Strong and uniform convergence in the teleportation simulation of bosonic Gaussian channels

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In the literature on the continuous-variable bosonic teleportation protocol due to Braunstein and Kimble [Phys. Rev. Lett. **80**, 869 (1998)], it is often loosely stated that this protocol converges to a perfect teleportation of an input state in the limit of ideal squeezing and ideal detection, but the exact form of this convergence is typically not clarified. In this paper, I explicitly clarify that the convergence is in the strong sense, and not the uniform sense, and furthermore that the convergence occurs for any input state to the protocol, including the infinite-energy Basel states defined and discussed here. I also prove, in contrast to the above result, that the teleportation simulations of pure-loss, thermal, pure-amplifier, amplifier, and additive-noise channels converge both strongly and uniformly to the original channels, in the limit of ideal squeezing and detection for the simulations. For these channels, I give explicit uniform bounds on the accuracy of their teleportation simulations. I then extend these uniform convergence results to particular multimode bosonic Gaussian channels. These convergence statements have important implications for mathematical proofs that make use of the teleportation simulation of bosonic Gaussian channels, some of which have to do with bounding their nonasymptotic secret-key-agreement capacities. As a by-product of the discussion given here, I confirm the correctness of the proof of such bounds from my joint work with Berta and Tomamichel from [Wilde, Tomamichel, and Berta, IEEE Trans. Inf. Theory **63**, 1792 (2017)]. Furthermore, I show that it is not necessary to invoke the energy-constrained diamond distance in order to confirm the correctness of this proof.

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I. INTRODUCTION

The quantum teleportation protocol is one of the most powerful primitives in quantum information theory [1]. By sharing entanglement and making use of a classical communication link, a sender can transmit an arbitrary quantum state to a receiver. In resource-theoretic language, the resources of a maximally entangled state of two qubits

$$|\Phi^+\rangle_{AB} = (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2} \tag{1}$$

and two classical bit channels can be used to simulate an ideal qubit channel from the sender to the receiver [2,3]. Generalizing this, a maximally entangled state of two qudits

$$|\Phi_d\rangle_{AB} = d^{-1/2}\sum_{i=0}^{d-1}|i\rangle_A|i\rangle_B \tag{2}$$

and two classical channels, each of dimension d, can be used to simulate an ideal d-dimensional quantum channel [1]. The teleportation primitive has been extended in multiple nontrivial ways, including a method to simulate an unideal channel using a noisy, mixed resource state [4, Sec. V] (see also [5–8]) and as a way to implement nonlocal quantum gates [9,10]. The former extension has been used to bound the rates at which quantum information can be conveyed over a quantum channel assisted by local operations and classical communication (LOCC) [4,8], and more generally, as a way to reduce a general LOCC-assisted protocol to one that consists of preparing a resource state followed by a single round of LOCC [4,8].

Due in part to the large experimental interest in bosonic continuous-variable quantum systems, given their practical applications [12,13], the teleportation protocol was extended to this paradigm [11] (see Fig. 1). The standard protocol begins with a sender and receiver sharing a two-mode squeezed vacuum state of the following form:

$$|\Phi(N_S)\rangle_{AB} \equiv \frac{1}{\sqrt{N_S+1}}\sum_{n=0}^{\infty}\left(\sqrt{\frac{N_S}{N_S+1}}\right)^n |n\rangle_A|n\rangle_B, \tag{3}$$

where $N_S \in [0,\infty)$ represents the squeezing strength and $\{|n\rangle\}_n$ denotes the photon-number basis. Suppose that the goal is to teleport a mode A'. The sender mixes, on a 50-50 beam splitter, the mode A' with the mode A of the state in (3). Afterward, the sender performs homodyne detection on the modes emerging from the beam splitter and forwards the measurement results over classical channels to the receiver, who possesses mode B of the above state. Finally, the receiver performs a unitary displacement operation on mode B. In the limit as $N_S \to \infty$ and in the limit of ideal homodyne detection, this continuous-variable teleportation protocol is often loosely stated in the literature to simulate an ideal channel on any state of the mode A', such that this state is prepared in mode B after the protocol is finished. Due to the lack of a precise notion of convergence being given, there is the potential for confusion regarding mathematical proofs that make use of the continuous-variable teleportation protocol.

With this in mind, one purpose of this paper is to clarify the precise kind of convergence that occurs in continuous-variable quantum teleportation, which is typically not discussed in

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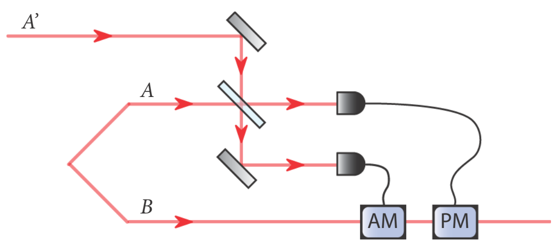

FIG. 1. Depiction of the bosonic continuous-variable teleportation protocol from [11], described in the main text. "AM" and "PM" denote amplitude and phase modulators, which implement the displacement operator needed in the teleportation protocol.

the literature on this topic. In particular, I prove that the convergence is in the strong topology sense, and not in the uniform topology sense (see, e.g., [14, Sec. 3] for discussions of these notions of convergence). I then show how to extend this strong convergence result to the teleportation simulation of n parallel ideal channels, and I also show how this strong convergence extends to the teleportation simulation of n ideal channels that could be used in any context.

Strong convergence and uniform convergence are then discussed for the teleportation simulation of bosonic Gaussian channels. For this latter case, and in contrast to the result discussed above for the continuous-variable teleportation protocol, I prove that the teleportation simulations of the pure-loss, thermal, pure-amplifier, amplifier, and additive-noise channels converge both strongly and uniformly to the original channels, in the limit of ideal squeezing and detection for the simulations. Here, I give explicit uniform bounds on the accuracy of the teleportation simulations of these channels, and I suspect that these bounds will be useful in future applications. After this development, I then extend these uniform convergence results to particular multimode bosonic Gaussian channels.

These convergence results are important, even if they might be implicit in prior works, as they provide meaningful clarification of mathematical proofs that make use of teleportation simulation, such as those given in recent work on bounding nonasymptotic secret-key-agreement capacities. In particular, one can employ these convergence statements to confirm the correctness of the proof of such bounds given in my joint work with Berta and Tomamichel from [15]. Furthermore, these strong convergence statements can be used to conclude that the energy-constrained diamond distance is *not necessary* to arrive at a proof of the bounds from [15]. Another by-product of the discussion given in this paper is that it is clarified that the methods of [15] allow for bounding secret-key rates of rather general protocols that make use of infinite-energy states, such as the Basel states in (20) and (124). Although there should be great skepticism concerning whether these infinite-energy Basel states could be generated in practice, this latter by-product is nevertheless of theoretical interest.

The rest of the paper proceeds as follows. In the next section, I discuss the precise form of convergence that occurs in continuous-variable quantum teleportation and then develop various extensions of this notion of convergence. I then prove that teleportation simulations of the pure-loss, thermal, pure-amplifier, amplifier, and additive-noise channels converge both strongly and uniformly to the original channels in the limit of ideal squeezing and detection for the simulations. The uniform convergence results are then extended to the teleportation simulations of particular multimode bosonic Gaussian channels. Section III gives a physical interpretation of the aforementioned convergence results, by means of the CV teleportation game. After that, Sec. IV briefly reviews what is meant by a secret-key-agreement protocol and nonasymptotic secret-key-agreement capacity. Finally, in Sec. V, I review the proof of [15, Theorem 24] and carefully go through some of its steps therein, confirming its correctness, while showing how the strong convergence of teleportation simulation applies. In Sec. VI, I conclude with a brief summary and a discussion.

II. NOTIONS OF QUANTUM CHANNEL CONVERGENCE, WITH APPLICATIONS TO TELEPORTATION SIMULATION

One main technical issue discussed in this paper is how the continuous-variable bosonic teleportation protocol from [11] converges to an identity channel in the limit of infinite squeezing and ideal detection. This issue is often not explicitly clarified in the literature on the topic, even though it has been implicit for some time in various works that the convergence is to be understood in the strong sense (topology of strong convergence), and *not* necessarily the uniform sense (topology of uniform convergence) (see, e.g., [14, Sec. 3]). For example, in the original paper [11], the following statement is given regarding this issue:

> "Clearly, for $r \to \infty$ the teleported state of Eq. (4) reproduces the original unknown state."

Although it is clear that this statement implies convergence in the strong sense, it could be helpful to clarify this point, and the purpose of this section is to do so.

In what follows, I first recall the definitions of strong and uniform convergence from [14, Sec. 3]. I then discuss the precise form of convergence that occurs in continuous-variable bosonic teleportation and show how strong convergence and uniform convergence are extremely different in the setting of continuous-variable teleportation. After that, I prove that strong convergence of a channel sequence implies strong convergence of n-fold tensor powers of these channels and follow this with a proof that strong convergence of a channel sequence implies strong convergence of n uses of these channels in any context in which they could be invoked. I also prove that the teleportation simulations of pure-loss, thermal, pure-amplifier, amplifier, and additive-noise channels converge both strongly and uniformly to the original channels, in the limit of ideal squeezing and detection for the simulations. The uniform convergence results are then extended to the teleportation simulations of particular multimode bosonic Gaussian channels.

A. Definitions of strong and uniform convergence

Before discussing the precise statement of convergence in the continuous-variable bosonic teleportation protocol, let us begin by recalling general definitions of strong and uniform convergence from [14, Sec. 3]. I adopt slightly different definitions from those given in [14, Sec. 3], in order to suit the needs of this paper, but note that they are equivalent to the original definitions as shown in [14] and [16, Lemma 2]. In this context, see also [17]. Let $\{\mathcal{N}^k_{A\to B}\}_k$ denote a sequence of quantum channels (completely positive, trace-preserving maps), which each accept as input a trace-class operator acting on a separable Hilbert space $\mathcal{H}_A$ and output a trace-class operator acting on a separable Hilbert space $\mathcal{H}_B$. This sequence converges *strongly* to a channel $\mathcal{N}_{A\to B}$ if for all density operators ρ_{RA} acting on $\mathcal{H}_R \otimes \mathcal{H}_A$, where $\mathcal{H}_R$ is an arbitrary, auxiliary separable Hilbert space, the following limit holds:

$$\lim_{k\to\infty} \varepsilon(k,\rho_{RA}) = 0, \tag{4}$$

where the infidelity is defined as

$$\varepsilon(k,\rho_{RA}) \equiv 1 - F\big(\big(\mathrm{id}_R \otimes \mathcal{N}^k_{A\to B}\big)(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(\rho_{RA})\big), \tag{5}$$

id_R denotes the identity map on the auxiliary space, and $F(\tau,\omega) \equiv \|\sqrt{\tau}\sqrt{\omega}\|_1^2$ is the quantum fidelity [18], defined for density operators τ and ω. The quantum fidelity obeys a data processing inequality, which is the statement that

$$F(\mathcal{M}(\tau),\mathcal{M}(\omega)) \geqslant F(\tau,\omega), \tag{6}$$

for states τ and ω and a quantum channel $\mathcal{M}$. We can summarize strong convergence more compactly as the following mathematical statement:

$$\sup_{\rho_{RA}} \lim_{k\to\infty} \varepsilon(k,\rho_{RA}) = 0. \tag{7}$$

Due to purification, the Schmidt decomposition theorem, and the data-processing inequality for fidelity, we find that for every mixed state ρ_{RA}, there exists a pure state $\psi_{R'A}$ with the auxiliary Hilbert space $\mathcal{H}_{R'}$ taken to be isomorphic to $\mathcal{H}_A$, such that

$$\varepsilon(k,\rho_{RA}) \leqslant \varepsilon(k,\psi_{R'A}). \tag{8}$$

Thus, when considering strong convergence, it suffices to consider only pure states $\psi_{R'A}$, so that

$$\sup_{\rho_{RA}} \lim_{k\to\infty} \varepsilon(k,\rho_{RA}) = \sup_{\psi_{R'A}} \lim_{k\to\infty} \varepsilon(k,\psi_{R'A}). \tag{9}$$

Strong convergence is strictly different from uniform convergence [14, Sec. 3], which amounts to a swap of the supremum and the limit in (7). That is, the channel sequence $\{\mathcal{N}^k_{A\to B}\}_k$ converges *uniformly* to the channel $\mathcal{N}_{A\to B}$ if the following holds:

$$\lim_{k\to\infty} \sup_{\rho_{RA}} \varepsilon(k,\rho_{RA}) = 0. \tag{10}$$

Even though this swap might seem harmless and is of no consequence in finite dimensions, the issue is important to consider in infinite-dimensional contexts, especially for bosonic channels. That is, a sequence of channels could converge in the strong sense, but be as far as possible from converging in the uniform sense, and an example of this behavior is given in the next subsection.

