leqslant 2$, which is the same as Eq. (3) for a qubit and is depicted in Fig. 2.

C. Amplitude damping channel

The Kraus operators for the amplitude damping channel are given by $K_1 = \sqrt{q}\,|0\rangle\langle 1|$ and $K_2 = |0\rangle\langle 0| + \sqrt{1-q}\,|1\rangle\langle 1|$. Under the amplitude channel the state transforms as

$$
\begin{aligned}
&\rho \rightarrow \mathcal{E}(\rho) \\
&= \frac{1}{2}\begin{bmatrix} 1+q & \sqrt{(1-q)}(\lambda_0-\lambda_1) \\ (\sqrt{(1-q)}+q)(\lambda_0-\lambda_1) & 1-q \end{bmatrix}.
\end{aligned}
$$

The disturbance of the amplitude damping channel is given by

$$
\begin{aligned}
D(\rho,\mathcal{E}) = & -\tfrac{1}{2}(1+\lambda_0-\lambda_1)\log_2\tfrac{1}{2}(1+\lambda_0-\lambda_1) \\
& -\tfrac{1}{2}(1-\lambda_0+\lambda_1)\log_2\tfrac{1}{2}(1-\lambda_0+\lambda_1) \\
& -(1-q\lambda_1)\log_2(1-q\lambda_1) - q\lambda_1\log_2 q\lambda_1 \\
& +\tfrac{1}{2}[1-\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}] \\
& \times \log_2\tfrac{1}{2}[1-\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}] \\
& +\tfrac{1}{2}[1+\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}] \\
& \times \log_2[1+\sqrt{q^2+(\lambda_0-\lambda_1)^2(1-q)}]. \qquad (13)
\end{aligned}
$$

For the amplitude damping channel also $D(\rho,\mathcal{E})$ and $C(\rho)$ follow the original relation (3). The trade-off relations derived above can be seen as given in Fig. 3.

Numerical data show that the trade-off relation for the coherence and disturbance given in Eq. (3) is satisfied for all the above channels for single-qubit systems. The amount of disturbance decreases as the measurement strength is decreased, which is expected in the case of all the channels. It can be also seen that the trade-off between coherence and disturbance is channel dependent. The trade-off relation obeyed for a single-qubit state in the case of a weak measurement channel is tighter than Eq. (3), while the amplitude damping and depolarizing channels follow the original relation for a single-qubit state.

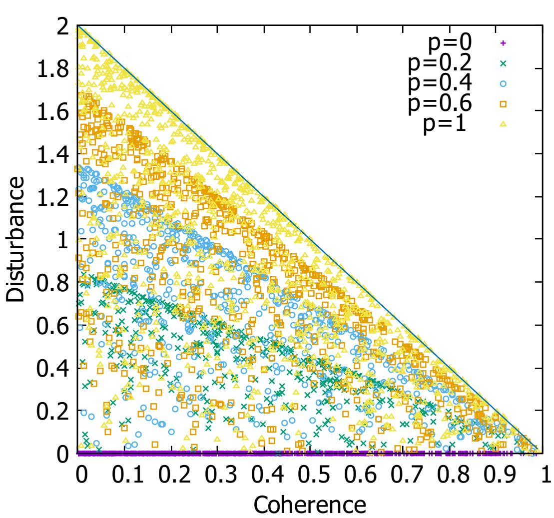

FIG. 2. Trade-off between coherence $C(\rho)$ and disturbance $D(\rho,\mathcal{E})$ for the depolarizing channel. The figure shows coherence along the X axis and disturbance along the Y axis. Random states were generated and coherence and entropy were calculated using the MATLAB package [55].

