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Universal bound states of one-dimensional bosons with two- and three-body attractions

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When quantum particles are confined into lower dimensions, an effective three-body interaction inevitably arises and may cause significant consequences. Here we study bosons in one dimension with weak two-body and three-body interactions, predict the existence of two three-body bound states when both interactions are attractive, and determine their binding energies as universal functions of the two-body and three-body scattering lengths. We also show that an infinitesimal three-body attraction induces an excited bound state only for 3, 39, or more bosons. Our findings herein have direct relevance to a broad range of quasi-one-dimensional systems realized with ultracold atoms.

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I. INTRODUCTION

Effective three- and higher-body interactions are ubiquitous and play important roles in various subfields of physics [1–5]. One such example is provided by quantum particles confined into lower dimensions even when their interaction in free space is purely pairwise. As far as low-energy physics relative to the transverse excitation energy is concerned, the system admits an effective low-dimensional description where multibody interactions inevitably arise from virtual transverse excitations. In particular, the three-body interaction in one-dimensional systems may cause significant consequences because it breaks the integrability [6–8] and is marginally relevant when attractive [9,10]. The purpose of this work is to elucidate possible consequences of the three-body interaction for bound states of bosons in one dimension.

Model and universality

Bosons in one dimension with two-body and three-body interactions are described by

$$H = \int dx\left[\frac{1}{2m}\frac{d\phi^\dagger(x)}{dx}\frac{d\phi(x)}{dx} + \frac{u_2}{2m}|\phi(x)|^4 + \frac{u_3}{6m}|\phi(x)|^6\right], \tag{1}$$

where we set $\hbar = 1$ and $|\phi(x)|^{2n} \equiv [\phi^\dagger(x)]^n[\phi(x)]^n$. When this system is realized by confining weakly interacting bosons with a two-dimensional harmonic potential [11], the two-body and three-body couplings are provided by

$$u_2 = 2\frac{a_{3D}}{l_\perp^2} \quad \text{and} \quad u_3 = -12\ln(4/3)\frac{a_{3D}^2}{l_\perp^2}, \tag{2}$$

respectively, for $|a_{3D}| \ll l_\perp$, where a_{3D} is the s-wave scattering length in free space and $l_\perp \equiv 1/\sqrt{m\omega_\perp}$ is the harmonic oscillator length [12,13].[1] While the two-body interaction can be either attractive or repulsive depending on the sign of a_{3D}, the three-body interaction is always attractive ($u_3 < 0$) because it arises from the second-order perturbation theory [8]. We note that four- and higher-body interactions also exist but are irrelevant to low-energy physics.

It is more convenient to parametrize the two-body and three-body couplings in terms of the scattering lengths. The two-body scattering length is introduced as $a_2 \equiv -2/u_2$. With this definition, the binding energy of a two-body bound state (dimer) is provided by $E_2 = -1/(ma_2^2)$ for $a_2 \gg l_\perp$ [11]. Similarly, the three-body scattering length is introduced so that the binding energy of a three-body bound state (trimer) is provided by $E_3 \equiv -1/(ma_3^2)$ for $a_3 \gg l_\perp$ when the two-body interaction is assumed to be absent [9]. This definition leads to $a_3 \sim e^{-\sqrt{3}\pi/u_3}l_\perp$ as we will see later in Eq. (7). While $a_3 \gg |a_2| \gg l_\perp$ is naturally realized for weakly interacting bosons with $|a_{3D}| \ll l_\perp$, we study the system with an arbitrary $-\infty < a_3/a_2 < +\infty$ because the two-body and three-body interactions are independently tunable in principle with ultracold atoms [14–17]. As far as both interactions are weak in the sense of $|a_2|,a_3 \gg l_\perp$, low-energy physics of the system at $|E| \ll 1/(ml_\perp^2)$ is universal, i.e., depends only on the two scattering lengths.