It was also stressed in [14] that the topology of uniform convergence is too strong for physical applications and should typically not be considered.

B. Strong and uniform convergence considerations for continuous-variable teleportation

We now turn our attention to convergence in the continuous-variable bosonic teleportation protocol and focus on [11, Eq. (9)], which states that an unideal continuous-variable bosonic teleportation protocol with input mode A realizes the following additive-noise quantum Gaussian channel $\mathcal{T}^{\bar{\sigma}}_A$ on an input density operator ρ_A:

$$\rho_A \to \mathcal{T}^{\bar{\sigma}}_A(\rho_A) \equiv \int d^2\alpha\, G_{\bar{\sigma}}(\alpha)\, D(\alpha)\rho_A D(-\alpha), \tag{11}$$

where $D(\alpha)$ is a displacement operator [13] and

$$G_{\bar{\sigma}}(\alpha) \equiv \frac{1}{\pi\bar{\sigma}} \exp\left(-\frac{|\alpha|^2}{\bar{\sigma}}\right) \tag{12}$$

is a zero-mean, circularly symmetric complex Gaussian probability density function with variance $\bar{\sigma} > 0$. To be clear, the integral in (11) is over the whole complex plane $\alpha \in \mathbb{C}$. For an explicit proof of (11), one can also consult [19,20]. The variance parameter $\bar{\sigma}$ quantifies unideal squeezing and unideal detection. Thus, for any $\bar{\sigma} > 0$, the teleportation channel $\mathcal{T}^{\bar{\sigma}}_A$ is unideal and intuitively becomes ideal in the limit $\bar{\sigma} \to 0$. However, it is this convergence that needs to be made precise. To examine this, we need a measure of the channel input-output dissimilarity, and the entanglement infidelity is a good choice, which is essentially the choice made in [11] for quantifying the performance of unideal bosonic teleportation. For a fixed pure state $\psi_{RA} \equiv |\psi\rangle\langle\psi|_{RA}$ of modes R and A, the entanglement infidelity of the channel $\mathcal{T}^{\bar{\sigma}}_A$ with respect to ψ_{RA} is defined as

$$\varepsilon(\bar{\sigma},\psi_{RA}) \equiv 1 - \langle\psi|_{RA}\big(\mathrm{id}_R \otimes \mathcal{T}^{\bar{\sigma}}_A\big)(|\psi\rangle\langle\psi|_{RA})|\psi\rangle_{RA}. \tag{13}$$

Examining [11, Eq. (11)], we see that the entanglement infidelity can alternatively be written as

$$\varepsilon(\bar{\sigma},\psi_{RA}) = 1 - \int d^2\alpha\, G_{\bar{\sigma}}(\alpha)|\chi_{\psi_A}(\alpha)|^2, \tag{14}$$

where $\chi_{\psi_A}(\alpha) = \mathrm{Tr}\{D(\alpha)\psi_A\}$ is the Wigner characteristic function of the reduced density operator ψ_A. By applying the Hölder inequality, we conclude that $\chi_{\psi_A}(\alpha)$ is bounded for all $\alpha \in \mathbb{C}$ because

$$|\mathrm{Tr}\{D(\alpha)\psi_A\}| \leqslant \|D(\alpha)\|_\infty \|\psi_A\|_1 = 1. \tag{15}$$

Exploiting the continuity of $\chi_{\psi_A}(\alpha)$ at $\alpha = 0$ and the fact that $\chi_{\psi_A}(0) = 1$ [21, Theorem 5.4.1], as well as invoking the boundedness of $\chi_{\psi_A}(\alpha)$ and [22, Theorem 9.8] regarding the convergence of nascent delta functions, we then conclude that for a given state ψ_{RA}, the following strong convergence holds:

$$\lim_{\bar{\sigma}\to 0} \varepsilon(\bar{\sigma},\psi_{RA}) = 0, \tag{16}$$

which can be written, as before, more compactly as

$$\sup_{\psi_{RA}} \lim_{\bar{\sigma}\to 0} \varepsilon(\bar{\sigma},\psi_{RA}) = 0. \tag{17}$$

Note that, as before and due to (9), Eq. (17) implies that

$$\sup_{\rho_{RA}} \lim_{\bar{\sigma}\to 0} \varepsilon(\bar{\sigma},\rho_{RA}) = 0 \tag{18}$$

for any mixed state ρ_{RA} where

$$\varepsilon(\bar{\sigma},\rho_{RA}) \equiv 1 - F\big(\rho_{RA},\big(\mathrm{id}_R \otimes \mathcal{T}^{\bar{\sigma}}_A\big)(\rho_{RA})\big) \tag{19}$$

and $\mathcal{H}_R$ is an arbitrary auxiliary separable Hilbert space.

One should note here that the convergence in (17) already calls into question any claim regarding the necessity of an energy constraint for the states that are to be teleported using the continuous-variable teleportation protocol. Clearly, the state ψ_{RA} to be teleported could be chosen as the following Basel state:

$$|\beta\rangle_{RA} = \sqrt{\frac{6}{\pi^2}}\sum_{n=1}^{\infty}\sqrt{\frac{1}{n^2}}|n\rangle_R|n\rangle_A, \tag{20}$$

which has mean photon number equal to ∞, but it also satisfies (16). Such a state is called a "Basel state," due to its normalization factor being connected with the well-known Basel problem, which establishes that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$. For $\hat{n} = \sum_{n=0}^{\infty} n|n\rangle\langle n|$ the photon-number operator, one can easily check that the mean photon number $\text{Tr}\{(\hat{n}_R + \hat{n}_A)\beta_{RA}\} = \infty$, due to the presence of the divergent harmonic series after $\hat{n}_A$ multiplies the reduced density operator β_A. Thus, the only constraint needed for the convergence in (17) is that the state to be teleported be a state (i.e., normalizable). Reference [23] claims that it is necessary for there to be an energy constraint for strong convergence in teleportation; however, the example of the Basel states given above proves that such an energy constraint is not necessary.

It is also important to note that an exchange of the limit and the supremum in (17) leads to a drastically different conclusion:

$$\lim_{\bar{\sigma}\to 0}\sup_{\psi_{RA}} \varepsilon(\bar{\sigma},\psi_{RA}) = 1. \tag{21}$$

The drastic difference is due to the fact that for all fixed $\bar{\sigma} > 0$, one can find a sequence of states $\{\psi^k_{RA}\}_k$ such that $\sup_k \varepsilon(\bar{\sigma},\psi^k_{RA}) = 1$, establishing (21). For example, one could pick each ψ^k_{RA} to be a two-mode squeezed vacuum state with squeezing parameter increasing with increasing k. In the limit of large squeezing, the ideal channel and the additive-noise channel for any $\bar{\sigma} > 0$ become perfectly distinguishable, having infidelity approaching one, implying (21). One can directly verify this calculation by employing the covariance matrix representation of the two-mode squeezed vacuum and the additive-noise channel, as well as the overlap formula in [13, Eq. (4.51)] to calculate entanglement fidelity.

I now give details of the aforementioned calculation, regarding how the continuous-variable bosonic teleportation protocol from [11] does not converge uniformly to an ideal channel. Consider a two-mode squeezed vacuum state $\Phi(N_S)$ with mean photon number N_S for one of its reduced modes, as defined in (3). Such a state has a Wigner-function covariance matrix [13] as follows:

$$\begin{bmatrix} 2N_S+1 & 2\sqrt{N_S(N_S+1)} \\ 2\sqrt{N_S(N_S+1)} & 2N_S+1 \end{bmatrix} \oplus \begin{bmatrix} 2N_S+1 & -2\sqrt{N_S(N_S+1)} \\ -2\sqrt{N_S(N_S+1)} & 2N_S+1 \end{bmatrix}. \tag{22}$$

After sending one mode of this state through an additive-noise channel with variance $\bar{\sigma}$ (corresponding to an unideal continuous-variable bosonic teleportation), the covariance matrix becomes as follows, corresponding to a state $\tau(N_S,\bar{\sigma})$:

$$\begin{bmatrix} 2N_S+1 & 2\sqrt{N_S(N_S+1)} \\ 2\sqrt{N_S(N_S+1)} & 2N_S+1+2\bar{\sigma} \end{bmatrix} \oplus \begin{bmatrix} 2N_S+1 & -2\sqrt{N_S(N_S+1)} \\ -2\sqrt{N_S(N_S+1)} & 2N_S+1+2\bar{\sigma} \end{bmatrix}. \tag{23}$$

The overlap $\text{Tr}\{\omega\sigma\}$ of two zero-mean, two-mode Gaussian states ω and σ is given by [13, Eq. (4.51)]

$$\text{Tr}\{\omega\sigma\} = 4/\sqrt{\det(V_\omega + V_\sigma)}, \tag{24}$$

where V_ω and V_σ are the Wigner-function covariance matrices of ω and σ, respectively. We can then employ this formula to calculate the overlap

$$\langle\Phi(N_S)|\tau(N_S,\bar{\sigma})|\Phi(N_S)\rangle = \text{Tr}\{\Phi(N_S)\tau(N_S,\bar{\sigma})\} \tag{25}$$

as

$$\langle\Phi(N_S)|\tau(N_S,\bar{\sigma})|\Phi(N_S)\rangle = \frac{1}{\bar{\sigma}+2\bar{\sigma}N_S+1}. \tag{26}$$

Thus, for a fixed $\bar{\sigma} > 0$ and in the limit as $N_S \to \infty$, we find that $1 - \langle\Phi(N_S)|\tau(N_S,\bar{\sigma})|\Phi(N_S)\rangle \to 1$, so that the continuous-variable bosonic teleportation protocol from [11] does not converge uniformly to an ideal channel.

To summarize, the kind of convergence considered in (17) is the strong sense (topology of strong convergence), whereas the kind of convergence considered in (21) is the uniform sense (topology of uniform convergence) (see, e.g., [14, Sec. 3]). That is, (17) demonstrates that unideal continuous-variable bosonic teleportation converges *strongly* to an ideal quantum channel in the limit of ideal squeezing and detection, whereas (21) demonstrates that it does *not* converge *uniformly*.

C. Strong and uniform convergence for tensor-power channels

A natural consideration to make in the context of quantum Shannon theory is the convergence of n uses of a channel on a general state of n systems, where n is a positive integer. To this end, suppose that the strong convergence in (7) holds for the sequence $\{\mathcal{N}^k_{A\to B}\}_k$ of channels. Then, it immediately follows that the sequence $\{(\mathcal{N}^k_{A\to B})^{\otimes n}\}_k$ converges strongly to $\mathcal{N}^{\otimes n}_{A\to B}$. Indeed, for an arbitrary density operator ρ_{RA^n} acting on $\mathcal{H}_R \otimes \mathcal{H}_A^{\otimes n}$, we are now interested in bounding the infidelity for the tensor-power channels $(\mathcal{N}^k_{A\to B})^{\otimes n}$ and $\mathcal{N}^{\otimes n}_{A\to B}$:

$$\varepsilon^{(n)}(k,\rho_{RA^n}) \equiv 1 - F\Big(\big(\text{id}_R \otimes \big(\mathcal{N}^k_{A\to B}\big)^{\otimes n}\big)(\rho_{RA^n}), \big(\text{id}_R \otimes \mathcal{N}^{\otimes n}_{A\to B}\big)(\rho_{RA^n})\Big), \tag{27}$$

in the limit as $k \to \infty$. By employing the fact that

$$P(\tau,\omega) \equiv \sqrt{1 - F(\tau,\omega)} \tag{28}$$

obeys the triangle inequality [24–27], we conclude that, for an arbitrary density operator ρ_{RA^n}, the following inequality holds:

$$P\Big(\big(\text{id}_R \otimes \big(\mathcal{N}^k_{A\to B}\big)^{\otimes n}\big)(\rho_{RA^n}), \big(\text{id}_R \otimes \mathcal{N}^{\otimes n}_{A\to B}\big)(\rho_{RA^n})\Big) \leqslant \sum_{i=1}^{n} P\Big(\big(\text{id}_R \otimes \big(\mathcal{N}^k_{A\to B}\big)^{\otimes i} \otimes \mathcal{N}^{\otimes n-i}_{A\to B}\big)(\rho_{RA^n}), \big(\text{id}_R \otimes \big(\mathcal{N}^k_{A\to B}\big)^{\otimes i-1} \otimes \mathcal{N}^{\otimes n-i+1}_{A\to B}\big)(\rho_{RA^n})\Big) \tag{29}$$

$$\leqslant \sum_{i=1}^{n} P\big(\big(\mathrm{id}_R \otimes \mathrm{id}_A^{\otimes i-1} \otimes \mathcal{N}^k_{A\to B} \otimes \mathcal{N}^{\otimes n-i}_{A\to B}\big)(\rho_{RA^n}),\ \big(\mathrm{id}_R \otimes \mathrm{id}_A^{\otimes i-1} \otimes \mathcal{N}_{A\to B} \otimes \mathcal{N}^{\otimes n-i}_{A\to B}\big)(\rho_{RA^n})\big). \tag{30}$$

