Finite-size scaling analysis in the two-photon Dicke model

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We perform a Schrieffer-Wolff transformation to the two-photon Dicke model by keeping the leading-order correction with a quartic term of the field, which is crucial for finite-size scaling analysis. Besides a spectral collapse as a consequence of two-photon interaction, the super-radiant phase transition is indicated by the vanishing of the excitation energy and the uniform atomic polarization. The scaling functions for the ground-state energy and the atomic pseudospin are derived analytically. The scaling exponents of the observables are the same as those in the standard Dicke model, indicating they are in the same universality class.

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I. INTRODUCTION

The Dicke model [1] describes a collection of N two-level atoms interacting with a single radiation mode via an atom-field coupling. Due to the spontaneous coherent radiation of the atomic ensemble, a super-radiant quantum phase transition (QPT) occurs [2,3] in the ultrastrong-coupling (USC) regime, where the atom-field coupling strength is comparable to the field frequency [4–7]. There is ongoing interest in the realization of the super-radiant phase in circuit quantum electrodynamics (QED) systems [8–11], where two-level qubits are strongly coupled to microwave cavities. Such experimental achievement has prompted a number of theoretical efforts for generalizations of the Dicke model, such as including anisotropic couplings [12–14] and two-photon interaction [15–17].

In particular, two-photon interaction usually describes a second-order process in different physical setups, such as Rydberg atoms in microwave superconducting cavities [18,19] and quantum dots [20,21]. For an atom coupling to the field via two-photon interaction, the interesting finding is a spectral collapse, for which all discrete system spectra collapse into a continuous band [22–25]. In a collective of atoms systems described by the two-photon Dicke model, the important finding besides a spectral collapse is a super-radiant phase transition [17], which is induced by coherent radiations of the atoms. However, the universal scaling and critical exponents of the super-radiant QPT in the two-photon Dicke model remain elusive. The finite-size correction in a many-body system has been shown to be crucial in the understanding of the universality class in the QPT [26–30]. Numerically, it is very challenging to give a convincing exact treatment of the finite-size two-photon Dicke model. So it is highly desirable to explore finite-size scaling exponents in the atomic ensemble, which are significant for distinguishing the universality class.

The main motivation of this paper is to investigate the universal critical exponents by the analytical scaling functions. We employ a Holstein-Primakoff expansion [2] and Schrieffer-Wolff (SW) transformation [31–34] to diagonalize the Hamiltonian beyond the mean-field approximation. In contrast to the mean-field analysis and second-order quantum fluctuations [17], a lower excitation energy is obtained in the super-radiant phase by our method. Moreover, as an improvement, a quartic potential for the field is added to the leading-order corrections to the effective Hamiltonian, which is crucial to study the quantum criticality. Critical exponents of the ground-state energy and the atomic pseudospin are extracted analytically from the universal finite-size scaling functions. We show that the super-radiant QPT in the two-photon Dicke model belongs to the same universality class as the standard Dicke model [26,27].

The paper is outlined as follows. In Sec. II, the Hamiltonian is diagonalized by a Holstein-Primakoff expansion and SW transformations in the normal and super-radiant phases, respectively. In Sec. III, analytical expressions for some observables are evaluated to show the super-radiant phase transition. In Sec. IV, we discuss the universal finite-size scaling in the critical regime, and the critical exponents are given analytically. Finally, a brief summary is given in Sec. V.

II. THERMODYNAMIC LIMIT

The Hamiltonian of the two-photon Dicke model, where N identical two-level atoms interact with a single bosonic mode via two-photon interaction, is

$$H = \Delta J_z + \omega a^\dagger a + \frac{2g}{N}(a^{\dagger 2} + a^2)J_x, \tag{1}$$

where $a^\dagger$ (a) is the creation (annihilation) operator of the single-mode cavity with frequency ω. The collective angular momentum operators $J_z = \sum_{i=1}^N \sigma_z^{(i)}/2$ and $J_x = \sum_{i=1}^N \sigma_x^{(i)}/2$ describe the ensemble of N two-level atoms with a pseudospin $j = N/2$. Here, Δ is the atomic transition frequency and g is the collective coupling strength of the two-photon interaction.

The Hamiltonian commutes with a generalized Z_4 parity operator Π, which is defined by $\Pi = (-1)^N \otimes_{n=1}^N \sigma_z^{(n)} e^{i\pi a^\dagger a/2}$. Π has four eigenvalues, ± 1 and $\pm i$, and is different from the Z_2 parity in the standard Dicke model [2,3]. The Z_4 parity

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symmetry in the ground state is expected to be spontaneously broken in the super-radiant phase transition.