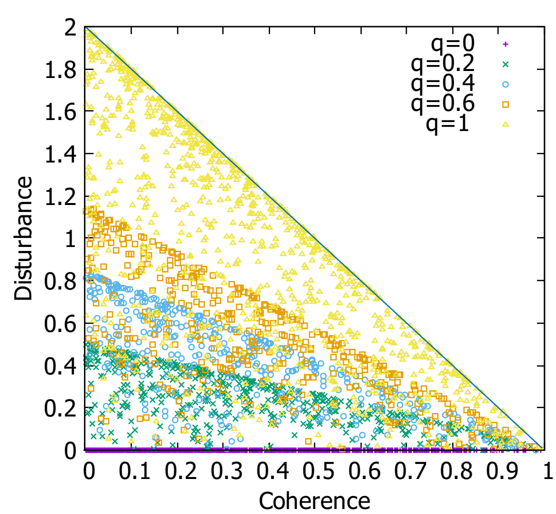

FIG. 3. Trade-off between coherence $C(\rho)$ and disturbance $D(\rho,\mathcal{E})$ for the amplitude damping channel. The figure shows coherence along the X axis and disturbance along the Y axis. Random states were generated and coherence and entropy were calculated using the MATLAB package [55].

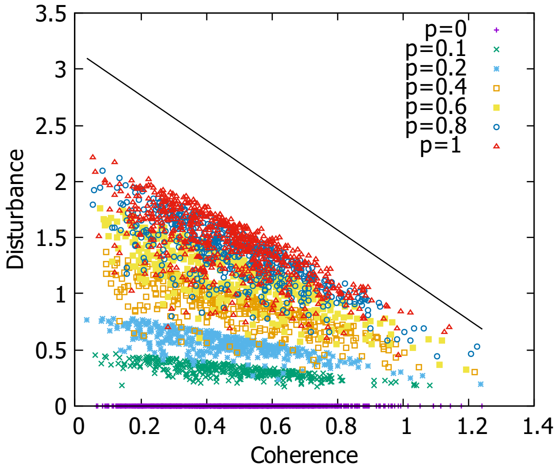

FIG. 4. Trade-off between coherence $C(\rho)$ and disturbance $D(\rho,\mathcal{E})$ for the depolarizing channel applied on a qutrit state. The figure shows coherence along the X axis and disturbance along the Y axis. The straight line corresponds to $2C_r(\rho) + D(\rho,\mathcal{E}) = 2\log_2 3$. Random states were generated and coherence and entropy were calculated using the MATLAB package [55].

D. Examples from the qutrit system

Consider a two-qutrit pure composite state of system and ancilla in the $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$ basis

$$|\Psi\rangle_{SR} = \sqrt{\lambda_0}\,|\alpha\rangle_S\,|\alpha\rangle_R + \sqrt{\lambda_1}\,|\beta\rangle_S\,|\beta\rangle_R + \sqrt{\lambda_2}\,|\gamma\rangle_S\,|\gamma\rangle_R\,,$$

where $|\alpha\rangle = \frac{|0\rangle-|1\rangle+|2\rangle}{\sqrt{3}}$, $|\beta\rangle = \frac{|0\rangle+|1\rangle-|2\rangle}{\sqrt{3}}$, and $|\gamma\rangle = \frac{-|0\rangle+|1\rangle+|2\rangle}{\sqrt{3}}$. For this composite state, the density matrix of the system in the computational basis is given by

$$\rho = \frac{1}{3}\begin{bmatrix} 1 & 2\lambda_0-1 & 2\lambda_1-1 \\ 2\lambda_0-1 & 1 & 2\lambda_2-1 \\ 2\lambda_1-1 & 2\lambda_2-1 & 1 \end{bmatrix}.$$

In this basis it has nonzero coherence, which is given by

$$C_r(\rho) = \log_2 3 + \lambda_0\log_2\lambda_0 + \lambda_1\log_2\lambda_1 + \lambda_2\log_2\lambda_2. \quad (14)$$