The first inequality follows from the triangle inequality for $P(\tau,\omega)$, and the second follows from data processing for the fidelity under the channel

$$\mathrm{id}_R \otimes \big(\mathcal{N}^k_{A\to B}\big)^{\otimes i-1} \otimes \mathrm{id}_A^{\otimes n-i+1} \tag{31}$$

acting on the states

$$\big(\mathrm{id}_R \otimes \mathrm{id}_A^{\otimes i-1} \otimes \mathcal{N}^k_{A\to B} \otimes \mathcal{N}^{\otimes n-i}_{A\to B}\big)(\rho_{RA^n}), \tag{32}$$

$$\big(\mathrm{id}_R \otimes \mathrm{id}_A^{\otimes i-1} \otimes \mathcal{N}_{A\to B} \otimes \mathcal{N}^{\otimes n-i}_{A\to B}\big)(\rho_{RA^n}). \tag{33}$$

The method used in (29) and (30) is related to the telescoping approach of [28], employed in the context of continuity of quantum channel capacities (see the proof of [28, Theorem 11] in particular). Now employing strong convergence of the channel sequence $\{\mathcal{N}^k_{A\to B}\}_k$ and the fact that $\mathcal{N}^{\otimes n-i}_{A\to B}(\rho_{RA^n})$ is a *fixed state* independent of k for each $i \in \{1,\dots,n\}$, we conclude that for all density operators ρ_{RA^n}

$$\lim_{k\to\infty} \varepsilon^{(n)}(k,\rho_{RA^n}) = 0. \tag{34}$$

This result can be summarized more compactly as

$$\sup_{\rho_{RA}} \lim_{k\to\infty} \varepsilon(k,\rho_{RA}) = 0 \implies \sup_{\rho_{RA^n}} \lim_{k\to\infty} \varepsilon^{(n)}(k,\rho_{RA^n}) = 0. \tag{35}$$

That is, the strong convergence of the channel sequence $\{\mathcal{N}^k_{A\to B}\}_k$ implies the strong convergence of the tensor-power channel sequence $\{(\mathcal{N}^k_{A\to B})^{\otimes n}\}_k$ for any finite n. For the case of continuous-variable teleportation, we have that n unideal teleportations with the same performance correspond to the tensor-power channel $(\mathcal{T}^{\bar{\sigma}}_A)^{\otimes n}$. By appealing to (17) and (35), or alternatively employing Wigner characteristic functions [21, Theorem 5.4.1] and [22, Theorem 9.8], the following convergence holds: for a pure state ψ_{RA^n}, we have that

$$\lim_{\bar{\sigma}\to 0} \varepsilon^{(n)}(\bar{\sigma},\psi_{RA^n}) = 0, \tag{36}$$

where

$$\varepsilon^{(n)}(\bar{\sigma},\psi_{RA^n}) \equiv 1 - \langle\psi|_{RA^n}\big(\mathrm{id}_R \otimes \big(\mathcal{T}^{\bar{\sigma}}_A\big)^{\otimes n}\big)(|\psi\rangle\langle\psi|_{RA^n})|\psi\rangle_{RA^n}. \tag{37}$$

Again, more compactly, this is the same as

$$\sup_{\psi_{RA^n}} \lim_{\bar{\sigma}\to 0} \varepsilon^{(n)}(\bar{\sigma},\psi_{RA^n}) = 0 \tag{38}$$

and drastically different from

$$\lim_{\bar{\sigma}\to 0} \sup_{\psi_{RA^n}} \varepsilon^{(n)}(\bar{\sigma},\psi_{RA^n}) = 1. \tag{39}$$

I end this subsection by noting the following proposition, having to do with the strong convergence of parallel compositions of strongly converging channel sequences. In fact, the parallel composition result in (35) could be proven by using only the following proposition and iterating.

Proposition 1. Let $\{\mathcal{N}^k_{A_1\to B_1}\}_k$ be a channel sequence that converges strongly to a channel $\mathcal{N}_{A_1\to B_1}$, and let $\{\mathcal{M}^k_{A_2\to B_2}\}_k$ be a channel sequence that converges strongly to a channel $\mathcal{M}_{A_2\to B_2}$. Then, the channel sequence $\{\mathcal{N}^k_{A_1\to B_1} \otimes \mathcal{M}^k_{A_2\to B_2}\}_k$ converges strongly to $\mathcal{N}_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2}$.

Proof. A proof is similar to what is given above, and I give it for completeness. Let $\rho_{RA_1A_2}$ be an arbitrary state. Consider that

$$\begin{aligned}
&P\big(\big(\mathrm{id}_R \otimes \mathcal{N}^k_{A_1\to B_1} \otimes \mathcal{M}^k_{A_2\to B_2}\big)(\rho_{RA_1A_2}),(\mathrm{id}_R \otimes \mathcal{N}_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2})(\rho_{RA_1A_2})\big)\\
&\leqslant P\big(\big(\mathrm{id}_R \otimes \mathcal{N}^k_{A_1\to B_1} \otimes \mathcal{M}^k_{A_2\to B_2}\big)(\rho_{RA_1A_2}),\big(\mathrm{id}_R \otimes \mathcal{N}^k_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2}\big)(\rho_{RA_1A_2})\big)\\
&\quad + P\big(\big(\mathrm{id}_R \otimes \mathcal{N}^k_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2}\big)(\rho_{RA_1A_2}),(\mathrm{id}_R \otimes \mathcal{N}_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2})(\rho_{RA_1A_2})\big)\\
&\leqslant P\big(\big(\mathrm{id}_{RA_1} \otimes \mathcal{M}^k_{A_2\to B_2}\big)(\rho_{RA_1A_2}),\mathcal{M}_{A_2\to B_2}(\rho_{RA_1A_2})\big) + P\big(\big(\mathrm{id}_R \otimes \mathcal{N}^k_{A_1\to B_1} \otimes \mathrm{id}_{A_2}\big)(\rho_{RA_1A_2}),(\mathrm{id}_R \otimes \mathcal{N}_{A_1\to B_1} \otimes \mathrm{id}_{A_2})(\rho_{RA_1A_2})\big).
\end{aligned} \tag{40}$$

The first inequality follows from the triangle inequality and the second from data processing. Using the strong convergence of $\{\mathcal{N}^k_{A_1\to B_1}\}_k$ and $\{\mathcal{M}^k_{A_2\to B_2}\}_k$, applying the inequality in (40), and taking the limit $k\to\infty$, we find that

$$\lim_{k\to\infty} P\big(\big(\mathcal{N}^k_{A_1\to B_1} \otimes \mathcal{M}^k_{A_2\to B_2}\big)(\rho_{RA_1A_2}),\ (\mathcal{N}_{A_1\to B_1} \otimes \mathcal{M}_{A_2\to B_2})(\rho_{RA_1A_2})\big) = 0. \tag{41}$$

Since the state $\rho_{RA_1A_2}$ was arbitrary, the proof is complete. ■

D. Strong and uniform convergence in arbitrary contexts

The most general way to distinguish n uses of two different quantum channels is by means of an adaptive protocol. Such adaptive channel discrimination protocols have been considered extensively in the literature in the context of finite-dimensional quantum channel discrimination (see, e.g., [29–34]). However, to the best of my knowledge, the issues of strong and uniform convergence have not yet been considered explicitly in the literature in the context of infinite-dimensional channel discrimination using adaptive strategies. The purpose of this section is to clarify these issues by defining strong

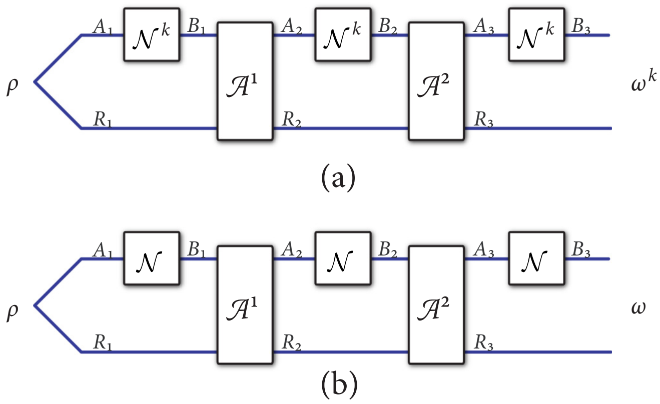

FIG. 2. Adaptive protocol for distinguishing three uses of the channel $\mathcal{N}^k_{A\to B}$ from three uses of $\mathcal{N}_{A\to B}$. The protocol is denoted by $\mathcal{P}^{(3)}$ and consists of state preparation $\rho_{R_1A_1}$, as well as the channels $\mathcal{A}^{(1)}_{R_1B_1\to R_2A_2}$ and $\mathcal{A}^{(2)}_{R_2B_2\to R_3A_3}$. (a) The protocol $\mathcal{P}^{(3)}$ is used with three uses of the channel $\mathcal{N}^k_{A\to B}$, and the final state is $\omega^k_{R_3B_3}$. (b) The protocol $\mathcal{P}^{(3)}$ is used with three uses of the channel $\mathcal{N}_{A\to B}$, and the final state is $\omega_{R_3B_3}$. Strong convergence of the channel sequence $\{\mathcal{N}^k_{A\to B}\}_k$ to $\mathcal{N}_{A\to B}$ implies that, given a fixed protocol $\mathcal{P}^{(3)}$, the infidelity of the states $\omega^k_{R_3B_3}$ and $\omega_{R_3B_3}$ converges to zero in the limit as $k\to\infty$.

and uniform convergence in this general context and then to prove explicitly that strong convergence of a channel sequence $\{\mathcal{N}^k_{A\to B}\}_k$ to $\mathcal{N}_{A\to B}$ implies strong convergence of n uses of each channel in $\{\mathcal{N}^k_{A\to B}\}_k$ to n uses of $\mathcal{N}_{A\to B}$ in the rather general sense described below.

To clarify what is meant by an adaptive protocol for channel discrimination, suppose that the task is to distinguish n uses of the channel $\mathcal{N}^k_{A\to B}$ from n uses of the channel $\mathcal{N}_{A\to B}$. The most general protocol for doing so begins with the preparation of a state $\rho_{R_1A_1}$, where system A_1 is isomorphic to the channel input system A and R_1 corresponds to an arbitrary auxiliary separable Hilbert space. The system A_1 is then fed in to the first channel use of $\mathcal{N}^k_{A\to B}$ or $\mathcal{N}_{A\to B}$, depending on which of these channels is chosen from the start. The resulting state is then either

$$\mathcal{N}^k_{A\to B}(\rho_{R_1A_1}) \quad \text{or} \quad \mathcal{N}_{A\to B}(\rho_{R_1A_1}), \tag{42}$$

depending on which channel is selected, and where I have omitted the identity map on R_1 for simplicity. After this, the discriminator applies a quantum channel $\mathcal{A}^{(1)}_{R_1B_1\to R_2A_2}$, where R_2 corresponds to another arbitrary separable Hilbert space, which need not be isomorphic to R_1, and A_2 corresponds to a separable Hilbert space isomorphic to the channel input A. The discriminator then calls the second use of $\mathcal{N}^k_{A\to B}$ or $\mathcal{N}_{A\to B}$, such that the state is now either

$$\left(\mathcal{N}^k_{A_2\to B_2}\circ\mathcal{A}^{(1)}_{R_1B_1\to R_2A_2}\circ\mathcal{N}^k_{A_1\to B_1}\right)(\rho_{R_1A_1}) \tag{43}$$

or

$$\left(\mathcal{N}_{A_2\to B_2}\circ\mathcal{A}^{(1)}_{R_1B_1\to R_2A_2}\circ\mathcal{N}_{A_1\to B_1}\right)(\rho_{R_1A_1}). \tag{44}$$

This process continues for n channel uses, and then the final state is either

$$\omega^k_{R_nB_n} \equiv \left(\mathcal{N}^k_{A_n\to B_n}\circ\left[\bigcirc_{j=1}^{n-1}\mathcal{A}^{(j)}_{R_jB_j\to R_{j+1}A_{j+1}}\circ\mathcal{N}^k_{A_j\to B_j}\right]\right)(\rho_{R_1A_1}) \tag{45}$$

or

$$\omega_{R_nB_n} \equiv \left(\mathcal{N}_{A_n\to B_n}\circ\left[\bigcirc_{j=1}^{n-1}\mathcal{A}^{(j)}_{R_jB_j\to R_{j+1}A_{j+1}}\circ\mathcal{N}_{A_j\to B_j}\right]\right)(\rho_{R_1A_1}). \tag{46}$$

Let $\mathcal{P}^{(n)}$ denote the full protocol, which consists of the state preparation $\rho_{R_1A_1}$ and the $n-1$ channels $\{\mathcal{A}^{(j)}_{R_jB_j\to R_{j+1}A_{j+1}}\}_{j=1}^{n-1}$. The infidelity in this case, for the fixed protocol $\mathcal{P}^{(n)}$, is then equal to

$$\varepsilon^{(n)}_{\rm ad}(k,\mathcal{P}^{(n)}) \equiv 1 - F\left(\omega^k_{R_nB_n},\omega_{R_nB_n}\right). \tag{47}$$

Figure 2 depicts these channels and states, which are used in a general adaptive strategy to discriminate three uses of $\mathcal{N}^k_{A\to B}$ from three uses of $\mathcal{N}_{A\to B}$.

