Differential and integral Jones matrices for a cholesteric

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Previous attempts to derive the differential Jones matrix (DJM, N) by Jones [Jones, J. Opt. Soc. Am. **38**, 671 (1948)] for a twisted crystal and the integral Jones matrix (IJM, J) by Chandrasekhar and Rao [Chandrasekhar and Rao, Acta Crystallogr. A **24**, 445 (1968)] for a cholesteric liquid crystal resulted in Jones matrices, which are valid for the spectral range except the selective light reflection band. We argue that the limitation of their validity is rooted in two key assumptions used in both approaches, namely, (1) local (nonrotated) DJM N^0 and the elementary IJM J^0 (to which the cholesteric is split) are those of a uniform nematic and (2) under rotation of the coordinate system, N^0 and J^0 obey the similarity transformation rule, namely, $N = RN^0R^{-1}$ and $J = RJ^0R^{-1}$, where R is the rotation matrix. We show that both of these assumptions are of limited applicability for a cholesteric, being justified only for weak twist. In our approach, the DJM and IJM are derived for a cholesteric without these assumptions. To derive the cholesteric DJM, we have established the relation between the diagonal form N^{0d} of N^0 and Mauguin solutions [Mauguin, Bull. Soc. Fr. Mineral. Crystallogr. **N° 3**, 71 (1911)] of Maxwell equations for eigenwaves propagating in the cholesteric. Namely, the eigenvalues of N^0 appear to be the wave numbers for the two eigenwaves propagating in the sample. Then the form of N^0 reconstructs from its diagonal form N^{0d}. Our DJM and IJM, derived for a general case of any ellipticity value of the eigenwaves, correspond to an optically anisotropic plate possessing gyrotropy, linear birefringence, and Jones dichroism. In the limiting approximations of circularly polarized eigenwaves and that corresponding to the Mauguin regime, the DJM and IJM reduce to those known from the literature. We found that the form of the transformation rule for the local DJM N^0 under rotation of the coordinate system depends on the regime of light propagation, being different from the similarity transformation rule alluded to above, but reduces to it at weak twist corresponding to the Mauguin regime.

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I. INTRODUCTION

A cholesteric is a liquid crystal, composed of elongated chiral molecules or molecular aggregates with the orientation of their long axes spontaneously helically twisted around a $\vec{Z}$ axis, which is perpendicular to them. In each plane perpendicular to $\vec{Z}$, long molecular or aggregate axes tend to align parallel to a common direction $\overleftrightarrow{n}$ and the molecules or aggregates freely migrate within the plane. Because its ends are undistinguishable, $\overleftrightarrow{n}$ is not a true vector and for this reason is called the director, in the same vein as it is for a nematic. For this reason, the notation $\overleftrightarrow{n}$ seems to be more appropriate for the director than the commonly accepted notation $\vec{n}$. In other words, a cholesteric is a spontaneously twisted chiral nematic with $\overleftrightarrow{n} \perp \vec{Z}$. The distance, at which the director makes a full turn around the $\vec{Z}$ axis, is called the pitch P.

The first description of optical properties for a cholesteric is due to Mauguin [1], who solved Maxwell equations in the framework of a model, according to which the azimuth of the tensor of dielectric permittivity $\hat{\varepsilon}$ of the cholesteric linearly varies along the Cartesian coordinate axis $\vec{Z}$, which is perpendicular to $\overleftrightarrow{n}$. Later the problem was revisited by de Vries [2], Kats [3], Oldano and co-workers [4–7], Relaix *et al.* [8], and other authors. Now the problem is well presented in textbooks [9–12].

Two different approaches have been developed in the framework of Jones matrix method [13–20] to describe the optical properties of a twisted crystal. Namely, Jones proposed two types of matrices, called the *integral* and *differential* matrices, respectively [19]. The *integral Jones matrix* (IJM), which describes an optical element as a whole, does not carry any information about the variation of optical parameters inside the element and was proposed by Jones for optical systems consisting of discrete optical elements. Although for the description of optically inhomogeneous anisotropic media, Jones specifically introduced the notion of a differential matrix, the IJM approach can also be used for this purpose and appears to be even more popular than the *differential Jones matrix* (DJM) approach.