E. Depolarizing channel

The Kraus operators for the single-qutrit depolarizing channel [56] are given by

$$K_1 = \sqrt{1-\frac{8p}{9}}I_3,\quad K_2 = \sqrt{\frac{p}{9}}Y,\quad K_3 = \sqrt{\frac{p}{9}}Z,$$
$$K_4 = \sqrt{\frac{p}{9}}YZ,\quad K_5 = \sqrt{\frac{p}{9}}Y^2Z,\quad K_6 = \sqrt{\frac{p}{9}}YZ^2,$$
$$K_7 = \sqrt{\frac{p}{9}}Y^2Z^2,\quad K_8 = \sqrt{\frac{p}{9}}Y^2,\quad K_9 = \sqrt{\frac{p}{9}}Z^2,$$

where

$$Y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\quad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix},$$

and $\omega = \exp^{2\pi i/3}$. Under the depolarizing channel the qutrit state changes as

$$\rho \to \mathcal{E}(\rho) = \begin{bmatrix} 1-\frac{2p}{3} & \frac{2\lambda_0-1}{3}p & \frac{2\lambda_1-1}{3}p \\ \frac{2\lambda_0-1}{3}p & 1-\frac{2p}{3} & \frac{2\lambda_2-1}{3}p \\ \frac{2\lambda_1-1}{3}p & \frac{2\lambda_2-1}{3}p & 1-\frac{2p}{3} \end{bmatrix}.$$

The disturbance for the depolarizing channel is given by

$$\begin{aligned}
D(\rho,\mathcal{E}) = {}& -\lambda_0\log_2\lambda_0 - \lambda_1\log_2\lambda_1 - \lambda_2\log_2\lambda_2 \\
& + \left(\frac{p}{3} + (1-p)\lambda_0\right)\log_2\left(\frac{p}{3} + (1-p)\lambda_0\right) \\
& + \left(\frac{p}{3} + (1-p)\lambda_1\right)\log_2\left(\frac{p}{3} + (1-p)\lambda_1\right) \\
& + \left(\frac{p}{3} + (1-p)\lambda_2\right)\log_2\left(\frac{p}{3} + (1-p)\lambda_2\right) \\
& - 2\frac{p\lambda_0}{3}\log_2\frac{p\lambda_0}{3} - \left(\lambda_0 - \frac{8p\lambda_0}{9}\right) \\
& \times \log_2\left(\lambda_0 - \frac{8p\lambda_0}{9}\right) - 2\frac{p\lambda_1}{3}\log_2\frac{p\lambda_1}{3} \\
& - \left(\lambda_1 - \frac{8p\lambda_1}{9}\right)\log_2\left(\lambda_1 - \frac{8p\lambda_1}{9}\right) \\
& - 2\frac{p\lambda_2}{3}\log_2\frac{p\lambda_2}{3} - \left(\lambda_2 - \frac{8p\lambda_2}{9}\right) \\
& \times \log_2\left(\lambda_2 - \frac{8p\lambda_2}{9}\right). \qquad (15)
\end{aligned}$$

From Eqs. (14) and (15) we plot coherence and disturbance in Fig. 4. We find that in this case the trade-off relation is stronger than our original relation (3).

F. Amplitude damping channel

The Kraus operators for the amplitude damping channel are given by

$$\begin{aligned}
K_1 &= \sqrt{q_0}\,|0\rangle\,\langle 1| + \sqrt{2q(1-q)}\,|1\rangle\,\langle 2|\,,\\
K_2 &= \sqrt{q}\,|0\rangle\,\langle 2|\,,\\
K_3 &= |0\rangle\,\langle 0| + \sqrt{1-q}\,|1\rangle\,\langle 1| + \sqrt{1-q}\,|2\rangle\,\langle 2|\,.
\end{aligned}$$

Under the amplitude channel the state transforms as

$$\rho \to \mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} \frac{1+q}{3} & \frac{\sqrt{1-q}}{3}(2\lambda_0+2q-1) & \frac{\sqrt{1-q}}{3}(2\lambda_1-1) \\ \frac{\sqrt{1-q}}{3}(2\lambda_0+2q-1) & \frac{1}{3}[1+2q(1-q)] & \frac{1-q}{3}(2\lambda_1-1) \\ \frac{\sqrt{1-q}}{3}(2\lambda_1-1) & \frac{1-q}{3}(2\lambda_1-1) & \frac{1-q}{3} \end{bmatrix}.$$