In this general context, strong convergence corresponds to the following statement: for a given protocol $\mathcal{P}^{(n)}$, the following limit holds:

$$\lim_{k\to\infty}\varepsilon^{(n)}_{\rm ad}(k,\mathcal{P}^{(n)}) = 0 \tag{48}$$

or, more compactly,

$$\sup_{\mathcal{P}^{(n)}}\lim_{k\to\infty}\varepsilon^{(n)}_{\rm ad}(k,\mathcal{P}^{(n)}) = 0. \tag{49}$$

Uniform convergence again corresponds to a swap of the supremum and limit

$$\lim_{k\to\infty}\sup_{\mathcal{P}^{(n)}}\varepsilon^{(n)}_{\rm ad}(k,\mathcal{P}^{(n)}) = 0, \tag{50}$$

and again, it should typically be avoided in physical applications as it is too strong and not needed for most purposes, following the suggestions of [14, Sec. 3].

I now explicitly show that strong convergence of the sequence $\{\mathcal{N}^k_{A\to B}\}_k$ implies strong convergence of n uses of each channel in this sequence in this general sense. The proof is elementary and similar to that in (29) and (30), making use of the triangle inequality and data processing of fidelity. It bears similarities to *numerous* prior results in the literature [33,35–41], in which adaptive protocols were analyzed. For simplicity, we can focus on the case of $n=3$ and then the proof is easily extended. Begin by considering a *fixed protocol* $\mathcal{P}^{(3)}$. Then, consider that

$$\begin{aligned}\sqrt{\varepsilon^{(3)}_{\rm ad}(k,\mathcal{P}^{(3)})} &= P\left[\omega^k_{R_3B_3},\omega_{R_3B_3}\right]\\ &= P[(\mathcal{N}^k\circ\mathcal{A}^{(2)}\circ\mathcal{N}^k\circ\mathcal{A}^{(1)}\circ\mathcal{N}^k)(\rho_{R_1A_1}),(\mathcal{N}\circ\mathcal{A}^{(2)}\circ\mathcal{N}\circ\mathcal{A}^{(1)}\circ\mathcal{N})(\rho_{R_1A_1})]\\ &\leqslant P[(\mathcal{N}^k\circ\mathcal{A}^{(2)}\circ\mathcal{N}^k\circ\mathcal{A}^{(1)}\circ\mathcal{N}^k)(\rho_{R_1A_1}),(\mathcal{N}^k\circ\mathcal{A}^{(2)}\circ\mathcal{N}^k\circ\mathcal{A}^{(1)}\circ\mathcal{N})(\rho_{R_1A_1})]\end{aligned}$$

$$
\begin{aligned}
&+ P[(\mathcal{N}^k \circ \mathcal{A}^{(2)} \circ \mathcal{N}^k \circ \mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1}),(\mathcal{N}^k \circ \mathcal{A}^{(2)} \circ \mathcal{N} \circ \mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1})] \\
&+ P[(\mathcal{N}^k \circ \mathcal{A}^{(2)} \circ \mathcal{N} \circ \mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1}),(\mathcal{N} \circ \mathcal{A}^{(2)} \circ \mathcal{N} \circ \mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1})] \\
\leqslant\ & P[\mathcal{N}^k(\rho_{R_1A_1}),\mathcal{N}(\rho_{R_1A_1})] + P[\mathcal{N}^k[(\mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1})],\mathcal{N}[(\mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1})]] \\
&+ P[\mathcal{N}^k[(\mathcal{A}^{(2)} \circ \mathcal{N} \circ \mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1})],\mathcal{N}[(\mathcal{A}^{(2)} \circ \mathcal{N} \circ \mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1})]], \qquad (51)
\end{aligned}
$$

where I have omitted some system labels for simplicity. The first inequality follows from the triangle inequality and the second from data processing of the fidelity under the channels

$$\mathcal{N}^k \circ \mathcal{A}^{(2)} \circ \mathcal{N}^k \circ \mathcal{A}^{(1)}, \qquad (52)$$

$$\mathcal{N}^k \circ \mathcal{A}^{(2)}. \qquad (53)$$

The inequality in (51) can be understood as saying that the overall distinguishability of the n uses of $\mathcal{N}^k$ and $\mathcal{N}$, as captured by $P[\omega^k_{R_3B_3},\omega_{R_3B_3}]$, is limited by the sum of the distinguishabilities at every step in the discrimination protocol (this is similar to the observations made in [33,35–41]). Now, employing the inequality in (51), the facts that

$$\rho_{R_1A_1}, \qquad (54)$$

$$(\mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1}), \qquad (55)$$

$$(\mathcal{A}^{(2)} \circ \mathcal{N} \circ \mathcal{A}^{(1)} \circ \mathcal{N})(\rho_{R_1A_1}) \qquad (56)$$

are *fixed states* independent of k, the strong convergence of $\{\mathcal{N}^k_{A\to B}\}_k$, and taking the limit $k \to \infty$ on both sides of the inequality in (51), we conclude that for any fixed protocol $\mathcal{P}^{(3)}$, the following limit holds:

$$\lim_{k\to\infty} \varepsilon^{(3)}_{\rm ad}(k,\mathcal{P}^{(3)}) = 0. \qquad (57)$$

By the same reasoning with the triangle inequality and data processing, the argument extends to any finite positive integer n, so that for any fixed protocol $\mathcal{P}^{(n)}$, the following limit holds:

$$\lim_{k\to\infty} \varepsilon^{(n)}_{\rm ad}(k,\mathcal{P}^{(n)}) = 0. \qquad (58)$$

We can summarize the above development more compactly as

$$
\begin{aligned}
&\sup_{\rho_{RA}} \lim_{k\to\infty} \varepsilon(k,\rho_{RA}) = 0 \\
&\qquad \Longrightarrow \sup_{\mathcal{P}^{(n)}} \lim_{k\to\infty} \varepsilon^{(n)}_{\rm ad}(k,\mathcal{P}^{(n)}) = 0. \qquad (59)
\end{aligned}
$$

That is, strong convergence of the channel sequence $\{\mathcal{N}^k_{A\to B}\}_k$ implies strong convergence of n uses of each channel $\mathcal{N}^k_{A\to B}$ in this sequence in any context in which the n uses of $\mathcal{N}^k_{A\to B}$ could be invoked.

I end this subsection by noting the following proposition, having to do with the strong convergence of serial compositions of channel sequences. In fact, the serial composition result in (59) for adaptive protocols could be proven by employing only the following proposition and iterating.

Proposition 2. Let $\{\mathcal{N}^k_{A\to B}\}_k$ be a channel sequence that converges strongly to a channel $\mathcal{N}_{A\to B}$, and let $\{\mathcal{M}^k_{B\to C}\}_k$ be a channel sequence that converges strongly to a channel $\mathcal{M}_{B\to C}$. Then, the channel sequence $\{\mathcal{M}^k_{B\to C} \circ \mathcal{N}^k_{A\to B}\}_k$ converges strongly to $\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B}$.

Proof. A proof is similar to what is given above, and I give it for completeness. Let ρ_{RA} be an arbitrary state. Consider that

$$
\begin{aligned}
&P\big((\mathcal{M}^k_{B\to C} \circ \mathcal{N}^k_{A\to B})(\rho_{RA}),(\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B})(\rho_{RA})\big) \\
&\leqslant P\big((\mathcal{M}^k_{B\to C} \circ \mathcal{N}^k_{A\to B})(\rho_{RA}),(\mathcal{M}^k_{B\to C} \circ \mathcal{N}_{A\to B})(\rho_{RA})\big) \\
&\quad + P\big((\mathcal{M}^k_{B\to C} \circ \mathcal{N}_{A\to B})(\rho_{RA}),(\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B})(\rho_{RA})\big) \\
&\leqslant P\big(\mathcal{N}^k_{A\to B}(\rho_{RA}),\mathcal{N}_{A\to B}(\rho_{RA})\big) \\
&\quad + P\big((\mathcal{M}^k_{B\to C} \circ \mathcal{N}_{A\to B})(\rho_{RA}),(\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B})(\rho_{RA})\big). \qquad (60)
\end{aligned}
$$

The first inequality follows from the triangle inequality and the second from data processing. Using the strong convergence of $\{\mathcal{N}^k_{A\to B}\}_k$ and $\{\mathcal{M}^k_{B\to C}\}_k$, the fact that $\mathcal{N}_{A\to B}(\rho_{RA})$ is a fixed state independent of k, applying the inequality in (60), and taking the limit $k \to \infty$, we find that

$$
\begin{aligned}
&\lim_{k\to\infty} P\big((\mathcal{M}^k_{B\to C} \circ \mathcal{N}^k_{A\to B})(\rho_{RA}), \\
&\quad (\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B})(\rho_{RA})\big) = 0. \qquad (61)
\end{aligned}
$$

Since the state ρ_{RA} was arbitrary, the proof is complete. ■

E. Strong convergence in the teleportation simulation of bosonic Gaussian channels

The teleportation simulation of a bosonic Gaussian channel is another important notion to discuss. As found in [7], single-mode, phase-covariant bosonic channels, such as the thermal, amplifier, or additive-noise channels, can be simulated by employing the bosonic teleportation protocol from [11]. More general classes of bosonic Gaussian channels can be simulated as well [6]. In this subsection, I exclusively discuss single-mode bosonic Gaussian channels and extend the results later to particular multimode bosonic Gaussian channels. Denoting the original channel by $\mathcal{G}$, an unideal teleportation simulation of it realizes the bosonic Gaussian channel $\mathcal{G}^{\bar{\sigma}} \equiv \mathcal{G} \circ \mathcal{T}^{\bar{\sigma}}$, where $\mathcal{T}^{\bar{\sigma}}$ is the additive-noise channel from (11). This unideal teleportation simulation is possible due to the displacement covariance of bosonic Gaussian channels. Again, it is needed to clarify the meaning of the convergence $\mathcal{G} = \lim_{\bar{\sigma}\to 0} \mathcal{G}^{\bar{\sigma}}$. Based on the previous discussions in this paper, it is clear that the convergence should be considered in the strong sense in most applications: for a state ρ_{RA}, we have that

$$\lim_{\bar{\sigma}\to 0}\big[1 - F\big((\mathrm{id}_R \otimes \mathcal{G}_A)(\rho_{RA}),\big(\mathrm{id}_R \otimes \mathcal{G}^{\bar{\sigma}}_A\big)(\rho_{RA})\big)\big] = 0, \qquad (62)$$

where F denotes the quantum fidelity. This equality follows as a consequence of (16) and the data-processing inequality for fidelity.

As a consequence of (36) and data processing, we also have the following convergence for a teleportation simulation of the tensor-power channel $\mathcal{G}^{\otimes n}$: for a state ρ_{RA^n}, we have that

$$\lim_{\bar{\sigma}\to 0}\left[1 - F(\mathcal{G}_A^{\otimes n}(\rho_{RA^n}),(\mathcal{G}_A^{\bar{\sigma}})^{\otimes n}(\rho_{RA^n}))\right] = 0, \tag{63}$$

where the identity map id_R is omitted for simplicity.

Finally, the argument from Sec. II D applies to the teleportation simulation of bosonic Gaussian channels as well. In more detail, the strong convergence of $\mathcal{G}^{\bar{\sigma}}$ to $\mathcal{G}$ in the limit $\bar{\sigma}\to 0$ implies strong convergence of n uses of $\mathcal{G}^{\bar{\sigma}}$ to n uses of $\mathcal{G}$ in the general sense discussed in Sec. II D. That is, as a consequence of (17) and (59), we have that

$$\sup_{\mathcal{P}^{(n)}}\lim_{\bar{\sigma}\to 0}\varepsilon_{\mathrm{ad}}^{(n)}(\bar{\sigma},\mathcal{P}^{(n)}) = 0, \tag{64}$$

where $\varepsilon_{\mathrm{ad}}^{(n)}(\bar{\sigma},\mathcal{P}^{(n)})$ is defined by replacing $\mathcal{N}_{A\to B}^k$ with $\mathcal{G}^{\bar{\sigma}}$ and $\mathcal{N}_{A\to B}$ with $\mathcal{G}$ in (45), (46), and (47).