It is convenient to describe two-photon interaction by introducing new operators $K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2})$, $K_+ = \frac{1}{2}a^{\dagger 2}$, and $K_- = \frac{1}{2}a^2$, which form the SU(1,1) Lie algebra and obey commutation relations $[K_0,K_\pm] = \pm K_\pm$ and $[K_+,K_-] = -2K_0$. Then, we use the Holstein-Primakoff transformation of the collective angular momentum operators defined as $J_+ = b^\dagger\sqrt{N - b^\dagger b}$, $J_- = \sqrt{N - b^\dagger b}b$, and $J_z = b^\dagger b - N/2$ with $[b,b^\dagger] = 1$. After that, the Hamiltonian takes the form

$$H = \Delta(b^\dagger b - N/2) + \omega\left(2K_0 - \frac{1}{2}\right) + \frac{2g}{\sqrt{N}}(K_+ + K_-)\left(b^\dagger\sqrt{1 - \frac{b^\dagger b}{N}} + \sqrt{1 - \frac{b^\dagger b}{N}}b\right). \quad (2)$$

We consider the two-photon Dicke model in the thermodynamic limit for infinite atoms, $N \to \infty$. By means of the boson expansion approach,we expand the Hamiltonian with respected to the bosonic operator $b^\dagger(b)$ as a power series in $1/N$.

A. Normal phase

We derive the Hamiltonian of the normal phase by simply neglecting terms of the order of $O(1/N^{3/2})$ in Eq. (2) as

$$H_{\rm np} = \frac{\omega_1}{N}b^\dagger b + 2\omega K_0 + \lambda(b^\dagger + b)(K_+ + K_-) - \frac{\omega + \omega_1}{2}, \quad (3)$$

where the parameters $\omega_1 = N\Delta$ and $\lambda = 2g/\sqrt{N}$.

Inspired by the SW transformation [31–34], we present a treatment of $H_{\rm np}$ based on the unitary transformation $U = e^R$ with the generator $R = \lambda R_1 + \lambda^3 R_3$. The aim of the SW transformation is to eliminate the block-off-diagonal interacting terms, such as $(b^\dagger + b)(K_+ + K_-)$, and to keep the block-diagonal coupling terms such as $(b + b^\dagger)^2 K_0$ (see Appendix A). Consequently, we keep the terms up to the order of $1/N^2$ and the higher-order terms can be neglected. It results in the transformed Hamiltonian $H'_{\rm np} = H'_1 + H'_2$, consisting of

$$H'_1 = \frac{\omega_1}{N}b^\dagger b - \frac{4g^2}{N\omega}(b + b^\dagger)^2 K_0 + 2\omega K_0 - \frac{\omega + \omega_1}{2} \quad (4)$$

and

$$H'_2 = -\frac{4g^4}{N^2\omega^3}(b + b^\dagger)^4 K_0 - \frac{\omega_1 g^2}{N^2\omega^2}(K_+ - K_-)^2. \quad (5)$$

The Hamiltonian is free of coupling terms between $b + b^\dagger$ and $K_+ - K_-$, and can be simply diagonalized in the subspace of K_0 with $\langle K_0\rangle = 1/4$. In particular, the terms H'_2 involve a quartic potential for the field, which plays a crucial role in the finite-size scaling ansatz. Equation (4) can be diagonalized to be $H_{\rm np} = \varepsilon_1(g)b^\dagger b + E_g^{(1)}$ by a squeezing operator $S = e^{\zeta(b^2 - b^{+2})/2}$ with $\zeta = -\ln(1 - \frac{4g^2}{N\omega\Delta})/4$. And the excitation energy is obtained as $\varepsilon_1(g) = \omega_1\sqrt{1 - g^2/g_c^2}/N$, which is real only when $g \le \sqrt{\omega\omega_1}/2 = g_c$. With the inclusion of the term H'_2, the ground-state energy in the normal phase is

$$E_g^{(1)} = -\frac{\omega_1}{2} + \frac{\omega_1}{2N}\left(\sqrt{1 - \frac{g^2}{g_c^2}} - 1\right) - \frac{g^2}{N^2}\left[\frac{\omega_1}{2\omega^2} + \frac{g^2 g_c^2}{\omega^3(g_c^2 - g^2)}\right]. \quad (6)$$

By comparing with the mean-field results [17], the ground-state energy is obtained by keeping terms of the order of $1/N^2$. Meanwhile, the ground state for the normal phase is $|\varphi_{\rm np}\rangle = U^\dagger S^\dagger|0\rangle_b|0\rangle_{K_0}$, where $|0\rangle_b$ is the vacuum state of the atom ensemble and $|0\rangle_{K_0}$ is the ground state of K_0. One can easily obtain the expectation value of the bosonic operator $\langle\hat{b}\rangle$, which is equal to zero in the normal phase.

B. Super-radiant phase

In the super-radiant phase, there occurs a uniform atomic polarization and the pseudospin J_z is polarized along the z axis. In the Holstein-Primakoff representation, the atomic operator b is expected to be shifted as

$$d = D^\dagger[-\beta\sqrt{N}]bD[-\beta\sqrt{N}] = \beta\sqrt{N} + b, \quad (7)$$

with a unitary transformation $D[-\beta\sqrt{N}] = e^{-\beta\sqrt{N}(\hat{b} - \hat{b}^\dagger)}$. As previously reported, the displacement β is obtained by the mean-field value [17]. We proceed to determine the variable β beyond the mean-field approximation.