In the IJM approach, an anisotropic medium, in which optical parameters vary along the light-propagation direction, is modeled by a stack of m parallel, thin enough lamellae, such that each elementary lamella can be considered as being

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uniform. The integral Jones matrix J of such a stack is the product of all elementary IJMs J_i,

$$J = \prod_{i=1}^{m} J_i. \tag{1}$$

An attempt to describe the optical properties of a cholesteric using Eq. (1) was made by Chandrasekhar and Rao [10,21]. A cholesteric was modeled by a stack of m spatially rotated plates, such that the plates are parallel, uniform, and uniaxial (read nematic) with the optic axis of each next elementary plate being parallel to the plate plane and helically rotated by a discrete angle around and along the $\vec{Z}$ axis, which is parallel to the plates normal. An inconvenience of this approach is that in Eq. (1), J is expressed symbolically, not via analytical functions and, thus, can be calculated only numerically. At limiting assumption of long pitches, J can be obtained in analytical forms [10,21,22], which appear to be in agreement with those obtained by Mauguin [1] and later by de Vries [2] for this regime. However, the main problem of the approach [21] is that the obtained matrix fails to describe optical properties of the cholesteric in the spectral region of selective reflection. In this paper, we show that the key hypotheses according to which (1) the elementary plate is optically equivalent to a uniform nematic layer and (2) the transformation rule for rotation of the IJM and DJM is used in [19,21], are of limited applicability for cholesterics.

Another attempt to describe the optics of cholesterics with the IJM approach using Eq. (1) was made by Yang and Mi [23], who accounted for the reflection between adjacent slices. The resulting IJM for an elementary lamella successfully describes selective light reflection and appears to have four nonzero components. The latter implies that the elementary lamella is not a nematic. We shall return to the discussion of this issue in Sec. IX.

A fruitful approach for the derivation of the integral Jones matrix for a cholesteric was developed by Oldano and coworkers [24–27]. Their approach is based on Berreman's 4×4 matrix, which by proper transformations is reduced to the 2×2 integral Jones matrix [25,26]. Berreman's 4×4 matrix method [28] remains one of the most powerful theoretical tools for description of light propagation in liquid crystals with a distorted director field. However, it implies complicated time-demanding computer calculations. The result is obtained in a numerical form and consequently is not suitable for analytical considerations. Modification of Berreman's approach, which can be used for analysis via analytical functions, was proposed in Ref. [24]. In Ref. [27], the 2×2 integral Jones matrix obtained by transformation from 4×4 Berreman's matrix is derived for the quasiadiabatic regime of light propagation.

Transmission and reflection integral Jones matrices in terms of eigenwaves accounting for boundary conditions for a cholesteric were constructed by Gevorgyan [29]. This approach is principally different from that in Ref. [21]. The cholesteric is not modeled by a stack of elementary lamella. The integral Jones matrix is derived based on the solutions of Maxwell equations and, thus, adequately describes optical properties of the cholesteric in a whole spectral range, including selective reflection band.

Our approach is different from those developed earlier in Refs. [19,21,23–27,29]. Starting from the Maxwell equations, we derive the DJM for a cholesteric. In a general case, the form of the DJM allows for the identification of optical phenomena contributing to light propagation in an anisotropic medium. A 2×2 DJM has four components, each of which can have a real and imaginary part. These eight parameters correspond to eight possible optical parameters: average refractive and absorption indices, linear birefringence and linear dichroism, circular birefringence (optical activity) and circular dichroism, Jones birefringence and Jones dichroism [19]. The obtained DJM is used for the derivation of the IJM for a cholesteric as explained below.

In the framework of the DJM approach, the IJM is of the form [19]

$$J = e^{\int N(z)dz} J_0, \tag{2}$$

where $N(z)$ is the DJM of the medium and J_0 is the IJM describing the properties of the entrance boundary of the medium. When considering bulk optical properties of the medium, J_0 can be replaced by the identity matrix. The matrix exponent in Eq. (2) is a symbolic representation; it can be represented via analytical functions if $N(z)$ in Eq. (2) is independent of z [19]. The latter thereby implies that for this, the power $\Phi = \int N(z)dz$ of the exponent can be replaced by its averaged value $\bar{\Phi} = \bar{N}\Delta z$, where $\bar{N} = \frac{1}{\Delta z}\int_{z_1}^{z_2} N(z)dz$ is the DJM, averaged on the sample thickness $\Delta z = z_2 - z_1$ between the coordinates z_1 and z_2. Then the matrix exponent can be written in analytical form in terms of hyperbolic functions of the components of the averaged matrix [see Eq. (3.26) in [9], Eqs. (2)–(7) in [22], or Eq. (55) below in this paper].