F. Uniform convergence in the teleportation simulations of pure-loss, thermal, pure-amplifier, amplifier, and additive-noise channels

I now prove that the teleportation simulations of pure-loss, thermal, pure-amplifier, amplifier, and additive-noise channels converge *uniformly* to the original channels, in the limit of ideal squeezing and detection for the simulations. The argument for uniform convergence is elementary, using the structure of these channels and their teleportation simulations, as well as a data-processing argument that is the same as that which was employed in [34,42]. Note that these uniform convergence results are in contrast to the teleportation simulation of the ideal channel, where the convergence occurs in the strong sense but *not* in the uniform sense.

To prove the uniform convergence of the teleportation simulations of the aforementioned channels, let us start with the thermal channel. Consider that the thermal channel $\mathcal{L}_{\eta,N_B}$ of transmissivity $\eta\in(0,1)$ and thermal photon number $N_B\geqslant 0$ is completely specified by its action on the 2×1 mean vector s and 2×2 covariance matrix V of a single-mode input [13]:

$$s \to Xs, \tag{65}$$

$$V \to XVX^T + Y, \tag{66}$$

where

$$X = \sqrt{\eta}I_2, \tag{67}$$

$$Y = (1-\eta)(2N_B+1)I_2, \tag{68}$$

and I_2 denotes the 2×2 identity matrix. An unideal teleportation simulation of a thermal channel is equivalent to the serial concatenation of the additive-noise channel $\mathcal{T}^{\bar{\sigma}}$ with variance $\bar{\sigma}>0$, followed by the thermal channel $\mathcal{L}_{\eta,N_B}$, as discussed in Sec. II E. Since the additive-noise channel has the same action as in (65) and (66), but with

$$X = I_2, \tag{69}$$

$$Y = 2\bar{\sigma}I_2, \tag{70}$$

we find, after composing and simplifying, that the simulating channel $\mathcal{L}_{\eta,N_B}\circ\mathcal{T}^{\bar{\sigma}}$ has the same action as in (65) and (66), but with

$$X = \sqrt{\eta}I_2, \tag{71}$$

$$Y = [\eta 2\bar{\sigma} + (1-\eta)(2N_B+1)]I_2 \tag{72}$$

$$= (1-\eta)\{2[N_B+\eta\bar{\sigma}/(1-\eta)]+1\}I_2. \tag{73}$$

This latter finding means that the simulating channel $\mathcal{L}_{\eta,N_B}\circ\mathcal{T}^{\bar{\sigma}}$ is equivalent to the thermal channel $\mathcal{L}_{\eta,N_B+\eta\bar{\sigma}/(1-\eta)}$, i.e.,

$$\mathcal{L}_{\eta,N_B}\circ\mathcal{T}^{\bar{\sigma}} = \mathcal{L}_{\eta,N_B+\eta\bar{\sigma}/(1-\eta)}. \tag{74}$$

Let us set

$$N_B' \equiv N_B + \eta\bar{\sigma}/(1-\eta). \tag{75}$$

Note that any thermal channel $\mathcal{L}_{\eta,N_B}$ can be realized in three steps:

(1) prepare an environment mode in a thermal state $\theta(N_B)$ of mean photon number $N_B\geqslant 0$, where

$$\theta(N_B) = \frac{1}{N_B+1}\sum_{n=0}^{\infty}\left(\frac{N_B}{N_B+1}\right)^n|n\rangle\langle n|; \tag{76}$$

(2) interact the channel input mode with the environment mode at a unitary beam splitter $\mathcal{B}_\eta$ of transmissivity η;

(3) discard the environment mode.

This observation and that in (74) are what led to uniform convergence of the simulating channel $\mathcal{L}_{\eta,N_B'}$ to the original channel $\mathcal{L}_{\eta,N_B}$ in the limit as $\bar{\sigma}\to 0$. Indeed, let ρ_{RA} be an arbitrary input state, with R a reference system corresponding to an arbitrary separable Hilbert space and system A the channel input. Then, we find that

$$\begin{aligned}
&P((\mathrm{id}_R\otimes\mathcal{L}_{\eta,N_B})(\rho_{RA}),(\mathrm{id}_R\otimes\mathcal{L}_{\eta,N_B'})(\rho_{RA}))\\
&\leqslant P((\mathrm{id}_R\otimes\mathcal{B}_\eta)[\rho_{RA}\otimes\theta(N_B)],(\mathrm{id}_R\otimes\mathcal{B}_\eta)[\rho_{RA}\otimes\theta(N_B')])\\
&= P(\rho_{RA}\otimes\theta(N_B),\rho_{RA}\otimes\theta(N_B'))\\
&= P(\theta(N_B),\theta(N_B')) \equiv e(N_B,\eta,\bar{\sigma}),
\end{aligned} \tag{77}$$

where $e(N_B,\eta,\bar{\sigma})$ explicitly evaluates to

$$\begin{aligned}
e(N_B,\eta,\bar{\sigma}) = [1 - [&\sqrt{(N_B+1)[N_B+\eta\bar{\sigma}/(1-\eta)+1]}\\
&-\sqrt{N_B[N_B+\eta\bar{\sigma}/(1-\eta)]}]^{-2}]^{1/2}.
\end{aligned} \tag{78}$$

The explicit evaluation in (78) is a direct consequence of [34, Eqs. (34) and (35)], found by evaluating the fidelity between two thermal states of respective mean photon numbers N_B and $N_B+\eta\bar{\sigma}/(1-\eta)$. See also [43,44] for formulas for the fidelity of single-mode Gaussian states. The first inequality in (77) follows from data processing. The first equality follows from unitary invariance of the metric P and the second from its invariance under tensoring in the same state ρ_{RA}. To summarize the inequality in (77), it is stating that the distinguishability of the channels $\mathcal{L}_{\eta,N_B}$ and $\mathcal{L}_{\eta,N_B'}$, when allowing for any input probe state ρ_{RA}, is limited by the distinguishability of the environment states $\theta(N_B)$ and $\theta(N_B')$, and this is similar to the observations made in [34,42]. Thus, the bound in (77) is a *uniform* bound, holding for all input states ρ_{RA}, and so we conclude that

$$\sup_{\rho_{RA}} P((\mathrm{id}_R\otimes\mathcal{L}_{\eta,N_B})(\rho_{RA}),(\mathrm{id}_R\otimes\mathcal{L}_{\eta,N_B'})(\rho_{RA})) \leqslant e(N_B,\eta,\bar{\sigma}). \tag{79}$$

Now, taking the limit $\bar{\sigma} \to 0$ and using the fact that $\lim_{\bar{\sigma}\to 0} e(N_B,\eta,\bar{\sigma}) = 0$ for $\eta \in (0,1)$ and $N_B \geqslant 0$, we find that

$$\lim_{\bar{\sigma}\to 0} \sup_{\rho_{RA}} P((\mathrm{id}_R \otimes \mathcal{L}_{\eta,N_B})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{L}_{\eta,N_B'})(\rho_{RA})) = 0. \tag{80}$$

Thus, the teleportation simulation $\mathcal{L}_{\eta,N_B} \circ \mathcal{T}^{\bar{\sigma}}$ of the thermal channel $\mathcal{L}_{\eta,N_B}$ of transmissivity $\eta \in (0,1)$ and thermal photon number $N_B \geqslant 0$ converges *uniformly* to the thermal channel.

The above uniform convergence result holds in particular for a pure-loss channel of transmissivity $\eta \in (0,1)$ because this channel is a thermal channel with $N_B = 0$. That is, the environment state for the pure-loss channel is a vacuum state, and its teleportation simulation is a thermal channel with the same transmissivity and environment state given by a thermal state of mean photon number $\eta\bar{\sigma}/(1-\eta)$. In this case, the uniform upper bound $e(N_B = 0,\eta,\bar{\sigma})$ simplifies to

$$e(N_B = 0,\eta,\bar{\sigma}) = \sqrt{1 - \frac{1}{\eta\bar{\sigma}/(1-\eta)+1}}, \tag{81}$$

for which we clearly have that $\lim_{\bar{\sigma}\to 0} e(N_B = 0,\eta,\bar{\sigma}) = 0$. Thus, the teleportation simulation of a pure-loss channel $\mathcal{L}_{\eta,N_B=0}$ converges *uniformly* to $\mathcal{L}_{\eta,N_B=0}$.

Similar results hold for the pure-amplifier and amplifier channels. Indeed, to see this, let us begin by considering a general amplifier channel $\mathcal{A}_{G,N_B}$ of gain $G > 1$ and thermal photon number $N_B \geqslant 0$. Such a channel has the action as in (65) and (66), but with

$$X = \sqrt{G} I_2, \tag{82}$$

$$Y = (G-1)(2N_B+1)I_2. \tag{83}$$

By similar reasoning as before, the teleportation simulation $\mathcal{A}_{G,N_B} \circ \mathcal{T}^{\bar{\sigma}}$ of the amplifier channel has the action as in (65) and (66), but with

$$X = \sqrt{G} I_2, \tag{84}$$

$$\begin{aligned} Y &= [G2\bar{\sigma} + (G-1)(2N_B+1)]I_2 &(85)\\ &= (G-1)\{2[N_B + G\bar{\sigma}/(G-1)]+1\}I_2. &(86)\end{aligned}$$

Thus, the teleportation simulation $\mathcal{A}_{G,N_B} \circ \mathcal{T}^{\bar{\sigma}}$ is equivalent to an amplifier channel $\mathcal{A}_{G,N_B+G\bar{\sigma}/(G-1)}$:

$$\mathcal{A}_{G,N_B} \circ \mathcal{T}^{\bar{\sigma}} = \mathcal{A}_{G,N_B+G\bar{\sigma}/(G-1)}. \tag{87}$$

An amplifier channel $\mathcal{A}_{G,N_B}$ can be realized by the following three steps:

(1) prepare an environment mode in a thermal state $\theta(N_B)$ of mean photon number $N_B \geqslant 0$;

(2) interact the channel input mode with the environment mode using a unitary two-mode squeezer $\mathcal{S}_G$ of gain G;

(3) discard the environment mode.

For an arbitrary state ρ_{RA}, we find the following upper bound by the same reasoning as in (77), but replacing the beam splitter $\mathcal{B}_\eta$ therein by the two-mode squeezer $\mathcal{S}_G$,

$$\begin{aligned} &P((\mathrm{id}_R \otimes \mathcal{A}_{G,N_B})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{A}_{G,N_B''})(\rho_{RA}))\\ &\quad\leqslant P(\theta(N_B),\theta(N_B'')) \equiv e(N_B,G,\bar{\sigma}), \end{aligned} \tag{88}$$

where

$$N_B'' \equiv N_B + G\bar{\sigma}/(G-1) \tag{89}$$

and

$$\begin{aligned} e(N_B,G,\bar{\sigma}) = [1 - [&\sqrt{(N_B+1)[N_B + G\bar{\sigma}/(G-1)+1]}\\ &- \sqrt{N_B[N_B + G\bar{\sigma}/(G-1)]}]^{-2}]^{1/2}. \end{aligned} \tag{90}$$

Again, the inequality in (88) is the statement that the distinguishability of the channels $\mathcal{A}_{G,N_B}$ and $\mathcal{A}_{G,N_B''}$, when allowing for any input probe state ρ_{RA}, is limited by the distinguishability of the channel environment states $\theta(N_B)$ and $\theta(N_B'')$. Given that the bound in (88) is a *uniform* bound holding for all input states ρ_{RA}, this implies that

$$\begin{aligned} &\sup_{\rho_{RA}} P((\mathrm{id}_R \otimes \mathcal{A}_{G,N_B})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{A}_{G,N_B''})(\rho_{RA}))\\ &\quad\leqslant e(N_B,G,\bar{\sigma}). \end{aligned} \tag{91}$$

We can then take the limit $\bar{\sigma} \to 0$ and use the fact that $\lim_{\bar{\sigma}\to 0} e(N_B,G,\bar{\sigma}) = 0$ for all $G > 1$ and $N_B \geqslant 0$ to find that

$$\lim_{\bar{\sigma}\to 0} \sup_{\rho_{RA}} P((\mathrm{id}_R \otimes \mathcal{A}_{G,N_B})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{A}_{G,N_B''})(\rho_{RA})) = 0. \tag{92}$$

Thus, the teleportation simulation $\mathcal{A}_{G,N_B} \circ \mathcal{T}^{\bar{\sigma}}$ of the amplifier channel $\mathcal{A}_{G,N_B}$ converges *uniformly* to it, for all $G > 1$ and thermal photon number $N_B \geqslant 0$.