Due to the shifted displacement of b, it is obvious that the expectation value of b in the super-radiant state is $\beta\sqrt{N}$, whereas the displacement of the field operator a equals zero due to the absence of linear interactions between atoms and cavity. As a consequence, the Hamiltonian of Eq. (2) becomes

$$H_{\rm sp} = \frac{\omega}{N}d^\dagger d + \frac{\omega_1}{\sqrt{N}}\beta(d^\dagger + d) + \frac{2g\beta_1\beta_2}{\sqrt{N}}(d^\dagger + d)(K_+ + K_-) - \frac{g\beta}{N\beta_1}[d^{\dagger 2} + d^2 + 4d^\dagger d](K_+ + K_-) + H_f + \beta_0 + O(N^{-3/2}), \quad (8)$$

where the field part in the Hamiltonian is $H_f = 2\omega K_0 + \lambda_\beta(K_+ + K_-)$, and the parameters are given by $\beta_1 = \sqrt{1 - \beta^2}$, $\beta_2 = 1 - \beta^2/(1 - \beta^2)$, $\beta_0 = \omega_1\beta^2 - (\omega_1 + \omega)/2$, and $\lambda_\beta = 4g\beta\beta_1$.

First, we apply a squeezing operator $S[r] = e^{-r(a^{\dagger 2} - a^2)/2}$ to diagonalize the field part of the above Hamiltonian H_f. The transformed Hamiltonian is derived as $H_2^{(0)} + V_1 + V_2 + V_3 + V_{\rm linear}$ in Appendix B. We now choose the displacement β to eliminate the term $V_{\rm linear}$ in Eq. (B4) that is linear in the bosonic operators. It gives

$$\omega_1\beta - g\beta_1\beta_2\frac{\lambda_\beta}{\sqrt{\omega^2 - \lambda_\beta^2}} = 0. \quad (9)$$

The $\beta = 0$ solution recovers the normal-phase Hamiltonian. The nontrivial solution gives

$$\beta = \frac{1}{\sqrt{2}}\left[1 - \sqrt{\frac{1 - 4g^2/\omega^2}{16g^4/(\omega\omega_1)^2 - 4g^2/\omega^2}}\right]^{1/2}, \quad (10)$$

which remains real, provided that $1-4g^2/\omega^2 \geq 0$ and $1-\sqrt{\frac{1-4g^2/\omega^2}{16g^4/(\omega\omega_1)^2-4g^2/\omega^2}} \geq 0$. It leads to the collapse point and the critical value of coupling strength, respectively,

$$g_{\text{collapse}} = \omega/2 \tag{11}$$

and

$$g_c = \frac{\sqrt{\omega\omega_1}}{2}. \tag{12}$$

Our solutions shows that the super-radiant QPT occurs at the critical point g_c, which is characterized by nonvanishing of the expectation value of b. Interestingly, the spectrum collapses at g_{collapse}, so that the Hamiltonian is not bounded from below and the model is not well defined. We focus on the parameter regime where the phase transition can be accessed in the validity coupling region $g < g_{\text{collapse}}$. Moreover, since the super-radiant phase transition occurs before the spectral collapse, one has the condition $\omega_1 = N\Delta < \omega$, requiring that the order of magnitude of Δ is ω/N. Hence, the scaled atom frequency $\omega_1 = N\Delta$ is introduced and is comparable to the field frequency ω.

Then, by eliminating the block-off-diagonal coupling terms V_1 in Eq. (B5) and V_2 in Eq. (B6), the Hamiltonian in the super-radiant phase H_{sp} can be diagonalized as

$$H_{\text{sp}} = \varepsilon_2(g)\left(d^\dagger d + \frac{1}{2}\right) + E_g^{(2)}, \tag{13}$$

where the excitation energy is

$$\varepsilon_2(g) = \frac{2\omega_1 - \lambda_3}{2N}\sqrt{1 - \frac{2\lambda_1^2/(2\sqrt{\omega^2-\lambda_\beta^2} + \lambda_3/N) + \lambda_3}{(\omega_1 - \lambda_3/2)}}, \tag{14}$$

and the ground-state energy is

$$E_g^{(2)} = \frac{1}{2}\varepsilon_2(g) - \frac{\omega_1 - \lambda_3}{2N} + \frac{\sqrt{\omega^2 - \lambda_\beta^2}}{2} + \beta_0, \tag{15}$$

with the parameters λ_1 and λ_3, as in Appendix B. Thus, we obtain the diagonal Hamiltonian H_{sp} for the super-radiant phase. If we choose the signs of the displacement as $-\beta$ in Eq. (7), we obtain an identical effective Hamiltonian. It is clear that the spectrum is doubly degenerate in the super-radiant phase.