II. THREE-BOSON SYSTEM

A. Formulation

We now focus on the system of three bosons whose Schrödinger equation reads

$$\left[-\frac{1}{2m}\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} + \frac{u_2}{m}\sum_{1\leqslant i<j\leqslant 3}\delta(x_{ij}) + \frac{u_3}{m}\delta(x_{12})\delta(x_{23})\right] \times \Psi(x_1,x_2,x_3) = E\,\Psi(x_1,x_2,x_3), \tag{3}$$

where $x_{ij} \equiv x_i - x_j$ is the interparticle separation. For a bound state with its binding energy $E \equiv -\kappa^2/m < 0$, the Schrödinger equation is formally solved in Fourier

[1]Our result for u_3 is four times smaller than that in Refs. [7,8] but agrees with Ref. [13].

space by

$$\tilde{\Psi}(p_1,p_2,p_3) = -\frac{\sum_{i=1}^{3}\tilde{\Psi}_2(P_{123}-p_i;p_i)+\tilde{\Psi}_3(P_{123})}{\kappa^2+\sum_{i=1}^{3}\frac{p_i^2}{2}}, \quad (4)$$

where $P_{123} \equiv p_1 + p_2 + p_3$ is the center-of-mass momentum and

$$\tilde{\Psi}_2(P;p) \equiv u_2 \int \frac{dq}{2\pi}\,\tilde{\Psi}(P-q,q,p), \quad (5a)$$

$$\tilde{\Psi}_3(P) \equiv u_3 \int \frac{dq\,dr}{(2\pi)^2}\,\tilde{\Psi}(P-q-r,q,r) \quad (5b)$$

are the Fourier transforms of $u_2\Psi(X,X,x)$ and $u_3\Psi(X,X,X)$, respectively. After rewriting $p_1 \to P-p-q$, $p_2 \to p$, and $p_3 \to q$ in Eq. (4), the integration over q leads to

$$\frac{1}{u_2}\tilde{\Psi}_2(P-p;p) = -\int\frac{dq}{2\pi}\frac{2\tilde{\Psi}_2(P-q;q)}{\kappa^2+\frac{(P-p-q)^2+p^2+q^2}{2}} - \frac{\tilde{\Psi}_2(P-p;p)+\tilde{\Psi}_3(P)}{2\sqrt{\kappa^2+\frac{(P-p)^2}{4}+\frac{p^2}{2}}}, \quad (6a)$$

while the integration over p and q leads to

$$\frac{1}{u_3}\tilde{\Psi}_3(P) = -\int\frac{dq}{2\pi}\frac{3\tilde{\Psi}_2(P-q;q)}{2\sqrt{\kappa^2+\frac{(P-q)^2}{4}+\frac{q^2}{2}}} - \frac{1}{\sqrt{3}\pi}\ln\left(\frac{\Lambda}{\sqrt{\kappa^2+\frac{P^2}{6}}}\right)\tilde{\Psi}_3(P), \quad (6b)$$

where $\Lambda \sim l_\perp^{-1}$ is the momentum cutoff and Eqs. (5) are used on the left-hand sides. Finally, by substituting the ansatz of $\tilde{\Psi}_2(P-p;p) \equiv 2\pi\delta(P)\tilde{\psi}_2(p)$ and $\tilde{\Psi}_3(P) \equiv 2\pi\delta(P)\tilde{\psi}_3$ (i.e., zero center-of-mass momentum) into Eqs. (6) as well as the two-body and three-body couplings parametrized as

$$u_2 = -\frac{2}{a_2} \quad \text{and} \quad u_3 = -\frac{\sqrt{3}\pi}{\ln(a_3\Lambda)}, \quad (7)$$

we obtain

$$\left(\frac{a_2}{2} - \frac{1}{2\sqrt{\kappa^2+\frac{3p^2}{4}}}\right)\tilde{\psi}_2(p) = \int\frac{dq}{2\pi}\frac{2\tilde{\psi}_2(q)}{\kappa^2+p^2+q^2+pq} + \frac{\tilde{\psi}_3}{2\sqrt{\kappa^2+\frac{3p^2}{4}}} \quad (8a)$$

and

$$\frac{\ln(a_3\kappa)}{\sqrt{3}\pi}\tilde{\psi}_3 = \int\frac{dq}{2\pi}\frac{3\tilde{\psi}_2(q)}{2\sqrt{\kappa^2+\frac{3q^2}{4}}}. \quad (8b)$$

Equation (8a) with $\tilde{\psi}_3$ eliminated by Eq. (8b) provides the closed one-dimensional integral equation for $\tilde{\psi}_2(p)$, which is to be solved numerically. We note that nontrivial solutions exist only in the even-parity channel where $\tilde{\psi}_2(p) = \tilde{\psi}_2(-p)$.