Jones modeled a twisted crystal by a spatially rotated differential matrix [19], obeying the *similarity transformation rule* [30],

$$N = R(qz)N_0R^{-1}(qz), \tag{3}$$

where

$$R = \begin{pmatrix} \cos(qz) & -\sin(qz) \\ \sin(qz) & \cos(qz) \end{pmatrix} \tag{4}$$

is the rotation matrix, $R^{-1}(qz)$ is the inverse of $R(qz)$, and q is the angular twist per unit length; according to Jones, N_0 is the differential matrix of the untwisted crystal. For the transformation rule, given by Eq. (3), Jones [19] derived the integral matrix of a twisted crystal in the form

$$J = R(qz)\exp\left[N_0 - qR\left(\frac{\pi}{2}\right)\right], \tag{5}$$

where $R(\pi/2) = [\{0,-1\},\{1,0\}]$ is the rotation matrix, given by Eq. (4), at $qz = \pi/2$. For a cholesteric, $q = 2\pi/P$ is the twist wave number. If N_0 is the DJM of the untwisted cholesteric (a *parent nematic*), then one can show [22] that in the limiting cases of long pitches, Eq. (5) is equivalent to the form obtained by Chandrasekhar and Rao [10,21] from Eq. (1). However, it turns out that for $N^0 = [\{n_\parallel,0\};\{0,n_\perp\}]$, Eq. (5) does not describe the phenomenon of selective light reflection as well ($n_\parallel$ and $n_\perp$ are refractive indices measured parallel and perpendicular to the director for the parent nematic). The question of why both the DJM and IJM approaches [19,21] do

not work for the cholesteric within the selective reflection band remains open; we answer it in this paper. Common to both approaches given in Refs. [19,21] is that the differential N_0 and elementary integral J_i matrices are assumed to be those of the uniform nontwisted parent nematic and obey the similarity transformation rule. Below, in Sec. II, we show that the latter assumptions are of limited validity for modeling the IJM and DJM of a cholesteric, being correct only for $P \gg \lambda$.

The goal of this paper is to find analytical expressions for IJM and DJM of a cholesteric without these assumptions. We achieve this by relating the DJM to the Mauguin solutions [1,2] of Maxwell equations. We show that in a general case, the elementary integral and differential matrices of a cholesteric are different from those for the parent nematic. The cholesteric DJM and IJM of elementary plates, to which a cholesteric is split in the framework of Jones matrix calculus, appear to be different for different regimes of light propagation in the cholesteric. To show this, in Sec. III we rewrite Maxwell equations to the matrix form in terms of the cholesteric DJM. To write the cholesteric DJM analytically, we find its eigenvalues and eigenvectors (Sec. IV), from which the DJM and IJM are then reconstructed (Sec. V). Transformation rules for DJM and IJM under rotation of the coordinate system for the propagation regimes of circularly polarized eigenwaves and Mauguin regime are derived in Secs. VI and VII. The DJM and IJM for the general case of light propagation for any light wavelength λ are considered in Sec. VIII. Obtained results are discussed in Sec. IX and conclusions are summarized in Sec. X.

II. JONES MATRICES UNDER ROTATION OF THE COORDINATE SYSTEM

Twist in a crystal can be modeled by rotation of the coordinate system around, say, the $\vec{Z}$ axis, which is along the normal to the sample plate and parallel to the light-propagation direction. In this section, we demonstrate that the dielectric permittivity tensor $\hat{\varepsilon}$ and the Jones matrices behave differently under rotation of the coordinate system.

Let the $\vec{X}$ axis of a nonrotated (original, superscript "0") coordinate system be oriented along a local axis of the dielectric permittivity tensor $\hat{\varepsilon}^0$. For a liquid crystal, the latter implies that the $\vec{X}$ axis can be set along the local director $\overleftrightarrow{n}$. In the nonrotated coordinate system, the dielectric displacement $\vec{D}^0$ and the electric field vector $\vec{E}^0$ of a light wave propagating in the sample are related in a linear fashion,

$$\vec{D}^0 = \varepsilon_0 \hat{\varepsilon}^0 \vec{E}^0, \tag{6}$$

where ε_0 is the dielectric constant. In this paper, we deal with transparent crystals and thus, for the undistorted crystal, $\hat{\varepsilon}^0$ is a diagonal matrix. For a twisted crystal, the $\vec{X}$ axis is helically rotated around the $\vec{Z}$ axis by an angle $\varphi = qz$, where q is the angular twist per unit length. In the rotated coordinate systems,