III. PHASE TRANSITION

After deriving the two effective Hamiltonians in the $N \to \infty$ limit, we now explore the properties of two phases. The excitation energies are given by $\varepsilon_1(g)$ in the normal phase and $\varepsilon_2(g)$ in the super-radiant phase. Figure 1 displays the behavior of the excitation energies as a function of coupling strength g/ω, which is lower than the mean-field result [17] in the super-radiant phase. As the coupling approaches the critical value $g \to g_c$, the excitation energy can be shown to vanish as

$$\varepsilon(\lambda \to \lambda_c) \sim \frac{\omega_1}{N}\sqrt{\frac{2}{g_c}}(g_c - g)^{1/2}. \tag{16}$$

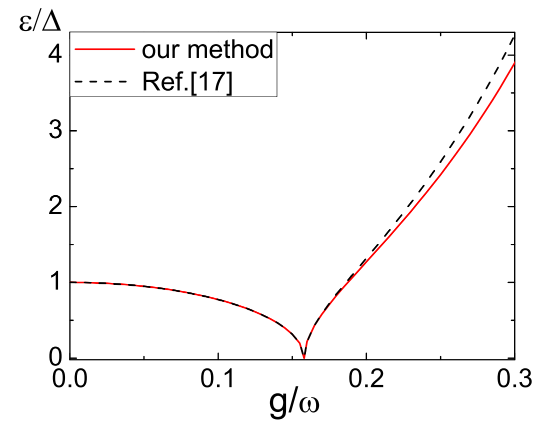

FIG. 1. Excitation energy $\varepsilon(g)/\Delta$ obtained by our method (red solid line) as a function of coupling g/ω for $\omega_1 = 0.1\omega$. For comparison, results obtained by mean-field analysis in Ref. [17] (black dashed line) are calculated.

The vanishing of the excitation energies at g_c reveals that a second-order phase transition occurs.

Figure 2(a) shows the scaled ground-state energy for the normal and super-radiant phases according to the analytical expression in Eqs. (6) and (15), which are consistent with the numerical ones for $N = 100$ atoms. In the thermodynamic limit $N \to \infty$, the scaled ground-state energy E_g/ω_1 at the critical point g_c equals $-1/2$, as shown in Table I.

We calculate the expectation value of the scaled pseudospin,

$$\langle J_z \rangle/N = \beta^2 - 1/2. \tag{17}$$

It makes the physical meaning of the displacement parameter β in Eq. (7) clear, which illustrates the uniform atomic polarization along the z axis. In Fig. 2(b), $\langle J_z \rangle/N$ becomes

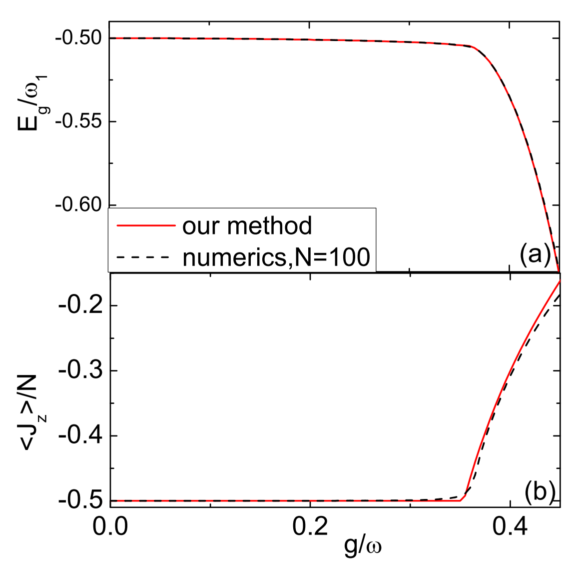

FIG. 2. (a) The scaled ground-state energy $E_g/(\omega_1)$ and (b) the expected value of the scaled pseudospin J_z/N obtained by our method as a function of coupling g/ω for $N = 100$ and $\omega_1/\omega = 0.5$. Solid lines denote our analytical results, whereas dashed lines correspond to exact-diagonalization ones.

TABLE I. Finite-size scaling exponents for the ground-state energy E_g/ω_1, and the scaled atomic angular momenta $\langle J_z\rangle/N$ and $\langle J_y^2\rangle/N^2$ for the two-photon Dicke model. We find that the corresponding scaling exponents are the same as those in the standard Dicke model [26,27].

Q_N	$\lim_{N\to\infty} Q_N$	Two-photon Dicke
E_g/ω_1	$-1/2$	$-4/3$
$\langle J_z\rangle/N$	$-1/2$	$-2/3$
$\langle J_y^2\rangle/N^2$	0	$-4/3$

larger than $-1/2$ when the coupling strength exceeds the critical point $g_c = \sqrt{2}\omega/4$ for $\omega_1/\omega = 0.5$.

As demonstrated above, the behavior of the excitation energies $\varepsilon(g)$, the scaled ground-state energy E_g/ω_1, and the pseudospin $\langle J_z\rangle/N$ are similar to those in the standard Dicke model in the thermodynamic limit [2,3]. It becomes interesting to explore the critical exponents and universality class of the two-photon Dicke model.