As we can see in Eq. (7), the positive (negative) two-body scattering length corresponds to the attractive (repulsive) two-body interaction. The two-body attraction increases with increasing $1/a_2$ from the strong repulsion $1/a_2 \to -\infty$ via no interaction $1/a_2 = 0$ to the strong attraction $1/a_2 \to +\infty$. On the other hand, the three-body scattering length is positive definite and the three-body attraction increases with increasing $1/a_3$ from the weak attraction $1/a_3 \to +0$ to the strong attraction $1/a_3 \to +\infty$. For later discussion, we identify the prefactor of $\tilde{\psi}_3$ in Eq. (8b) as $-1/\bar{u}_3(\kappa)$, where

$$\bar{u}_3(\kappa) \equiv -\frac{\sqrt{3}\pi}{\ln(a_3\kappa)} \quad (9)$$

is the renormalized three-body coupling with logarithmic energy dependence [9].

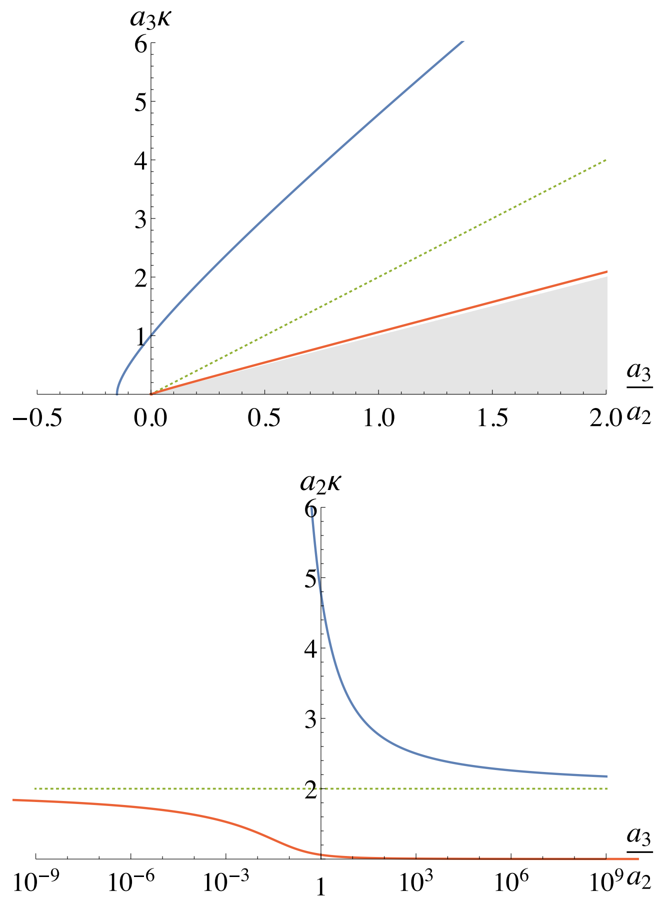

FIG. 1. Binding energies of three-body bound states $E = -\kappa^2/m$ in the forms of $a_3\kappa$ (top panel) and $a_2\kappa$ (bottom panel) as functions of the three-body to two-body scattering length ratio a_3/a_2. The upper (lower) solid curve corresponds to the ground (excited) state and the dotted line indicates $\kappa = 2/a_2$ for the McGuire trimer. The shaded region in the top panel indicates the atom-dimer continuum where $\kappa < \theta(a_2)/a_2$.

B. Binding energies

The numerical solutions for $\kappa > \theta(a_2)/a_2$ are plotted as functions of a_3/a_2 in Fig. 1 with the different normalizations.[2] Here we find that the ground state trimer appears at

[2]Their analytical expressions were recently obtained in Ref. [25].

$a_3/a_2 \approx -0.149\,218$. Its binding energy is $\kappa = 1/a_3$ at $a_3/a_2 = 0$ by the definition of a_3 and asymptotically approaches $\kappa = 2/a_2$ as

$$\kappa \to \frac{2}{a_2} + \frac{2\pi}{\sqrt{3}\,a_2 \ln(a_3/a_2)} \quad \text{toward} \quad \frac{a_3}{a_2} \to +\infty. \tag{10}$$

On the other hand, we find that the excited state trimer appears right at $a_3/a_2 = 0$ where the dimer state also appears. Its binding energy asymptotically approaches $\kappa = 2/a_2$ as

$$\kappa \to \frac{2}{a_2} + \frac{2\pi}{\sqrt{3}\,a_2 \ln(a_3/a_2)} \quad \text{toward} \quad \frac{a_3}{a_2} \to +0, \tag{11}$$

while it asymptotically approaches $\kappa = 1/a_2$ as

$$\kappa \to \frac{1}{a_2} + \frac{\pi^2}{18\,a_2 \ln^2(a_3/a_2)} \quad \text{toward} \quad \frac{a_3}{a_2} \to +\infty. \tag{12}$$