$$\vec{D} = \varepsilon_0 \hat{\varepsilon} \vec{E}. \tag{7}$$

Under rotation of the coordinate system, the dielectric displacement and electric field vectors of a light wave obey the same transformation rule,

$$\vec{D} = R(qz)\vec{D}^0, \tag{8}$$

$$\vec{E} = R(qz)\vec{E}^0. \tag{9}$$

The matrix $\hat{\varepsilon}$ is a tensor and thus, by definition, transforms according to the rule

$$\hat{\varepsilon} = R(qz)\hat{\varepsilon}^0 R^{-1}(qz). \tag{10}$$

Equation (10) can be obtained by substitution of Eqs. (8) and (9) in Eq. (7), taking into account that $\hat{\varepsilon}^0$ is a diagonal matrix. Two important statements follow from Eqs. (8)–(10). First, rotation of the coordinate system by an angle $\varphi = qz$ leads to the rotation of vectors $\vec{D}^0$, $\vec{E}^0$ and the tensor $\hat{\varepsilon}^0$ by the same angle. Second, back transformation $\hat{\varepsilon}^0 = R^{-1}(qz)\hat{\varepsilon}R(qz)$ gives the tensor of dielectric permittivity for a nontwisted crystal. Below we show that both of these statements are not applicable for both integral and differential Jones matrices if the rotation angle is dependent on z.

The IJM J^0 relates the electric field vector $\vec{E}^{0i}$ of the incident (superscript i) light wave at the sample plate entrance to the corresponding vector $\vec{E}^0$ at its exit in a linear fashion,

$$\vec{E}^0 = J^0 \vec{E}^{0i}. \tag{11}$$

Throughout this paper, the superscript 0 corresponds to the original (nonrotated) coordinate system, in which the X axis is along the local nematic director $\overleftrightarrow{n}$. After rotation of the coordinate system around the $\vec{Z}$ axis by an angle qz, the vectors $\vec{E}^0$ and $\vec{E}^{0i}$ transform into $\vec{E}$ and $(\vec{E}^{0i})'$ according to Eq. (9) and thus one finds a relation between the IJMs J^0 and J, respectively, in the original and rotated coordinate systems,

$$[R^{-1}(qz)JR(qz) - J^0]\vec{E}^{0i} = 0. \tag{12}$$

Matrix equation (12) is a system of two ordinary linear equations with four unknown components J_{ij}. The particular solution is of the form $R^{-1}(qz)JR(qz) - J^0 = 0$. It corresponds to the case when $J^0 = J^{0d}$ is a diagonal matrix. Throughout this paper, the superscript d denotes a diagonal matrix. For such a particular (superscript p) case, from Eq. (12), one finds

$$J^p = R(qz)J^{0d}R^{-1}(qz). \tag{13}$$

However, in a general case of nondiagonal J^0, Eq. (12) cannot be solved with respect to four unknown components without further assumptions and, thus, there is no evidence that in a general case under rotation of the coordinate system the IJM transforms by the same rule as that for the dielectric permittivity tensor $\hat{\varepsilon}$, given by Eq. (10).

Concerning the DJM, Jones had shown that for a constant rotation angle, the DJM transforms in the same way as its corresponding IJM [19]. However, as we show below, this is not true if the rotation angle is z dependent, even for $J^0 = J^{0d}$ being diagonal. In the rotated coordinate system for a particular (superscript p) case of the diagonal $J^0 = J^{0d}$, the DJM N^p is related to its corresponding IJM J^p by the equation [19]

$$N^p = \frac{dJ^p}{dz}(J^p)^{-1}. \tag{14}$$

Substituting Eq. (13) into (14), one finds

$$N^p = R(qz)N^0R^{-1}(qz) + q\left\{R\left(\frac{\pi}{2}\right) - R(qz)J^{0d}R\left(\frac{\pi}{2}\right)(J^{0d})^{-1}R^{-1}(qz)\right\}, \tag{15}$$

which shows that under rotation of the coordinate system, the DJM N^0 transforms differently from its corresponding IJM J^{0d} [compare to Eq. (13)]. As a result, the reverse rotation $R^{-1}(qz)N^pR(qz)$ does not transform the matrix N^p back to the original matrix N^0 of untwisted crystal. This is different from the transformation of the tensor $\hat{\varepsilon}^0$ [Eq. (10)].