The disturbance for the amplitude damping channel is given by

$$D(\rho,\mathcal{E}) = -2\lambda_0 \log_2 \lambda_0 - \lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2 + (\lambda_0 + q\lambda_1)\log_2(\lambda_0 + q\lambda_1) + [(1-q)\lambda_1 + (3-2q)q\lambda_2]\log_2[(1-q)\lambda_1 + (3-2q)q\lambda_2] - q\lambda_1 \log_2 q\lambda_1 - [(1-q)\lambda_1]\log_2[(1-q)\lambda_1] - (3-2q)q\lambda_2 \log_2(3-2q)q\lambda_2. \tag{16}$$

For the amplitude damping channel $D(\rho,\mathcal{E})$ and $C(\rho)$ follow the original relation (3). The trade-off relations derived above can be verified with that given in Fig. 5.

V. CONCLUSION

To summarize, we have shown that there exists a trade-off relation between the coherence of a state and disturbance caused by a CPTP map or a measurement channel on a quantum system. For the measurement channel we found a tighter trade-off relation. Moreover, we obtained a trade-off relation for the quantum coherence, relative entropy of entanglement, and disturbance for a bipartite system. A similar relation was also obtained for the quantum coherence, quantum discord, and disturbance for a bipartite state. The trade-off relation for the coherence and disturbance has been illustrated for a weak measurement channel and other quantum channels. Our results capture the intuition that coherence, entanglement, and quantum discord for a quantum system should respect a trade-off relation with disturbance. Our results provide a deep physical meaning about the relation between quantum coherence and disturbance which can be widely applied in various contexts. We hope that these results will find interesting applications in sending single or composite systems under noisy channels that tend to lose quantum coherence and entanglement. If we wish to maintain coherence or entanglement or both, then we need to send the quantum states through a channel that does not disturb the system to a greater extent. In the future it will be interesting to see if other measures of coherence and entanglement respect the trade-off relation with disturbance.

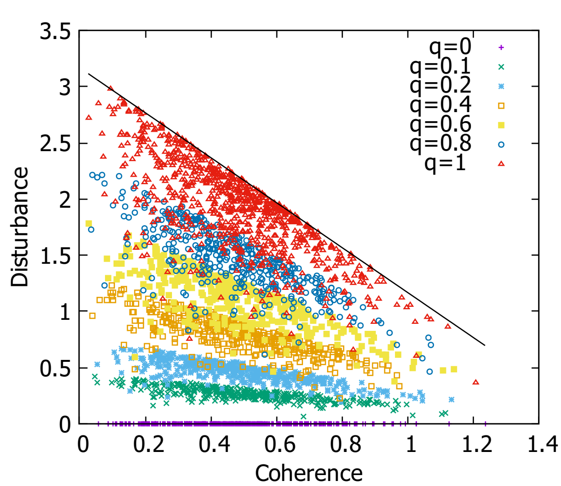

FIG. 5. Trade-off between coherence $C(\rho)$ and disturbance $D(\rho,\mathcal{E})$ for the amplitude damping channel applied on a qutrit state. The figure shows coherence along the X axis and disturbance along the Y axis. Random states were generated and coherence and entropy were calculated using the MATLAB package [55].

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APPENDIX

The choice of disturbance measure as given in Eq. (2) was motivated through a set of axioms and also was justified as that helps us to study how the quantum coherence may be degraded by a noisy quantum channel. However, one may ask if there exists a trade-off between other definitions of coherence and disturbance. Indeed, we find that the geometric measure of coherence defined in Ref. [37] obeys a trade-off with the fidelity-based disturbance measure given in Refs. [1,3,43]. For a state evolving under measurement as $\rho \rightarrow \mathcal{E}(\rho)$, disturbance can be defined using fidelity between initial and final states as