The pure-amplifier channel is a special case of the amplifier channel $\mathcal{A}_{G,N_B}$ with $N_B = 0$, so that the above analysis applies and the teleportation simulation of the pure-amplifier channel $\mathcal{A}_{G,N_B=0}$ converges *uniformly* to it. Indeed, the uniform upper bound $e(N_B = 0,G,\bar{\sigma})$ simplifies as

$$\begin{aligned} &P((\mathrm{id}_R \otimes \mathcal{A}_{G,N_B=0})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{A}_{G,G\bar{\sigma}/(G-1)})(\rho_{RA}))\\ &\quad\leqslant e(N_B = 0,G,\bar{\sigma}) = \sqrt{1 - \frac{1}{G\bar{\sigma}/(G-1)+1}}, \end{aligned} \tag{93}$$

and so it is clear that

$$\lim_{\bar{\sigma}\to 0} \sup_{\rho_{RA}} P(\mathcal{A}_{G,0}(\rho_{RA}),\mathcal{A}_{G,G\bar{\sigma}/(G-1)}(\rho_{RA})) = 0, \tag{94}$$

where I have omitted the identity maps id_R acting on system R for simplicity.

A similar argument establishes that the teleportation simulation of the additive-noise channel $\mathcal{T}^{\xi}$ with variance $\xi > 0$ converges uniformly to it. To see this, let us begin by noting that any additive-noise channel $\mathcal{T}^{\xi}$ can be realized by the following three steps:

(1) Prepare a continuous classical environment register according to the complex Gaussian distribution $G_\xi(\alpha)$, as defined in (12).

(2) Based on the classical value α in the environment register, apply a unitary displacement operation $D(\alpha)$ to the channel input. This step can be described as an interaction channel $\mathcal{C}$ between the channel input and the environment register.

(3) Finally, discard the environment register.

Also, note that the fidelity between two complex Gaussian distributions of variances $\xi_1,\xi_2 > 0$ is given by

$$F(G_{\xi_1},G_{\xi_2}) = \frac{4\xi_1\xi_2}{(\xi_1+\xi_2)^2}, \tag{95}$$

which one may verify directly or consult [45]. Then, proceeding as in the previous proofs, we have for an arbitrary input state ρ_{RA} that

$$\begin{aligned} &P((\mathrm{id}_R \otimes \mathcal{T}^{\xi})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{T}^{\xi+\bar{\sigma}})(\rho_{RA})) \\ &\quad\leqslant P((\mathrm{id}_R \otimes \mathcal{C})(\rho_{RA}\otimes G_{\xi}),(\mathrm{id}_R \otimes \mathcal{C})(\rho_{RA}\otimes G_{\xi+\bar{\sigma}})) \\ &\quad\leqslant P(\rho_{RA}\otimes G_{\xi},\rho_{RA}\otimes G_{\xi+\bar{\sigma}}) \\ &\quad= P(G_{\xi},G_{\xi+\bar{\sigma}}) \\ &\quad= \sqrt{1-\frac{4\xi(\xi+\bar{\sigma})}{(2\xi+\bar{\sigma})^2}}. \end{aligned} \tag{96}$$

The first and second inequalities follow from data processing. The first equality follows because the metric P is invariant under tensoring in the same state ρ_{RA}. The final equality follows from the definition of the metric P and the formula in (95). As in the other proofs, the inequality in (96) is intuitive, indicating that the distinguishability of the channels $\mathcal{T}^{\xi}$ and $\mathcal{T}^{\xi+\bar{\sigma}}$ is limited by the distinguishability of the underlying classical distributions G_{ξ} and $G_{\xi+\bar{\sigma}}$. The bound in (96) is a *uniform* bound, holding for all input states ρ_{RA}, and so we conclude that

$$\begin{aligned} &\sup_{\rho_{RA}} P((\mathrm{id}_R \otimes \mathcal{T}^{\xi})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{T}^{\xi+\bar{\sigma}})(\rho_{RA})) \\ &\quad\leqslant \sqrt{1-\frac{4\xi(\xi+\bar{\sigma})}{(2\xi+\bar{\sigma})^2}}. \end{aligned} \tag{97}$$

Finally, we take the limit as $\bar{\sigma} \to 0$ to establish the uniform convergence of the teleportation simulation of the additive-noise channel of variance $\xi > 0$ to itself:

$$\lim_{\bar{\sigma}\to 0}\sup_{\rho_{RA}} P((\mathrm{id}_R \otimes \mathcal{T}^{\xi})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{T}^{\xi+\bar{\sigma}})(\rho_{RA})) = 0. \tag{98}$$

Remark 1. By the results of [34], the uniform upper bounds in (79), (91), and (97) are all achievable and are thus equalities. The achievable strategy consists of taking the input states ρ_{RA} to be a sequence of two-mode squeezed vacuum states with photon number N_S tending to infinity.

Remark 2. On the one hand, the thermal, amplifier, and additive-noise channels are the single-mode bosonic Gaussian channels that are of major interest in applications, as stressed in [46, Sec. 3.5] and [47, Sec. 12.6.3]. On the other hand, one could consider generalizing the results of this section to arbitrary single-mode bosonic Gaussian channels. In doing so, one should consider the Holevo classification of single-mode bosonic Gaussian channels [48]. However, there is little reason to generalize the contents of this section to other channels in the Holevo classification. The thermal and amplifier channels form the class C discussed in [48], and the additive-noise channels form the class B_2 from [48], which I have already considered in this section. The classes that remain are labeled A, B_1, and D. The channels in classes A and D are *entanglement breaking*, as proved in [49]. Thus, the channels in classes A and D can be directly realized by the action of an LOCC on the input state (without any need for an entangled resource state), and thus we would never have any reason to be interested in the teleportation simulation of these channels. The final remaining class is B_1, but channels in the class B_1 do not seem to be interesting in physical applications.

Remark 3. The class B_1 has been considered in [50], where it was shown that the teleportation simulation of a channel in this class does not converge uniformly, similar to what occurs for the identity channel. Based on the above, and the fact that the ideal channel and channels in the class B_1 have unconstrained quantum capacity equal to infinity [48], as well as the fact that channels in the classes C and B_2 have finite unconstrained quantum capacity [42], we can conclude that, among the single-mode bosonic Gaussian channels that are not entanglement breaking, their teleportation simulations converge uniformly if and only if their unconstrained quantum capacity is finite. This establishes a nontrivial link between teleportation simulation and unconstrained quantum capacity.

G. Generalization of uniform convergence results to multimode bosonic Gaussian channels

In this section, I discuss a generalization of the results of the previous section to the case of multimode bosonic Gaussian channels [51]. Before doing so, I give a brief review of bosonic Gaussian states and channels (see [52] for a more comprehensive review). Let

$$\hat{R} \equiv [\hat{q}_1,\ldots,\hat{q}_m,\hat{p}_1,\ldots,\hat{p}_m] \equiv [\hat{x}_1,\ldots,\hat{x}_{2m}] \tag{99}$$

denote a row vector of position- and momentum-quadrature operators, satisfying the canonical commutation relations

$$[\hat{R}_j,\hat{R}_k] = i\Omega_{j,k}, \quad \text{where} \quad \Omega \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_m, \tag{100}$$

and I_m denotes the $m\times m$ identity matrix. We take the annihilation operator for the jth mode as $\hat{a}_j = (\hat{q}_j + i\hat{p}_j)/\sqrt{2}$. For z a column vector in $\mathbb{R}^{2m}$, we define the unitary displacement operator $D(z) = D^{\dagger}(-z) \equiv \exp(i\hat{R}z)$. Displacement operators satisfy the following relation:

$$D(z)D(z') = D(z+z')\exp\left(-\frac{i}{2}z^T\Omega z'\right). \tag{101}$$

Every state $\rho \in \mathcal{D}(\mathcal{H})$ has a corresponding Wigner characteristic function, defined as

$$\chi_{\rho}(z) \equiv \mathrm{Tr}\{D(z)\rho\}, \tag{102}$$

and from which we can obtain the state ρ as

$$\rho = \int \frac{d^{2m}z}{(2\pi)^m}\, \chi_{\rho}(z)\, D^{\dagger}(z). \tag{103}$$

A quantum state ρ is Gaussian if its Wigner characteristic function has a Gaussian form as

$$\chi_{\rho}(\xi) = \exp\left(-\tfrac{1}{4}z^T V^{\rho} z + i[\mu^{\rho}]^T z\right), \tag{104}$$

where μ^{ρ} is the $2m\times 1$ mean vector of ρ, whose entries are defined by $\mu^{\rho}_j \equiv \langle \hat{R}_j\rangle_{\rho}$ and V^{ρ} is the $2m\times 2m$ covariance matrix of ρ, whose entries are defined as

$$V^{\rho}_{j,k} \equiv \langle\{\hat{R}_j - \mu^{\rho}_j,\hat{R}_k - \mu^{\rho}_k\}\rangle_{\rho}. \tag{105}$$

The following condition holds for a valid covariance matrix: $V \geqslant i\Omega$, which is a manifestation of the uncertainty principle.

A $2m \times 2m$ matrix S is symplectic if it preserves the symplectic form $S\Omega S^T = \Omega$. According to Williamson's theorem [53], there is a diagonalization of the covariance matrix V^ρ of the form

$$V^\rho = S^\rho(D^\rho \oplus D^\rho)(S^\rho)^T, \tag{106}$$

where S^ρ is a symplectic matrix and $D^\rho \equiv \mathrm{diag}(\nu_1, \dots, \nu_m)$ is a diagonal matrix of symplectic eigenvalues such that $\nu_i \geqslant 1$ for all $i \in \{1, \dots, m\}$. Computing this decomposition is equivalent to diagonalizing the matrix $iV^\rho\Omega$ [54, Appendix A].

The Hilbert-Schmidt adjoint of a Gaussian quantum channel $\mathcal{N}_{X,Y}$ from m modes to m modes has the following effect on a displacement operator $D(z)$ [52]:

$$D(z) \longmapsto D(Xz)\exp\left(-\tfrac{1}{4}z^T Y z + i z^T d\right), \tag{107}$$

where X is a real $2m \times 2m$ matrix, Y is a real $2m \times 2m$ positive-semidefinite matrix, and $d \in \mathbb{R}^{2m}$, such that they satisfy

$$Y \geqslant i[\Omega - X^T \Omega X]. \tag{108}$$

The effect of the channel on the mean vector μ^ρ and the covariance matrix V^ρ is thus as follows:

$$\mu^\rho \longmapsto X^T \mu^\rho + d, \tag{109}$$

$$V^\rho \longmapsto X^T V^\rho X + Y. \tag{110}$$

All Gaussian channels are covariant with respect to displacement operators, and this is the main reason why they are teleportation simulable, as noted in [6,7]. That is, the following relation holds:

$$\mathcal{N}_{X,Y}(D(z)\rho D^\dagger(z)) = D(X^T z)\mathcal{N}_{X,Y}(\rho)D^\dagger(X^T z). \tag{111}$$

Just as every quantum channel can be implemented as a unitary transformation on a larger space followed by a partial trace [55], so can Gaussian channels be implemented as a Gaussian unitary on a larger space with some extra modes prepared in the vacuum state, followed by a partial trace [52]. Given a Gaussian channel $\mathcal{N}_{X,Y}$ with Z such that $Y = ZZ^T$ we can find two other matrices X_E and Z_E such that there is a symplectic matrix

$$S = \begin{bmatrix} X^T & Z \\ X_E^T & Z_E \end{bmatrix}, \tag{112}$$

which corresponds to the Gaussian unitary transformation on a larger space.

Alternatively, for certain Gaussian channels, there is a realization that is analogous to those discussed in the previous section for the thermal, amplifier, and additive-noise channels. In particular, consider a Gaussian channel with X and Y matrices as discussed above, and suppose that the mean vector $d = 0$ (note that the condition $d = 0$ is not particularly restrictive because it just corresponds to a unitary displacement at the input or output of the channel, and capacities or distinguishability measures do not change under such unitary actions). Whenever the matrix $\Omega - X^T\Omega X$ is full rank, implying then by (108) that Y is full rank, the Gaussian channel can be realized as follows [52, Theorem 1]:

$$\rho_A \to \mathcal{N}_{X,Y}(\rho_A) = \mathrm{Tr}_E\{U_{AE}(\rho_A \otimes \gamma_E(Y))U_{AE}^\dagger\}, \tag{113}$$

where ρ_A is the m-mode input, $\gamma_E(Y)$ is a zero-mean, m-mode Gaussian state with covariance matrix given by KYK^T, where K is the invertible matrix discussed around [52, Eqs. (27) and (28)], and U_{AE} is a Gaussian unitary acting on $2m$ modes.