The subleading term in Eq. (12) indicates that the atom-dimer scattering length is provided by $\alpha_{1,2} \to 3\sqrt{3}\,a_2 \ln(a_3/a_2)/(2\pi) \gg a_2$ for $\ln(a_3/a_2) \to +\infty$. This is consistent with the one obtained from the expectation value of the renormalized three-body interaction energy $V_3 = [\bar{u}_3(\kappa)/m]\delta(x_{12})\delta(x_{23})$ with respect to the wave function right at the atom-dimer threshold; $\Psi(x_1,x_2,x_3) = \sqrt{\frac{1}{3a_2L^2}}[\sum_{1\leqslant i<j\leqslant 3} e^{-|x_{ij}|/a_2} - 4e^{-\sum_{1\leqslant i<j\leqslant 3}|x_{ij}|/(2a_2)}]$ [18]. We note that the wave functions here and below are all normalized on a line of length $L \gg a_2$.

When the three-body interaction is assumed to be absent, McGuire predicted a single trimer state with its binding energy $\kappa = 2/a_2$ [19]. We find above that an infinitesimal three-body attraction immediately induces another trimer state appearing from the atom-dimer threshold at $\kappa = 1/a_2$ as in Eq. (12). While our ground state trimer unsurprisingly reduces to the McGuire trimer in the limit of strong two-body or weak three-body attraction [Eq. (10)], it is interesting that our excited state trimer also reduces to the McGuire trimer in the opposite limit of weak two-body or strong three-body attraction [Eq. (11)]. This is because the renormalized three-body coupling in Eq. (9) turns out to be positive and vanishingly small toward the three-boson threshold $a_3\kappa \to +0$. Indeed, the subleading terms in Eqs. (10) and (11) for $\ln(a_3/a_2) \to \pm\infty$ can both be obtained from the expectation value of the renormalized three-body interaction energy $V_3 = [\bar{u}_3(\kappa)/m]\delta(x_{12})\delta(x_{23})$ with respect to the wave function of the McGuire trimer; $\Psi(x_1,x_2,x_3) = \sqrt{\frac{8}{3a_2^2L}}e^{-\sum_{1\leqslant i<j\leqslant 3}|x_{ij}|/a_2}$ [20].

III. N-BOSON SYSTEM

While we have so far focused on the system of three bosons, it is straightforward to generalize our formulation and some results to an arbitrary N number of bosons. In particular, when the three-body interaction is assumed to be absent, McGuire also predicted a single N-body bound state for every N with its binding energy $E_N^{(\mathrm{MG})} \equiv -N(N^2-1)/(6ma_2^2)$ [19]. Its wave function in the domain of $x_1 < x_2 < \cdots < x_N$ is provided by

$$\Psi_N(\boldsymbol{x}) = \sqrt{\frac{(N-1)!}{NL}\left(\frac{2}{a_2}\right)^{N-1}} \exp\left(\sum_{i=1}^{N} \frac{N+1-2i}{a_2} x_i\right), \tag{13}$$

where $\boldsymbol{x} \equiv (x_1,x_2,\ldots,x_N)$ [20]. Then, the expectation value of the renormalized three-body interaction energy $V_3 = [\bar{u}_3(\kappa)/m]\sum_{1\leqslant i<j<k\leqslant N}\delta(x_{ij})\delta(x_{jk})$ with respect to the wave function in Eq. (13) leads to the binding-energy shift induced by an infinitesimal three-body attraction, which is found to be

$$\Delta E_N \equiv E_N - E_N^{(\mathrm{MG})} \to -\frac{\sqrt{3}\pi N(N^2-1)(N^2-4)}{45ma_2^2 \ln(a_3/a_2)} \tag{14}$$

for $\ln(a_3/a_2) \to +\infty$.