$$D(\rho,\mathcal{E}) = 1 - F(\rho,\mathcal{E}(\rho)), \tag{A1}$$

where $F(\rho,\mathcal{E}(\rho)) = \mathrm{Tr}\sqrt{\rho^{1/2}\mathcal{E}(\rho)\rho^{1/2}}$. It should be noted the disturbance in Eq. (A1) is reversible and is nonzero even for unitary evolution of a state. The geometric measure of coherence is given by

$$C_g(\rho) = 1 - \max_{\delta \epsilon I} F(\rho,\delta), \tag{A2}$$

where I is the set of all incoherent states. If the measurement process always leads to an incoherent state, i.e., $\mathcal{E}(\rho)\epsilon I$, then using Eqs. (A1) and (A2), we get the relation

$$C_g(\rho) \leqslant D(\rho,\mathcal{E}). \tag{A3}$$

This relation tells us that if the final state is an incoherent state, then the disturbance caused to the state will be at least equal to the amount of coherence in the system. If the system is in the eigenstate or the dephased state, then the coherence will be zero and the disturbance will also be zero as the state will no longer be disturbed. We present the relation (A3) using an example of the measurement channel acting on a single-qubit state. The Kraus operators for the measurement channel are projection operators Π_0 and Π_1 in the computational basis. Under the measurement channel, the system evolves as $\rho \rightarrow \mathcal{E}(\rho) = \Pi_0\rho\Pi_0 + \Pi_1\rho\Pi_1$.

By using Bloch vector representation the states ρ and $\mathcal{E}(\rho)$ can be represented as

$$\rho = \frac{\mathcal{I} + \vec{r}\cdot\vec{\sigma}}{2}, \quad \mathcal{E}(\rho) = \frac{\mathcal{I} + \vec{s}\cdot\vec{\sigma}}{2}.$$

Then the fidelity between ρ and $\mathcal{E}(\rho)$ has the form

$$F(\rho,\mathcal{E}(\rho)) = \tfrac{1}{2}[1 + \vec{r}\cdot\vec{s} + \sqrt{(1-|\vec{r}\,|^2)(1-|\vec{s}\,|^2)}].$$

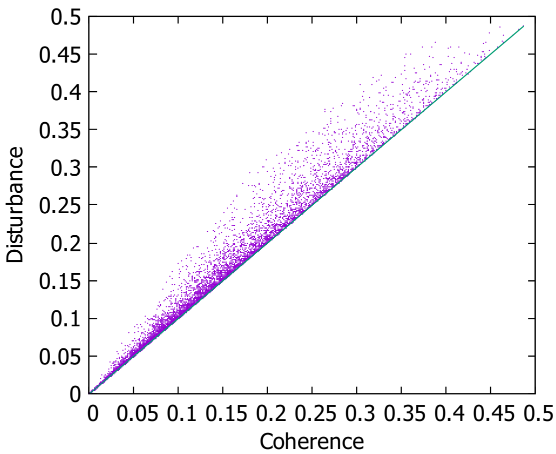

FIG. 6. Trade-off between geometric measure of coherence and fidelity-based disturbance measure plotted obeying the relation (A3). Random states were generated using the MATLAB package [55].

Therefore, we can write the disturbance using the Bloch vectors as

$$D(\rho,\mathcal{E}) = \tfrac{1}{2}[1 - \vec{r}\cdot\vec{s} - \sqrt{(1-|\vec{r}\,|^2)(1-|\vec{s}\,|^2)}]. \quad \text{(A4)}$$

The maximum value of fidelity of a qubit with an incoherent state is given by [57]

$$\max_{\delta\epsilon I} F(\rho,\delta) = \tfrac{1}{2}\big[1 + \sqrt{1 - r_x^2 - r_y^2}\big].$$

Now we can also express the geometric coherence using the Bloch vectors of the state ρ as

$$C_g(\rho) = \tfrac{1}{2}\big[1 - \sqrt{1 - r_x^2 - r_y^2}\big]. \quad \text{(A5)}$$

In Fig. 6 we see that the trade-off relation is respected.

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