IV. FINITE-SIZE SCALING

It is well known that different systems can exhibit similar behavior in the critical regime, giving rise to the universality. Finite-size scaling is a topic of major interest in the QPT system and has solid foundations since the formulation of a general theory [35,36]. As shown in previous studies [26,37,38], the $1/N$ corrections to physical observables such as order parameters display some singularities at the critical point. We now proceed to derive finite-size scaling functions analytically for some observables in the two-photon Dicke model.

We start with the Hamiltonian $H'_{\rm np} = H'_1 + H'_2$ in Eqs. (4) and (5) by including the quartic term for the field. By projecting the Hamiltonian to the subspace $|0\rangle_{K_0}$, we obtain

$$H'_{\rm np} = \frac{\omega_1}{N}b^\dagger b - \frac{\omega_1 g'^2}{4N}(b+b^\dagger)^2 - \frac{\omega_1^2 g'^4}{16N^2\omega}(b+b^\dagger)^4 + c, \tag{18}$$

where $g' = g/g_c$ and a constant term $c = -\omega_1/2 - \omega_1/(2N) - \omega_1 g^2/(2N^2\omega^2)$. To understand the properties of the phase transition, we rewrite $H'_{\rm np}$ by the introduction of coordinate and momentum operators for the bosonic mode, $x = 1/\sqrt{2\omega_1/N}(b^\dagger + b)$ and $p = i\sqrt{\frac{\omega_1}{2N}}(b^\dagger - b)$, as follows:

$$H'_{\rm np} = \frac{1}{2}p^2 + \frac{\omega_1^2}{2N^2}(1-g'^2)x^2 - \frac{\omega_1^4 g'^4}{4N^4\omega}x^4 - \frac{\omega_1}{2N}. \tag{19}$$

It is helpful to rescale the coordinate by $x = \tilde{x}N^\alpha$ and the corresponding momentum by $p = -i\partial/\partial x = \tilde{p}N^{-\alpha}$. Then the Hamiltonian becomes

$$H'_{\rm np} = \frac{1}{2}\tilde{p}^2 N^{-2\alpha} + \frac{\omega_1^2}{2}(1-g'^2)\tilde{x}^2 N^{2\alpha-2} - \frac{\omega_1^4 g'^4}{4\omega}\tilde{x}^4 N^{4\alpha-4}. \tag{20}$$

By setting $\alpha = 2/3$, we obtain the scaling variable

$$\eta = \frac{\omega_1^2}{2}(1-g'^2)N^{2/3} \tag{21}$$

and $\tilde{x} = xN^{-2/3}$. The renormalized Hamiltonian is written as

$$H'_{\rm np} = N^{-4/3}\left[-\frac{\partial^2}{2\partial\tilde{x}^2} + \eta\tilde{x}^2 - \frac{\omega_1^4 g'^4}{4\omega}\tilde{x}^4\right], \tag{22}$$

which is crucial to reveal the universal properties of the second-order QPT.

The ground-state wave function $\varphi_0(\tilde{x},\eta)$ is described straightforwardly by the following equation in terms of $\tilde{x}$ and η:

$$\left[-\frac{\partial^2}{2\partial\tilde{x}^2} + \eta\tilde{x}^2 - \frac{\omega_1^4 g'^4}{4\omega}\tilde{x}^4\right]\varphi_0(\tilde{x},\eta) = E_0(\eta)\varphi_0(\tilde{x},\eta), \tag{23}$$

where $E_0(\eta)$ gives the ground-state energy as

$$E_g = -\frac{\omega_1}{2} - \frac{\omega_1}{2N} + \frac{1}{N^{4/3}}E_0(\eta). \tag{24}$$

From the leading-order correction for the ground-state energy in the above equation, the finite-size scaling exponent of E_g is found to be $-4/3$, which is the same as that for the Dicke model [26,27], as shown in Table I.

Meanwhile, the scaling law of the atomic ensemble angular momenta $\langle J_z\rangle/N = \langle b^\dagger b - N/2\rangle/N$ and $\langle J_y^2\rangle/N^2$ can be derived as

$$\langle J_z\rangle/N = -\frac{1}{2} + \frac{\omega_1}{2}N^{-2/3}X(\eta) + \frac{1}{2\omega_1}N^{-4/3}P(\eta) \tag{25}$$

and

$$\langle J_y^2\rangle/N^2 = \frac{1}{2\omega_1}N^{-4/3}P(\eta), \tag{26}$$

where the universal functions $X(\eta)$ and $P(\eta)$ are the expectation values of $\tilde{x}^2$ and $\tilde{p}^2$ over the ground state $\varphi(\tilde{x},\eta)$. One can see that the leading-order corrections for $\langle J_z\rangle/N$ and $\langle J_y^2\rangle/N^2$ scale as $N^{-2/3}$ and $N^{-4/3}$, respectively. The finite-size scaling exponents are identical to those in the standard Dicke model [26,27] in Table I, providing evidence of the same universality class.