Similarly, regarding the scattering state consisting of an atom with momentum k and an $(N-1)$-body bound state at rest, its wave function in the domain of $x_1 < x_2 < \cdots < x_N$ is provided by

$$\Psi_{1,N-1}(\boldsymbol{x}) = \sum_{j=1}^{N} \frac{(N-2-ika_2)(N-ika_2)}{(N-2j-ika_2)(N-2j+2-ika_2)} \times \frac{e^{ikx_j}}{\sqrt{NL}}\Psi_{N-1}(\boldsymbol{x}\backslash\{x_j\}), \tag{15}$$

where $\boldsymbol{x}\backslash\{x_j\}$ refers to $\boldsymbol{x}$ with x_j excluded. Because the wave function factorizes as $\Psi_{1,N-1}(\boldsymbol{x}) \to \frac{e^{ikx_j}}{\sqrt{NL}}\Psi_{N-1}(\boldsymbol{x}\backslash\{x_j\})$ at a large separation $x_j \ll \boldsymbol{x}\backslash\{x_j\}$, the scattering length between the atom and the $(N-1)$-body bound state is divergent, i.e., noninteracting [20–22]. Then, the expectation value of the renormalized three-body interaction energy $V_3 = [\bar{u}_3(\kappa)/m]\sum_{1\leqslant i<j<k\leqslant N}\delta(x_{ij})\delta(x_{jk})$ with respect to the wave function in Eq. (15) at $k \to 0$ is found to be

$$\lim_{k\to 0}\langle V_3\rangle_{1,N-1} = \Delta E_{N-1} - \frac{N}{(N-1)m\alpha_{1,N-1}L}, \tag{16}$$

where the leading term is just the binding-energy shift in Eq. (14) but the subleading term reflects the interaction between the atom and the $(N-1)$-body bound state induced by an infinitesimal three-body attraction. The extracted scattering length $\alpha_{1,N-1} \equiv a_2\ln(a_3/a_2)/(\sqrt{3}\pi\beta_{1,N-1})$ is plotted in Fig. 2 and turns out to be positive for $N = 3$ and $N \geqslant 39$ but negative for $4 \leqslant N \leqslant 38$, which correspond to the attractive and repulsive interactions between the atom and the $(N-1)$-body bound state, respectively. Therefore, they in the former case with $\alpha_{1,N-1} \gg a_2$ constitute another N-body bound state

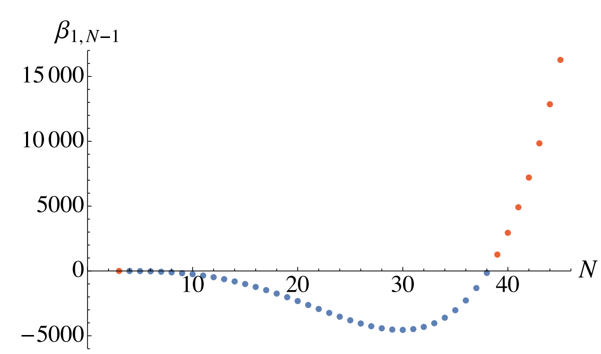

FIG. 2. Scattering length $\alpha_{1,N-1}$ between an atom and an $(N-1)$-body bound state induced by an infinitesimal three-body attraction in the form of $\beta_{1,N-1} \equiv a_2\ln(a_3/a_2)/(\sqrt{3}\pi\alpha_{1,N-1})$. It turns out to be positive for $N = 3$ and $N \geqslant 39$ but negative for $4 \leqslant N \leqslant 38$ as indicated by the different colors.

TABLE I. Values of $\beta_{1,N-1}$ for some selected boson numbers N.

N	$\beta_{1,N-1}$	N	$\beta_{1,N-1}$
3	2/9	20	-2.32241×10^3
4	-3	30	-4.54773×10^3
5	$-184/15$	40	2.94072×10^3
6	$-275/9$	50	4.06680×10^4
7	$-19162/315$	100	2.32605×10^6
8	$-1589/15$	200	6.36300×10^7
9	$-22744/135$	300	3.99017×10^8
10	$-6269/25$	400	1.43180×10^9

induced by the infinitesimal three-body attraction, whose binding energy measured from the threshold at $E=E_{N-1}$ reads

$$-\frac{N}{2(N-1)m\alpha_{1,N-1}^2}=-\frac{3\pi^2N\beta_{1,N-1}^2}{2(N-1)ma_2^2\ln^2(a_3/a_2)}\tag{17}$$

for $\ln(a_3/a_2)\to+\infty$. The values of $\beta_{1,N-1}$ for some selected N are presented in Table I.