We, therefore, are led to conclude that under rotation of the coordinate system, the IJM and DJM transform differently with respect to each other and to $\hat{\varepsilon}$. On one hand, if one assumes that the DJM transforms according to Eq. (3), then the IJM transforms according to Eq. (5). On the other hand, if one assumes that the IJM transforms according to Eq. (13), then its corresponding DJM transforms according to Eq. (15), from which it is seen that at $q \neq 0$, the reverse transformation $R^{-1}(qz)N^pR(qz)$ does not result in the matrix of untwisted crystal.

The consideration given above in this section shows that neither the form of differential matrix N^0 nor its transformation law can be hypothesized in the way as it can be done for the dielectric permittivity tensor. We derive the form of the differential matrix N^0 for a cholesteric from Maxwell equations in the framework of the model of spatially rotated dielectric tensor and then find the transformation law for the DJM under rotation of the coordinate system. It turns out that the transformation law for the DJM under rotation of the coordinate system is different not only from those for $\hat{\varepsilon}$ and J, but also for different regimes of the light propagation through the cholesteric.

III. MATRIX MAXWELL EQUATION FOR A CHOLESTERIC IN TERMS OF DJM

Maxwell equations for propagation of an electromagnetic wave through a flat transparent optically anisotropic plate along the plate normal can be written in the matrix form as follows [2]:

$$\frac{1}{c^2}\hat{\varepsilon}^{XY}\frac{\partial^2\vec{E}^{XY}(z,t)}{\partial t^2} = \frac{\partial^2\vec{E}^{XY}(z,t)}{\partial z^2}, \tag{16}$$

where $\hat{\varepsilon}^{XY}$ is the 2×2 tensor of dielectric permittivity for the cholesteric in the XY plane of the Cartesian coordinate system with the Z axis chosen parallel to the sample plate normal which is along the light wave vector and along the helical axis of the cholesteric; the XY plane is parallel to the plane of a cholesteric layer; and the X axis is arbitrarily oriented with respect to the local director $\overleftrightarrow{n}$. Assuming that the frequency of the wave does not change during the propagation through the crystal, the matrix Maxwell equation (16) reduces to the form

$$\hat{\varepsilon}^{XY}\vec{E}^{XY}(z) = -\lambda\!\!\!^{-2}\frac{d^2\vec{E}^{XY}(z)}{dz^2}, \tag{17}$$

with

$$\lambda\!\!\!^{-} = \frac{\lambda}{2\pi}, \quad \vec{E}^{XY}(z) = \begin{bmatrix} E_x(z) \\ E_y(z) \end{bmatrix}. \tag{18}$$

To solve Eq. (17), it is convenient to transform it to the internal local Cartesian coordinate system $(\vec{\eta}, \vec{\xi}, \vec{Z})$, for which the $\vec{Z}$ axis remains along the light wave vector, the $\vec{\eta}$ axis is parallel, and the $\vec{\xi}$ axis is perpendicular to the local director $\overleftrightarrow{n}$ in each cholesteric layer such that

$$\hat{\varepsilon}^{XY} = R(qz)\hat{\varepsilon}^0R^{-1}(qz), \quad \vec{E}^{XY} = R(qz)\vec{E}^0, \tag{19}$$

where the dielectric tensor $\hat{\varepsilon}^0$ is of the form

$$\hat{\varepsilon}^0 = \begin{bmatrix} \varepsilon_\parallel & 0 \\ 0 & \varepsilon_\perp \end{bmatrix}, \tag{20}$$

with $\varepsilon_\parallel, \varepsilon_\perp$ being the values of dielectric permittivity measured for the light wave polarizations, respectively, parallel and perpendicular to the local director. Below, values defined in the internal coordinate system will be called the local values. Local values $\varepsilon_\parallel, \varepsilon_\perp$ describe the corresponding dielectric permittivity components of the *parent* nematic. The electric field vector in the local coordinate system is of the form

$$\vec{E}^0 = \begin{bmatrix} E_\eta \\ E_\xi \end{bmatrix}, \tag{21}$$

where E_η, E_ξ are projections of the electric field vector on the corresponding axes. Then the matrix equation (17) transforms in the local coordinate system to the form

$$\frac{d^2\vec{E}^0(z)}{dz^2} + 2qR\left(\frac{\pi}{2}\right)\frac{d\vec{E}_0}{dz} + \left(\frac{\hat{\varepsilon}^0}{\lambda\!\!\!^{-2}} - q^2I\right)\vec{E}^0 = 0, \tag{22}$$

where $R(\pi/2) = [\{0,-1\},\{1,0\}]$ is the rotation matrix, given by Eq. (4), at $qz = \pi/2$. According to Jones [19], the following relation is valid for $\vec{E}^0$:

$$\frac{d\vec{E}^0}{dz} = N^0\vec{E}^0, \tag{23}$$

where N^0 is the the DJM in the local coordinate system (local DJM) of a cholesteric, which implies its independence of z and, thereby, $dN^0/dz = 0$. It should be noticed that here N^0 is an unknown matrix, which is not restricted by the form of Eq. (3). Substituting Eq. (23) in Eq. (22), one finds the Maxwell equation written in the local coordinate system in terms of the cholesteric DJM,

$$\left\{(N^0)^2 + 2qR\left(\frac{\pi}{2}\right)N^0 + \frac{1}{\lambda\!\!\!^{-2}}\hat{\varepsilon} - q^2I\right\}\vec{E}^0 = 0. \tag{24}$$

Equation (24) has a nontrivial solution with $\vec{E}^0 \neq 0$, when

$$\det\left\|(N^0)^2 + 2qR\left(\frac{\pi}{2}\right)N^0 + \frac{1}{\lambda\!\!\!^{-2}}\hat{\varepsilon} - q^2I\right\| = 0. \tag{25}$$

In a general case, Eq. (25) contains four unknown components of N^0 and cannot be solved without further assumptions. As we show below, N^0 can be reconstructed from its eigenvalues and eigenvectors with the assumption of an elliptically polarized light wave propagating in the sample.

IV. EIGENVALUES AND EIGENVECTORS FOR N^0

In a general case, the electric field of a light wave propagating in an anisotropic medium can be written in the form

$$\vec{E}^0 = \vec{A}e^{-i\frac{n^0}{\lambda\!\!\!^{-}}z}, \tag{26}$$

where $\vec{A} = [A_\eta, iA_\xi]^{\mathrm{Tr}}$ is the Jones vector of an elliptically polarized wave in the local coordinate system and n^0 is the local refractive index; the superscript Tr denotes the transpose operation such that $\vec{A}$ is a vector column. Substituting Eq. (26) in Eq. (23), we find

$$N^0 \vec{E}^0 = -i\frac{n^0}{\lambda}\vec{E}^0. \tag{27}$$

It is seen from Eq. (27) that by definition, $-i\frac{n^0}{\lambda}$ and $\vec{E}^0$ are, respectively, eigenvalues and eigenvectors of the DJM. Substituting Eqs. (26) and (27) in Eq. (25), one finds

$$(n^0)^4 - 2(\bar{\varepsilon} + \lambda^2 q^2)(n^0)^2 + (\varepsilon_\parallel - \lambda^2 q^2)(\varepsilon_\perp - \lambda^2 q^2) = 0, \tag{28}$$

where $\bar{\varepsilon} = (\varepsilon_\parallel + \varepsilon_\perp)/2$ is the average dielectric permittivity of the parent nematic. Solutions of Eq. (28) are of the form

$$(n^0_\pm)^2 = \bar{n}^2 + \frac{\Delta n^2}{4} + \left(\frac{\lambda}{P}\right)^2 \pm \sqrt{\bar{n}^2\Delta n^2 + 4\left(\bar{n}^2 + \frac{\Delta n^2}{4}\right)\left(\frac{\lambda}{P}\right)^2}, \tag{29}$$

where

$$\bar{\varepsilon} = \frac{\varepsilon_\parallel + \varepsilon_\perp}{2} = \bar{n}^2 + \frac{\Delta n^2}{4}, \quad \bar{n} = \frac{n_\parallel + n_\perp}{2} = \frac{\sqrt{\varepsilon_\parallel} + \sqrt{\varepsilon_\perp}}{2},$$
$$\Delta\varepsilon = \varepsilon_\parallel - \varepsilon_\perp = 2\bar{n}\Delta n, \quad \Delta n = n_\parallel - n_\perp = \sqrt{\varepsilon_\parallel} - \sqrt{\varepsilon_\perp}. \tag{30}$$

Correspondingly, the amplitudes of the eigenvectors of the DJM N^0 are of the form

$$\vec{A}^0_\pm = \begin{bmatrix} 1 \\ if_\pm \end{bmatrix}, \tag{31}$$

where

$$f_\pm = \frac{A^\eta_\pm}{A^\xi_\pm} = \frac{(n_\perp)^2 - (n^0_\pm)^2 - \lambda^2 q^2}{2n^0_\pm q\lambda}. \tag{32}$$

It has to be noticed that apart from notations, the solutions given by Eq. (29) are identical to those obtained by Mauguin [1] and later rediscovered by de Vries [2] from the Maxwell equations, given by Eq. (16). In our approach, we have rewritten Eq. (17) in terms of the DJM [Eq. (24)]. Using the assumption of an elliptically polarized wave [Eq. (26)] and employing Eq. (27), which relates the DJM to its eigenvalues, we have obtained the same solution [Eq. (29)] for the refractive index of the cholesteric as that obtained by Mauguin [1] and de Vries [2]. The latter confirms the compatibility of the conception of differential Jones matrices with Maxwell equations.