Given the above result, we can then generalize the argument from the previous section to argue that the teleportation simulations of these channels converge uniformly. Indeed, as discussed in [6] and as a generalization of the single-mode case discussed previously, the teleportation simulation of a Gaussian channel $\mathcal{N}_{X,Y}$ realizes the Gaussian channel $\mathcal{N}_{X,Y+\bar{\sigma}I}$, where $\bar{\sigma} > 0$ is a parameter characterizing the squeezing strength and the unideal detections involved in the teleportation simulation. Thus, as before, the teleportation simulation of a Gaussian channel simply acts as an additive-noise channel concatenated with the original channel, and the effect is that the noise matrix for the channel realized from the teleportation simulation is $Y + \bar{\sigma}I$, while the X matrix is unaffected. Thus, invoking [52, Theorem 1], the teleportation simulation $\mathcal{N}_{X,Y+\bar{\sigma}I}$ can be realized as the following transformation:

$$\rho_A \to \mathcal{N}_{X,Y}(\rho_A) = \mathrm{Tr}_E\{U_{AE}(\rho_A \otimes \gamma_E(Y + \bar{\sigma}I))U_{AE}^\dagger\}. \tag{114}$$

Then, we are led to the following theorem.

Theorem 1. Let $\mathcal{N}_{X,Y}$ be a multimode quantum Gaussian channel of the form in (107)–(110), such that $\Omega - X^T\Omega X$ is full rank. Then, its teleportation simulation converges uniformly, in the sense that

$$\begin{aligned}\sup_{\rho_{RA}} P((\mathrm{id}_R \otimes \mathcal{N}_{X,Y})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{N}_{X,Y+\bar{\sigma}I})(\rho_{RA}))\\ \leqslant P(\gamma_E(Y),\gamma_E(Y + \bar{\sigma}I)),\end{aligned} \tag{115}$$

where $\gamma_E(Y)$ is defined in (113),

$$\lim_{\bar{\sigma}\to 0} P(\gamma_E(Y),\gamma_E(Y + \bar{\sigma}I)) = 0, \tag{116}$$

and one can use the explicit formula [56, Sec. IV] for the fidelity of multimode, zero-mean Gaussian states to find an analytical expression for

$$P(\gamma_E(Y),\gamma_E(Y + \bar{\sigma}I)) = \sqrt{1 - F(\gamma_E(Y),\gamma_E(Y + \bar{\sigma}I))}, \tag{117}$$

for $\bar{\sigma} > 0$.

Proof. The proof of this theorem follows the same strategy given in the previous section, which in turn was used in [34,42]. In detail, letting ρ_{RA} be an arbitrary state, we have that

$$\begin{aligned}&P((\mathrm{id}_R \otimes \mathcal{N}_{X,Y})(\rho_{RA}),(\mathrm{id}_R \otimes \mathcal{N}_{X,Y+\bar{\sigma}I})(\rho_{RA}))\\ &\quad\leqslant P(U_{AE}(\rho_{RA} \otimes \gamma_E(Y))U_{AE}^\dagger, U_{AE}(\rho_{RA} \otimes \gamma_E(Y^{\bar{\sigma}}))U_{AE}^\dagger)\\ &\quad= P(\rho_{RA} \otimes \gamma_E(Y), \rho_{RA} \otimes \gamma_E(Y + \bar{\sigma}I))\\ &\quad= P(\gamma_E(Y),\gamma_E(Y + \bar{\sigma}I)),\end{aligned} \tag{118}$$

where $Y^{\bar{\sigma}} \equiv Y + \bar{\sigma}I$. The justification of these steps is the same as before, namely, data processing and unitary invariance. The above bound is clearly a uniform bound, holding for all states ρ_{RA}, and so we conclude (115). ■

III. PHYSICAL INTERPRETATION WITH THE CV TELEPORTATION GAME

We can interpret the results in the previous sections of this paper in a game-theoretic way, in order to further elucidate the physical meaning of the two kinds of convergence that we have considered in this paper. Let us consider a competitive game (call it the CV teleportation game) between a distinguisher and a teleporter, while an independent referee determines who wins the game. At the outset, the referee flips an unbiased coin and tells the outcome to the teleporter. If the coin outcome is heads, then the teleporter will apply the ideal channel. If it is tails, then he will apply the CV teleportation protocol. The game is such that either the distinguisher reveals his strategy to the teleporter beforehand, or vice versa. Furthermore, the referee always learns the strategies of both the distinguisher and the teleporter. Everyone involved plays honestly. A particular instance of the game is depicted in Fig. 3.

I now outline the full game in the case that the distinguisher reveals his strategy to the teleporter. In this case, the distinguisher picks a pure state ψ_{RA} and sends mode A to the teleporter and mode R to the referee. The distinguisher also reveals a classical description of the state ψ_{RA} to the referee, and also to the teleporter in this case. Based on this, the teleporter can compute the entanglement infidelity $\varepsilon(\bar{\sigma},\psi_{RA})$ from (13) and can adjust the teleportation imperfection $\bar{\sigma} > 0$ of his setup accordingly such that $\varepsilon(\bar{\sigma},\psi_{RA}) \approx 0$. The referee then reports the coin flip outcome to the teleporter. If heads, then the teleporter does nothing to mode A (ideal channel); if tails, the teleporter applies the continuous-variable bosonic teleportation protocol to mode A. The teleporter sends the output mode to the referee. The referee then performs the optimal binary measurement [57–59] to distinguish the two possible resulting states. If the measurement outcome is "heads," then the channel applied by the teleporter is decided to be the ideal channel. If the measurement outcome is "tails," then the channel applied by the teleporter is decided to be the teleportation channel. This procedure is then repeated a large number of times. If the fraction of rounds in which the coin flips match exceeds $\frac{3}{4}$, then the distinguisher wins. Otherwise, the teleporter wins.

To analyze the above physical setup, consider that the probability of distinguishing the channels in any single round is given by [57–59]

$$\Pr\{X = Y\} = \tfrac{1}{2}\big(1 + \tfrac{1}{2}\big\|\psi_{RA} - \mathcal{T}_A^{\bar{\sigma}}(\psi_{RA})\big\|_1\big), \tag{119}$$

where X is a Bernoulli random variable modeling the coin flip and Y is a Bernoulli random variable modeling the measurement outcome. In the above-described case in which the distinguisher reveals his strategy, this means that the teleporter can follow and choose $\bar{\sigma}$ as small as needed to guarantee that $\Pr\{X = Y\} < \frac{3}{4}$. Thus, with this structure to the game, the teleporter wins with high probability after a large number of repetitions. In fact, based on well-known relations between trace distance and fidelity [60,61], and the result given in (17), we have that

$$\sup_{\psi_{RA}} \inf_{\bar{\sigma}>0} \big\|\psi_{RA} - \mathcal{T}_A^{\bar{\sigma}}(\psi_{RA})\big\|_1 = 0. \tag{120}$$

Now, suppose that the opposite scenario occurs in which the teleporter reveals his strategy (and commits to it). This means that the teleportation imperfection $\bar{\sigma} > 0$ is fixed at the outset. Then, the distinguisher can choose his input state to be the two-mode squeezed vacuum state $\Phi(N_S)_{RA}$ such that $\varepsilon(\bar{\sigma},\Phi(N_S)_{RA}) \approx 1$, as considered in (21). This in turn means that the distinguisher can guarantee that $\Pr\{X = Y\} > \frac{3}{4}$. Thus, in this case, the distinguisher wins with high probability after a large number of repetitions. In fact, in this latter case, as a consequence of (21), we have that

$$\inf_{\bar{\sigma}>0} \sup_{\psi_{RA}} \big\|\psi_{RA} - \mathcal{T}_A^{\bar{\sigma}}(\psi_{RA})\big\|_1 = 2. \tag{121}$$

One may criticize whether the above game is truly physical. Indeed, it is never possible in practice to apply the ideal channel. Depending on the physical situation, the actual channel might be a pure-loss, thermal, pure-amplifier, amplifier, or additive-noise channel, for example (one could further criticize "pure-loss" or "pure-amplifier," but let us leave that). Let $\mathcal{G}_A$ denote one of these single-mode, phase-insensitive bosonic Gaussian channels. Suppose instead that the game changes in the following way: If the coin outcome is heads, then the teleporter will apply the channel $\mathcal{G}$. If it is tails, then he will apply the teleportation simulation $\mathcal{G}_A \circ \mathcal{T}_A^{\bar{\sigma}}$ of $\mathcal{G}_A$.

There is a striking, physically observable difference in this case. No matter whether the distinguisher reveals his strategy to the teleporter, or the other way around, the teleporter always wins with high probability! This is a direct consequence of the inequalities in (79), (91), and (97). Indeed, independent of the revealing, the teleporter can simply compute the teleportation imperfection $\bar{\sigma} > 0$, while incorporating his knowledge of the channel parameters, in order to always guarantee that $\Pr\{X = Y\} < \frac{3}{4}$. Thus, as a consequence of the mathematical fact that the teleportation simulations of these Gaussian channels converge both uniformly and strongly, so that

$$\inf_{\bar{\sigma}>0} \sup_{\psi_{RA}} \big\|\mathcal{G}_A(\psi_{RA}) - \mathcal{G}_A \circ \mathcal{T}_A^{\bar{\sigma}}(\psi_{RA})\big\|_1$$
$$= \sup_{\psi_{RA}} \inf_{\bar{\sigma}>0} \big\|\mathcal{G}_A(\psi_{RA}) - \mathcal{G}_A \circ \mathcal{T}_A^{\bar{\sigma}}(\psi_{RA})\big\|_1 = 0, \tag{122}$$

the *physically observable consequence* is that the teleporter always has the advantage in this modified (and physically more realistic) version of the CV teleportation game.

Note that other variations of the CV teleportation game are possible. One could allow for the distinguisher to employ

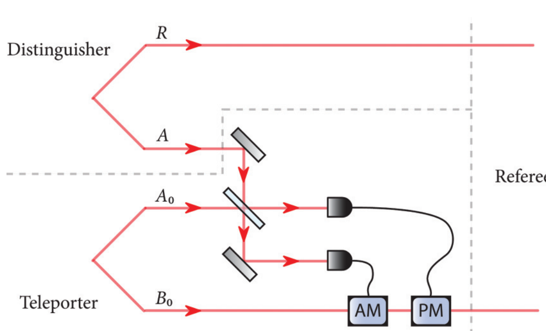

FIG. 3. Depiction of the CV teleportation game, in the case that the coin outcome is tails, so that the teleporter applies the continuous-variable bosonic teleportation protocol to the mode A that the distinguisher sends.

entangled strategies among the different rounds, or even adaptive channels in-between rounds. The results given in Propositions 1 and 2 can be used to analyze these other, richer variations of the CV teleportation game, but here I will not go into the details.

IV. NONASYMPTOTIC SECRET-KEY-AGREEMENT CAPACITIES

I now briefly review secret-key-agreement capacities of a quantum channel and some results from [15]. The secret-key-agreement capacity of a quantum channel is equal to the optimal rate at which a sender and receiver can use a quantum channel many times, as well as a free assisting classical channel, in order to establish a reliable and secure secret key. It is relevant in the context of quantum key distribution [62,63]. More generally, since capacity is a limiting notion that can never be reached in practice, one can consider a fixed $(n,P^{\leftrightarrow},\varepsilon)$ secret-key-agreement protocol that uses a channel n times and has ε error, while generating a secret key at the rate $P^{\leftrightarrow}$ [15]. Such protocols were explicitly discussed in [15,39,40], as well as the related developments in [41,64–70]. For a fixed integer n and a fixed error $\varepsilon \in (0,1)$, the nonasymptotic secret-key-agreement capacity of a quantum channel $\mathcal{N}$ is written as $P^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$ and is equal to the optimal secret-key rate P subject to these constraints [15]. That is,

$$P^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon) \equiv \sup\{P^{\leftrightarrow} \mid (n,P^{\leftrightarrow},\varepsilon) \text{ is achievable using } \leftrightarrow\}, \tag{123}$$

where $\leftrightarrow$ indicates the free use of LOCC between every use of the quantum channel.