In general, the $1/N$ expansion of a physical quantity $Q_N(g)$ in the vicinity of the critical point of the QPT can be decomposed in a regular and a singular function as follows [38,39]:

$$Q_N(g) = Q_N^{\rm reg}(g) + Q_N^{\rm sing}(g), \tag{27}$$

where $Q_N^{\rm reg}(g)$ and $Q_N^{\rm sing}(g)$ are regular and singular functions at $g = g_c$. With the scaling variable η in Eq. (21), the singular function for an observable in the two-photon Dicke model is given explicitly as

$$Q_N^{\rm sing}(g) = F_Q\left[\frac{\omega_1^2\left(1 - g^2/g_c^2\right)}{2}N^{2/3}\right], \tag{28}$$

where F_Q is a scaling function depending only on the scaling variable $\omega_1^2(1 - g^2/g_c^2)N^{2/3}/2$.

Figure 3 shows the finite-size scaling for the scaled ground-state energy for different sizes $N = 5, 10, 30, 50$, and 100. The singular parts of the ground-state energy $E_g + \omega_1/2 + \omega_1/(2N)$ for different sizes all collapse into a single curve in the critical regime. The numerical results confirm the validity of the universal function $E_0(\eta)$ in Eq. (24), which is independent of N. We also calculate the singular part of $\langle J_z\rangle/N + 1/2$ in

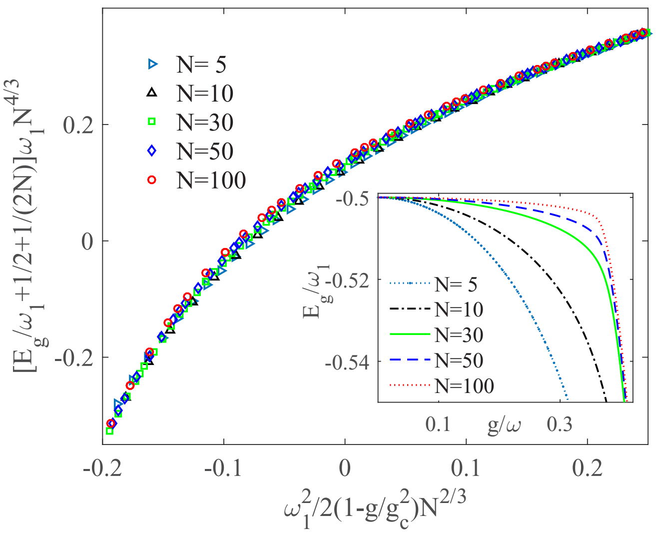

FIG. 3. Finite-size scaling for the scaled ground-state energy in the two-photon Dicke model. Points corresponding to different N collapse on the same curve. Inset: The ground-state energy E_g/ω_1 as a function of the coupling strength g/ω for different N.

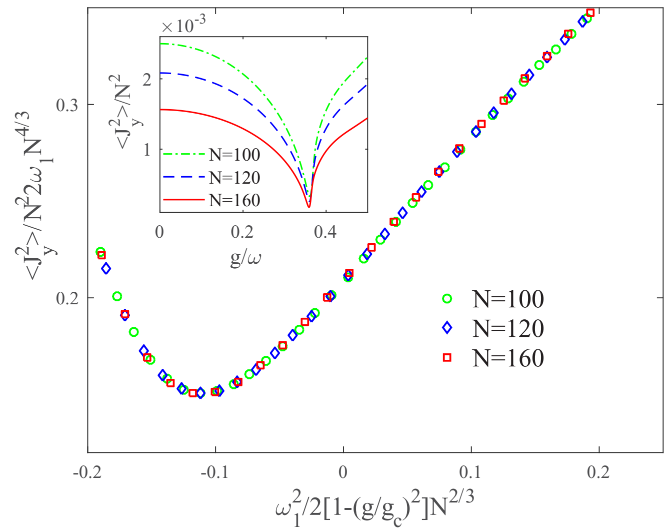

FIG. 5. Finite-size scaling for the scaled pseudospin $\langle J_y^2\rangle/N^2$ in the two-photon Dicke model. Points corresponding to different N collapse on the same curve. Inset: $\langle J_y^2\rangle/N^2$ as a function of the coupling strength g/ω for different N.

Fig. 4 and $\langle J_y^2\rangle/N^2$ in Fig. 5. Excellent collapses in the critical regime are also achieved. The numerical scaling results agree with the universal scaling functions $X(\eta)$ in Eq. (25) and $P(\eta)$ in Eq. (26). The above results demonstrate that the finite-size scaling functions by our treatment capture the universal laws of different observables.

V. CONCLUSIONS

In this paper, by combining the Schrieffer-Wolff transformation with the Holstein-Primakoff expansion, we diagonalize the Hamiltonian of the two-photon Dicke model in the normal and super-radiant phases in the thermodynamic limit, respectively. In the super-radiant phase, the uniform atomic polarization is characterized by the nonzero displacement of the atomic operator, which is obtained beyond the mean-field approximation. The vanishing of the excitation energy at the critical coupling strength illustrates the second-order super-radiant phase transition.

Since a convincing exact treatment of the finite-size two-photon Dicke model is lacking, our approach provides an efficient technique to derive the Hamiltonian by keeping the leading-order correction with the quartic term for the field. Consequently, the leading-order corrections and universal scaling functions for the ground-state energy and the atomic angular momenta are derived analytically, giving the finite-size scaling exponents precisely. We find that the two-photon Dicke model and standard Dicke model are in the same universality class of QPT.