V. DJM FOR A CHOLESTERIC IN THE LOCAL COORDINATE SYSTEM

Substituting Eqs. (31) and (32) in Eq. (27), we find eigenvectors of the matrix N^0:

$$\vec{E}^0_\pm = \begin{bmatrix} A^\eta_\pm \\ iA^\xi_\pm \end{bmatrix} \exp\left(-i\frac{n^0_\pm}{\lambda}z\right). \tag{33}$$

The matrix N^0 can be reconstructed from its eigenvalues and eigenvectors as follows:

$$N^0 = T^{0f} N^{0d} (T^{0f})^{-1}, \tag{34}$$

where

$$N^{0d} = -\frac{i}{\lambda}\begin{bmatrix} n^0_+ & 0 \\ 0 & n^0_- \end{bmatrix} \tag{35}$$

is the diagonal form of the DJM N^0, composed of its eigenvalues, given by Eq. (29), and

$$T^{0f} = \begin{bmatrix} 1 & 1 \\ if_+ & if_- \end{bmatrix} \tag{36}$$

is the transform matrix, composed of eigenvectors. From Eq. (34), we find the DJM for a cholesteric in the form

$$N^{0f} = -\frac{1}{\lambda}\begin{bmatrix} i\frac{n^0_+ f_- - n^0_- f_+}{f_- - f_+}, & \frac{n^0_- - n^0_+}{f_- - f_+} \\ \frac{n^0_- - n^0_+}{f_- - f_+} f_+ f_-, & i\frac{n^0_- f_- - n^0_+ f_+}{f_- - f_+} \end{bmatrix}. \tag{37}$$

To check whether Eqs. (29) and (31) satisfy Eq. (27) for eigenvalues and eigenvectors of the matrix N^0, we substitute Eqs. (31) and (37) in the left part of Eq. (27) and obtain the right part of Eq. (27), which confirms that Eqs. (29) and (31) define the eigenvalues and eigenvectors of the differential matrix of a cholesteric.

Equation (37) can be rewritten in the form of the sum of symmetrical and antisymmetrical parts, which shows the physical sense of the matrix components,

$$N^{0f} = \begin{bmatrix} -\frac{i}{\lambda}(\bar{n}^0 + LB), & JD + OA \\ JD - OA, & -\frac{i}{\lambda}(\bar{n}^0 - LB) \end{bmatrix}, \tag{38}$$

where $\bar{n}^0 = (n^0_+ + n^0_-)/2 = i\lambda(N^{0f}_{11} + N^{0f}_{22})/2$ is the average refractive index for the two eigenwaves, $LB = i\lambda(N^{0f}_{11} - N^{0f}_{22})/2 = \bar{n}^0(\varepsilon_\perp - \lambda^2 q^2 - n^0_+ n^0_-)/(\varepsilon_\perp - \lambda^2 q^2 + n^0_+ n^0_-)$ is the linear birefringence, $JD = (N^{0f}_{12} + N^{0f}_{21})/2 = -qLB/\bar{n}^0$ is the Jones dichroism, and $OA = (N^{0f}_{12} - N^{0f}_{21})/2 = q$ is the optical activity.

Equation (38) is a general form of the cholesteric DJM in the local coordinate system. For some extreme values of the ellipticities $f_\pm$ of the eigenwaves, the matrix N^{0f} can be reduced to simpler forms. There are two extreme regimes of the propagation of the electromagnetic light waves in a cholesteric, which correspond to the conditions of small linear birefringence at high twist and of small twist at high enough linear birefringence, respectively [3,9,11,12].