Beyond the limit of infinitesimal three-body attraction, the binding energies of N bosons are to be determined by generalizing Eqs. (8) as

$$\left[\frac{a_2}{2}-\frac{1}{2\sqrt{\kappa^2+\frac{1}{4}\left(\sum_{i=3}^N p_i\right)^2+\sum_{i=3}^N\frac{p_i^2}{2}}}\right]\tilde\psi_2(\boldsymbol{p}\backslash\{p_1,p_2\})$$
$$=\int\frac{dp_2}{2\pi}\frac{1}{\kappa^2+\frac{1}{2}\left(\sum_{i=2}^N p_i\right)^2+\sum_{i=2}^N\frac{p_i^2}{2}}\left[\sum_{1\leqslant i<j\leqslant N}^{(i,j)\neq(1,2)}\tilde\psi_2(\boldsymbol{p}\backslash\{p_i,p_j\})+\sum_{1\leqslant i<j<k\leqslant N}\tilde\psi_3(\boldsymbol{p}\backslash\{p_i,p_j,p_k\})\right]_{p_1\to-\sum_{i=2}^N p_i}\tag{18a}$$

and

$$\left[\frac{1}{\sqrt{3}\pi}\ln\left(a_3\sqrt{\kappa^2+\frac{1}{6}\left(\sum_{i=4}^N p_i\right)^2+\sum_{i=4}^N\frac{p_i^2}{2}}\right)\right]\tilde\psi_3(\boldsymbol{p}\backslash\{p_1,p_2,p_3\})$$
$$=\int\frac{dp_2dp_3}{(2\pi)^2}\frac{1}{\kappa^2+\frac{1}{2}\left(\sum_{i=2}^N p_i\right)^2+\sum_{i=2}^N\frac{p_i^2}{2}}\left[\sum_{1\leqslant i<j\leqslant N}\tilde\psi_2(\boldsymbol{p}\backslash\{p_i,p_j\})+\sum_{1\leqslant i<j<k\leqslant N}^{(i,j,k)\neq(1,2,3)}\tilde\psi_3(\boldsymbol{p}\backslash\{p_i,p_j,p_k\})\right]_{p_1\to-\sum_{i=2}^N p_i}.\tag{18b}$$

While elaborate analyses of these coupled integral equations are deferred to a future work, we note that Eq. (18b) without $\tilde\psi_2$ was solved numerically for $N=4$ in the absence of the two-body interaction $a_3/a_2=0$ [9]. Here three four-body bound states (tetramers) were found with their binding energies provided by $\kappa=873.456/a_3$, $11.7181/a_3$, and $1.45739/a_3$. On the other hand, in the opposite limit $a_3/a_2\to+\infty$ where the three-body attraction is infinitesimal, we find above that there exists only one tetramer state with its binding energy $\kappa\to\sqrt{10}/a_2$. Therefore, the bound-state spectrum of four or more bosons as a function of a_3/a_2 is rather nontrivial and should be elucidated in the future work.

In particular, when $a_{3\mathrm{D}}<0$ and $|a_{3\mathrm{D}}|\ll l_\perp$, the N-body to dimer binding-energy ratios predicted from Eqs. (2), (7), (14), and (17) read

$$\frac{E_N}{E_2}=\frac{E_N^{(\mathrm{MG})}}{E_2}+\frac{4N(N^2-1)(N^2-4)\ln(4/3)}{15}\left(\frac{a_{3\mathrm{D}}}{l_\perp}\right)^2\tag{19}$$

for the ground state and

$$\frac{E_N^*}{E_2}=\frac{E_{N-1}}{E_2}+\frac{72N\beta_{1,N-1}^2\ln^2(4/3)}{N-1}\left(\frac{a_{3\mathrm{D}}}{l_\perp}\right)^4\tag{20}$$

for the excited state with $N=3$ or $N\geqslant39$,[3] which may be observable in ultracold atom experiments.

IV. CONCLUSION

In this work, we studied bosons in one dimension with weak two-body and three-body interactions, predicted the existence of two trimer states when both interactions are attractive, and determined their binding energies as universal functions of the two-body and three-body scattering lengths. We also showed that an infinitesimal three-body attraction induces an excited bound state only for 3, 39, or more bosons. Because the effective three-body attraction inevitably arises by confining weakly interacting bosons into lower dimensions, our findings herein have direct relevance to a broad range of quasi-one-dimensional systems realized with ultracold atoms [11,22–24].

ACKNOWLEDGMENT

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[3]The existence and binding energies of these bound states for $N=3$ in quasi-one dimension were first presented in Ref. [26].

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