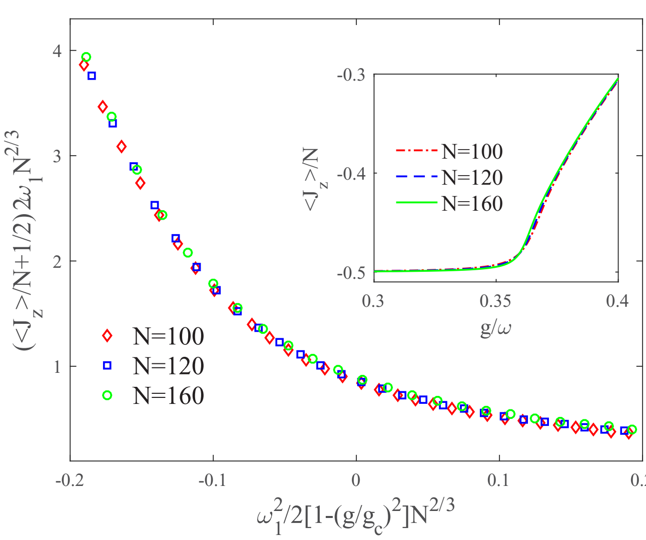

FIG. 4. Finite-size scaling for the scaled pseudospin $\langle J_z\rangle/N$ in the two-photon Dicke model. Points corresponding to different N collapse on the same curve. Inset: $\langle J_z\rangle/N$ as a function of the coupling strength g/ω for different N.

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APPENDIX A: DERIVATION OF THE EFFECTIVE HAMILTONIAN IN THE NORMAL PHASE

The Hamiltonian in the normal phase is written as $H_{\rm np} = H_0 + \lambda V$, consisting of

$$H_0 = \Delta b^\dagger b + 2\omega K_0 - \frac{\omega + \Delta N}{2}, \tag{A1}$$

$$V = (b^\dagger + b)(K_+ + K_-). \tag{A2}$$

We consider a unitary transformation $U = e^R$ with the generator $R = \lambda R_1 + \lambda^3 R_3$. The transformed Hamiltonian

$H'_{\rm np} = e^{-R}H_{\rm np}e^{R}$ is written as

$$\begin{aligned} H'_{\rm np} &= H_0 + \lambda V + \lambda[H_0,R_1] + \frac{\lambda^2}{2}[V,R_1] \\ &+ \lambda^3\left\{[H_0,R_3] + \frac{1}{3}[[V,R_1],R_1]\right\} \\ &+ \lambda^4\left\{\frac{1}{2}[V,R_3] - \frac{1}{24}[[[V,R_1],R_1],R_1]\right\}. \end{aligned} \tag{A3}$$

According to the SW transformation, the off-diagonal coupling terms such as V are required to be eliminated. One obtains

$$[H_0,R_1] = -V, \tag{A4}$$

$$[H_0,R_3] = -\tfrac{1}{3}[[V,R_1],R_1]. \tag{A5}$$

And the generators are determined as

$$R_1 = -\frac{1}{2\omega}(b+b^\dagger)(K_+ - K_-), \tag{A6}$$

$$R_3 = -\frac{1}{6\omega^3}(b+b^\dagger)^3(K_+ - K_-). \tag{A7}$$

Making use of the choice for the generators R_1 and R_3, the transformed Hamiltonian becomes

$$\begin{aligned} H'_{\rm np} &= \Delta b^\dagger b - \frac{4g^2}{N\omega}(b+b^\dagger)^2K_0 \\ &- \frac{\Delta g^2}{N\omega^2}(K_+ - K_-)^2 - \frac{4g^4}{N^2\omega^3}(b+b^\dagger)^4K_0 \\ &+ 2\omega K_0 - \frac{\omega + N\Delta}{2} + O\left(\frac{1}{N\sqrt{N}}\right). \end{aligned} \tag{A8}$$

APPENDIX B: DERIVATION OF THE EFFECTIVE HAMILTONIAN IN THE SUPER-RADIANT PHASE

Let us now consider the Hamiltonian $H_{\rm sp}$ in Eq. (8) in the super-radiant phase. First, the field part of the Hamiltonian $H_f = \lambda_\beta/2(a^{\dagger 2} + a^2) + \omega(a^\dagger a + 1/2)$ can be easily diagonalized by a squeezing transformation $S[r] = e^{-r(a^{\dagger 2}-a^2)/2}$. It leads to

$$\begin{aligned} S[r]H_fS^\dagger[r] &= [\omega\cosh 2r + \lambda_\beta\sinh 2r]\left(a^\dagger a + \frac{1}{2}\right) \\ &+ \frac{1}{2}[\omega\sinh 2r + \lambda_\beta\cosh 2r]\left(a^{\dagger 2} + a^2\right). \end{aligned} \tag{B1}$$