If an $(n,P^{\leftrightarrow},\varepsilon)$ protocol takes place over a bosonic channel, the stated definition of a secret-key-agreement protocol makes no restriction on the photon number of the channel input states and, as such, it is called an unconstrained protocol. The corresponding capacity is called the unconstrained secret-key-agreement capacity. For example, in such a scenario, a sender and receiver could freely make use of the following classically correlated Basel state with mean photon number equal to ∞:

$$\overline{\beta}_{AB} \equiv \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}|n\rangle\langle n|_A \otimes |n\rangle\langle n|_B, \tag{124}$$

which represents a dephased version of the state in (20). Even though it is questionable whether such states are physically realizable in practice, they are certainly normalizable, and thus allowed to be used in principle in an unconstrained secret-key-agreement protocol.

One of the main results of [15] is the following bound on $P^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon)$ when the channel is taken to be a pure-loss channel $\mathcal{L}^\eta$ of transmissivity $\eta \in (0,1)$:

$$P^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leqslant -\log_2(1-\eta) + C(\varepsilon)/n, \tag{125}$$

where

$$C(\varepsilon) \equiv \log_2 6 + 2\log_2([1+\varepsilon]/[1-\varepsilon]). \tag{126}$$

This bound was established by proving that $P^{\leftrightarrow} \leqslant -\log_2(1-\eta) + C(\varepsilon)/n$ for any fixed $(n,P^{\leftrightarrow},\varepsilon)$ unconstrained protocol. As a consequence of this *uniform* bound, one can then take a supremum over all $P^{\leftrightarrow}$ such that there exists an $(n,P^{\leftrightarrow},\varepsilon)$ protocol and conclude (125), as was done in the proof of [15, Theorem 24]. Similar reasoning was employed in [15] in order to arrive at bounds on the unconstrained capacities of other bosonic Gaussian channels.

A critical tool used to establish (125) is the simulation of a quantum channel via teleportation [4, Sec. V] (see also [8, Theorem 14 and Remark 11]), which, as discussed previously, has been extended to bosonic states and channels [6,7], by making use of the well-known bosonic teleportation protocol from [11]. More generally, one can allow for general local operations and classical communication (LOCC) when simulating a quantum channel from a resource state [5, Eq. (11)], known as LOCC channel simulation. This tool is used to reduce any arbitrary LOCC-assisted protocol over a teleportation-simulable channel to one in which the LOCC assistance occurs after the final channel use. Another critical idea is the reinterpretation of a three-party secret-key-agreement protocol as a two-party private-state generation protocol and employing entanglement measures such as the relative entropy of entanglement as a bound for the secret-key rate [71,72]. Finally, one can employ the Chen formula for the relative entropy of Gaussian states [73], as well as a formula from [54] for the relative entropy variance of Gaussian states. These tools were foundational for the results of [15, Theorem 24] in order to argue for bounds on secret-key-agreement capacities of bosonic Gaussian channels.

A key point mentioned above, which is critical to and clearly stated in the proof of [15, Theorem 24], is as follows: the proof begins by considering a fixed $(n,P^{\leftrightarrow},\varepsilon)$ protocol and then establishes a uniform bound on $P^{\leftrightarrow}$, independent of the details of the particular protocol. The discussion given in this paper clarifies that strong convergence in teleportation simulation suffices for the proof of [15, Theorem 24]. Furthermore, there is no need to invoke the energy-constrained diamond distance [16,74] in order to establish the correctness of the proof.

V. DETAILED REVIEW OF THE PROOF OF THEOREM 24 IN [15]

We can now step through the relevant parts of the proof of [15, Theorem 24] carefully, in order to clarify its correctness. I highlight, in italics, quotations from the proof of [15, Theorem 24] for clarity and follow each quotation with a brief discussion. The proof begins by stating the following:

"First, consider an arbitrary $(n,P^{\leftrightarrow},\varepsilon)$ protocol for the thermalizing channel $\mathcal{L}_{\eta,N_B}$. It consists of using the channel n times and interleaving rounds of LOCC between every channel use. Let $\zeta^n_{\hat{A}\hat{B}}$ denote the final state of Alice and Bob at the end of this protocol."

It is crucial to note here that this is saying, as it is written, that we should really start with a particular, fixed $(n,P^{\leftrightarrow},\varepsilon)$ protocol. This does not mean that we should be considering a sequence of such protocols or protocols involving unnormalizable states. Thus, proceeding by fixing an $(n,P^{\leftrightarrow},\varepsilon)$ protocol, the proof continues with the following:

"By the teleportation reduction procedure [...], such a protocol can be simulated by preparing n two-mode squeezed vacuum (TMSV) states each having energy $\mu - \frac{1}{2}$ (where we think of $\mu \geqslant \frac{1}{2}$ as a very large positive real), sending one mode of each TMSV through each channel use, and then performing continuous-variable quantum teleportation [11] to delay all of

the LOCC operations until the end of the protocol. Let ρ^{μ}_{η,N_B} denote the state resulting from sending one share of the TMSV through the thermalizing channel, and let $\zeta'_{\hat{A}\hat{B}}(n,\mu)$ denote the state at the end of the simulation."

This means, as it states, that the fixed protocol can be simulated with some error by replacing each channel use of $\mathcal{L}_{\eta,N_B}$ with the simulating channel $\mathcal{L}^{\mu}_{\eta,N_B}$, for $\mu \in [0,\infty)$. Here, the original channel $\mathcal{L}_{\eta,N_B}$ is in correspondence with $\mathcal{G}_A$ from Sec. II E, $\mathcal{L}^{\mu}_{\eta,N_B}$ with $\mathcal{G}^{\bar{\sigma}}_A$, and μ with $\bar{\sigma}$, in the sense that the limit $\mu \to \infty$ is in correspondence with the limit $\bar{\sigma} \to 0$. Furthermore, since this is an LOCC simulation and the original protocol consists of channel uses of $\mathcal{L}_{\eta,N_B}$ interleaved by LOCC, it is possible to write the simulating protocol as one that consists of a single round of LOCC on the state $[\rho^{\mu}_{\eta,N_B}]^{\otimes n}$, as observed in [4,7,8]. Continuing,

"Let $\varepsilon_{\mathrm{TP}}(n,\mu)$ denote the "infidelity" of the simulation:

$$\varepsilon_{\mathrm{TP}}(n,\mu) \equiv 1 - F\big(\zeta^{n}_{\hat{A}\hat{B}},\zeta'_{\hat{A}\hat{B}}(n,\mu)\big)." \tag{127}$$

In the context of the proof, this infidelity clearly corresponds to the infidelity of the simulation of the fixed protocol. One might argue that the notation $\varepsilon_{\mathrm{TP}}(n,\mu)$ somehow hides this dependence on a fixed protocol, but the dependence of $\varepsilon_{\mathrm{TP}}(n,\mu)$ on a fixed protocol is clear from the context and the fact that the quantity $1 - F(\zeta^{n}_{\hat{A}\hat{B}},\zeta'_{\hat{A}\hat{B}}(n,\mu))$ itself clearly depends on a fixed protocol as stated. This infidelity is similar to that in (47), except the initial state of the protocol is constrained to be a separable, unentangled state shared between the sender and receiver, and each channel $\mathcal{A}^{(j)}$ in the protocol is constrained to be an LOCC channel between the sender and receiver. Continuing,

"Due to the fact that continuous-variable teleportation induces a perfect quantum channel when infinite energy is available [11], the following limit holds for every n":

$$\limsup_{\mu\to\infty} \varepsilon_{\mathrm{TP}}(n,\mu) = 0. \tag{128}$$

This is indeed a key step for the proof. In the context of the proof given, the convergence is as it is written and is to be understood in the strong sense of (64): for a *fixed protocol* used in conjunction with the n teleportation simulations, the infidelity converges to zero in the limit of ideal squeezing and ideal detection (as $\mu \to \infty$ or, equivalently, as $\bar{\sigma} \to 0$). Note that here we are employing the notion of strong convergence in (64), given that the original protocol has LOCC channels interleaved between every use of $\mathcal{L}_{\eta,N_B}$.

Continuing the proof, *"By using that $\sqrt{1-F(\rho,\sigma)}$ is a distance measure for states ρ and σ (and thus obeys a triangle inequality) [...], the simulation leads to an $(n,P^{\leftrightarrow},\varepsilon(n,\mu))$ protocol for the thermalizing channel, where*

$$\varepsilon(n,\mu) \equiv \min\{1,[\sqrt{\varepsilon} + \sqrt{\varepsilon_{\mathrm{TP}}(n,\mu)}]^2\}. \tag{129}$$

Observe that $\limsup_{\mu\to\infty} \varepsilon(n,\mu) = \varepsilon$, *so that the simulated protocol has equivalent performance to the original protocol in the infinite-energy limit."*

This last part that I have recalled is saying how the error of the simulating protocol is essentially equal to the sum of the error of the original protocol and the error from substituting the original n channels with n unideal teleportations. It is a straightforward consequence of the triangle inequality for the metric $\sqrt{1-F}$, as recalled there, and thus captures a correct propagation of errors.

From this point on in the proof, using other techniques, the following bound is concluded on the rate $P^{\leftrightarrow}$ of the fixed $(n,P^{\leftrightarrow},\varepsilon)$ protocol by invoking the metaconverse in [15, Theorem 11]:

$$P^{\leftrightarrow} \leqslant D\big(\rho^{\mu}_{\eta,N_B}\|\sigma^{\mu}_{\eta,N_B}\big) + \sqrt{\frac{2V\big(\rho^{\mu}_{\eta,N_B}\|\sigma^{\mu}_{\eta,N_B}\big)}{n(1-\varepsilon(n,\mu))}} + C(\varepsilon(n,\mu))/n. \tag{130}$$

This bound holds for all μ sufficiently large, $\varepsilon \in (0,1)$, and positive integers n, and as such, it is a uniform bound. Given the uniform bound, the limit $\mu \to \infty$ is then taken to arrive at

$$P^{\leftrightarrow} \leqslant -\log[(1-\eta)\eta^{N_B}] - g(N_B) + \sqrt{\frac{2V_{\mathcal{L}_{\eta,N_B}}}{n(1-\varepsilon)}} + \frac{C(\varepsilon)}{n}. \tag{131}$$

Since this latter bound is itself a uniform bound, holding for all $(n,P^{\leftrightarrow},\varepsilon)$ protocols, it is then concluded that

$$P^{\leftrightarrow}_{\mathcal{L}_{\eta,N_B}}(n,\varepsilon) \leqslant -\log[(1-\eta)\eta^{N_B}] - g(N_B) + \sqrt{\frac{2V_{\mathcal{L}_{\eta,N_B}}}{n(1-\varepsilon)}} + \frac{C(\varepsilon)}{n}. \tag{132}$$

Further arguments are given in the proof of [15, Theorem 24] to establish the bound in (125) for the pure-loss channel.

To summarize, the original proof of [15, Theorem 24] as given there is correct as it is written. The proof of [15, Theorem 24] as written there establishes that for a fixed $(n,P^{\leftrightarrow},\varepsilon)$ protocol, the bound in (131) holds. Furthermore, one might think that a proof may *only* be given by employing the energy-constrained diamond distance, but this is clearly not the case either, as demonstrated above.

VI. CONCLUSION

The continuous-variable, bosonic quantum teleportation protocol from [11] is often loosely stated to simulate an ideal quantum channel in the limit of infinite squeezing and ideal homodyne detection. The precise form of convergence is typically not clarified in the literature and, as a consequence, this has the potential to lead to confusion in mathematical proofs that employ this protocol.

This paper has clarified various notions of channel convergence, with applications to the continuous-variable bosonic teleportation protocol from [11], and extended these notions to various contexts. This paper provided an explicit proof that the continuous-variable bosonic teleportation protocol from [11] converges *strongly* to an ideal quantum channel in the limit of ideal squeezing and detection. At the same time, this paper proved that this protocol does *not* converge *uniformly* to an ideal quantum channel, and this highlights the role of this paper in providing a precise clarification of the convergence that occurs in the continuous-variable bosonic teleportation protocol from [11]. I also proved that the teleportation simulations of the pure-loss, thermal, pure-amplifier, amplifier, and additive-noise channels converge both *strongly* and *uniformly* to the original channels, which is in contrast to what occurs in the teleportation simulation of the ideal channel. I suspect that the explicit uniform bounds in (79), (91), and (97), on the accuracy of the teleportation simulations of these channels, will be useful in future applications. The uniform convergence results were then generalized to the teleportation simulations of particular multimode bosonic Gaussian channels. Finally, I gave a physical interpretation of the convergence results discussed in this paper, by means of the CV teleportation game of Sec. III.

I also reviewed the proof of [15, Theorem 24] and confirmed its correctness as it is written there. One might think that it is necessary to use the energy-constrained diamond distance to arrive at a proof of (125), but this is clearly not the case.

It would be interesting in future work to explore physical scenarios of interest in which different topologies of convergence lead to physically distinct outcomes, as was the case in the CV teleportation game. Recent work in this direction is available in [17,74].

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