The squeezing parameter r is determined by the vanishing of the $a^{\dagger 2} + a^2$ terms,

$$r = \frac{1}{4}\ln\frac{\omega - \lambda_\beta}{\omega + \lambda_\beta}. \tag{B2}$$

We perform the squeezing transformation to the Hamiltonian $H_{\rm sp}$ in Eq. (8) as $S[r]H_{\rm sp}S^\dagger[r] = H_2^{(0)} + V_{\rm linear} + V_1 + V_2 + V_3$. They are

$$H_2^{(0)} = \frac{\omega_1 - \lambda_3K_0}{N}d^\dagger d + \left(2\sqrt{\omega^2 - \lambda_\beta^2} + \frac{\lambda_3}{N}\right)K_0 + \beta_0, \tag{B3}$$

$$V_{\rm linear} = \frac{1}{\sqrt{N}}[\omega_1\beta + 4g\beta_1\beta_2\sinh(2r)K_0](d^\dagger + d), \tag{B4}$$

$$V_1 = \frac{\lambda_1}{\sqrt{N}}(d^\dagger + d)(K_+ + K_-), \tag{B5}$$

$$V_2 = -\frac{\lambda_2}{N}(4d^\dagger d + d^{\dagger 2} + d^2)(K_+ + K_-), \tag{B6}$$

$$V_3 = -\frac{\lambda_3}{N}(d^\dagger + d)^2K_0, \tag{B7}$$

where $\lambda_1 = 2g\beta_1\beta_2\cosh(2r)$, $\lambda_2 = g\beta\cosh(2r)/\beta_1$, and $\lambda_3 = 2g\beta\sinh(2r)/\beta_1$. Here, we choose the value of β to make the linear term $V_{\rm linear}$ vanish. Then, we employ a transformation $U = e^{\frac{1}{\sqrt{N}}P + \frac{1}{N}Q}$ with the generators P and Q to eliminate the block-off-diagonal terms V_1 and V_2. It leads to

$$\frac{1}{\sqrt{N}}\left[H_2^{(0)},P\right] = -V_1, \tag{B8}$$

$$\frac{1}{N}\left[H_2^{(0)},Q\right] = -V_2, \tag{B9}$$

which give the generators as

$$P = -\frac{\lambda_1}{2\sqrt{\omega^2 - \lambda_\beta^2} + \lambda_3/N}(d^\dagger + d)(K_+ - K_-), \tag{B10}$$

$$Q = \frac{\lambda_2}{2\sqrt{\omega^2 - \lambda_\beta^2} + \lambda_3/N}(4d^\dagger d + d^{\dagger 2} + d^2)(K_+ - K_-). \tag{B11}$$

After that, the transformed Hamiltonian becomes

$$\begin{aligned} H'_{\rm sp} &= \frac{1}{N}(\omega_1 - 2\lambda_3K_0)d^\dagger d + \left(2\sqrt{\omega^2 - \lambda_\beta^2} + \frac{\lambda_3}{N}\right)K_0 \\ &- \frac{1}{N}\left(\frac{2\lambda_1^2}{2\sqrt{\omega^2 - \lambda_\beta^2} + \lambda_3/N} + \lambda_3\right)(d^\dagger + d)^2K_0 + \beta_0. \end{aligned} \tag{B12}$$

By applying a squeezing transformation $S[r_1] = \exp[r_1^2(d^{\dagger 2} - d^2)/2]$, we have

$$\begin{aligned} H''_{\rm sp} &= S^\dagger[r_1]H'_{\rm sp}S[r_1] \\ &= \frac{1}{N}\Bigg[(\omega_1 - 2\lambda_3K_0)\cosh 2r_1 \\ &\quad - 2\left(\frac{2\lambda_1^2}{2\sqrt{\omega^2 - \lambda_\beta^2} + \lambda_3/N} + \lambda_3\right)e^{2r_1}K_0\Bigg] \\ &\quad \times\left(d^\dagger d + \frac{1}{2}\right) - \frac{\omega_1 - 2\lambda_3}{2N} + \left(2\sqrt{\omega^2 - \lambda_\beta^2} + \frac{\lambda_3}{N}\right) \\ &\quad \times K_0 + \beta_0 + \lambda_4(d^{\dagger 2} + d^2), \end{aligned} \tag{B13}$$

with $\lambda_4 = \frac{1}{2N}[(\omega_1 - 2\lambda_3K_0)\sinh 2r_1 - 2(\frac{2\lambda_1^2}{2\sqrt{\omega^2-\lambda_\beta^2}+\lambda_3/N} + \lambda_3)e^{2r_1}K_0]$. Making the $(d^{\dagger 2} + d^2)$ term vanish in the subspace

$|0\rangle_{K_0}$, we obtain the squeezing parameter

$$r_1 = -\frac{1}{4}\ln\left[1 - \frac{2\lambda_1^2/\left(2\sqrt{\omega^2-\lambda_\beta^2}+\lambda_3/N\right)+\lambda_3}{(\omega_1-\lambda_3/2)}\right]. \tag{B14}$$

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