VI. REGIME OF CIRCULARLY POLARIZED EIGENWAVES

A. Local DJM for circularly polarized eigenwaves

At $\Delta\varepsilon \ll q\lambda$, which corresponds to low linear birefringence Δn at small pitch P, from Eq. (32) one finds $f_\pm \to \mp 1$, which corresponds to circularly (superscript c below) polarized waves of opposite handedness. Such a regime realizes in cholesterics with the submicron pitch. For $f_\pm \to \mp 1$, the transform matrix

T^{0f}, given by Eq. (36), takes the form

$$T^{0c} = [\vec{E}^{0+,x}; \quad \vec{E}^{0-,x}] = \begin{bmatrix} 1; & 1 \\ -i; & i \end{bmatrix}, \tag{39}$$

where the superscript x indicates that the x component of the eigenvector is normalized to 1 and the sign $+$ or $-$ indicates which of the two eigenvalues n_+^0 or n_-^0 corresponds to the eigenvector. The transform matrix T^{0c} is composed of the two eigenvectors: $\vec{E}^{0+,x} = [1, -i]^{\mathrm{Tr}}$ and $\vec{E}^{0-,x} = [1,i]^{\mathrm{Tr}}$, where the superscript Tr denotes the transpose operation and indicates that the components, in the square brackets, form a vector column.

In principle, the number of eigenvectors is infinite since their components are related through each other, such that one of the components can be any number. Then, the number of possible transform matrices is also infinite. To resolve this issue, one normalizes the components of eigenvectors by setting one of the two components equal to 1. For example, the components of the eigenvectors $\vec{E}^{0+,x} = [1, -i]^{\mathrm{Tr}}$ and $\vec{E}^{0-,x} = [1,i]^{\mathrm{Tr}}$, which are columns of the matrix T^{0c} in Eq. (39), are related by the equations

$$\frac{E_y^{0\pm}}{E_x^{0\pm}} = \mp i, \tag{40}$$

in which $E_x^{0\pm} = 1$. However, in the same vein, one can set $E_y^{0\pm} = 1$ and, from the same Eq. (40), one has $\vec{E}^{0\pm,y} = [\pm i, 1]^{\mathrm{Tr}}$, where, respectively, the superscript y indicates that now the y component is normalized to 1. Therefore, with the two possibilities for normalization, we have four eigenvectors: $\vec{E}^{0+,x} = [1,-i]^{\mathrm{Tr}}$, $\vec{E}^{0-,x} = [1,i]^{\mathrm{Tr}}$, $\vec{E}^{0+,y} = [i,1]^{\mathrm{Tr}}$, and $\vec{E}^{0-,y} = [-i,1]^{\mathrm{Tr}}$, and thus one can construct four transform matrices in the case of circularly polarized eigenwaves,

$$T^{0c,+x,-x} = [\vec{E}^{0+,x}; \quad \vec{E}^{0-,x}] = \begin{bmatrix} 1; & 1 \\ -i; & i \end{bmatrix}, \quad \det[T^{0c,+x,-x}] = 2i, \tag{41}$$

$$T^{0c,+x,-y} = [\vec{E}^{0+,x}; \quad \vec{E}^{0-,y}] = \begin{bmatrix} 1; & -i \\ -i; & 1 \end{bmatrix}, \quad \det[T^{0c,+x,-y}] = 2, \tag{42}$$

$$T^{0c,+y,-x} = [\vec{E}^{0+,y}; \quad \vec{E}^{0-,x}] = \begin{bmatrix} i; & 1 \\ 1; & -i \end{bmatrix}, \quad \det[T^{0c,+y,-x}] = 0, \tag{43}$$

$$T^{0c,+y,-y} = [\vec{E}^{0+,y}; \quad \vec{E}^{0-,y}] = \begin{bmatrix} i; & -i \\ 1; & 1 \end{bmatrix}, \quad \det[T^{0c,+y,-y}] = 2i. \tag{44}$$

The matrix T^{0c} from Eq. (39) is among the above four matrices, being denoted by $T^{0c,+x,-x}$ [Eq. (41)]. Since the matrix $T^{0c,+y,-x}$ [Eq. (43)] is singular, it cannot be used as a transform matrix. By substituting any of the three nonsingular transform matrices given by Eqs. (41), (42), and (44) into Eq. (34), one obtains the same form of the cholesteric local DJM for the regime of circularly polarized eigenwaves,

$$N^{0c} = \begin{bmatrix} -\frac{i}{\lambda}\bar{n}^0, & -\frac{1}{\lambda}\frac{\Delta n^0}{2} \\ \frac{1}{\lambda}\frac{\Delta n^0}{2}, & -\frac{i}{\lambda}\bar{n}^0 \end{bmatrix}. \tag{45}$$

It is seen from Eq. (45) that in this regime locally a cholesteric slice acts on a propagating light wave as an optically isotropic medium with the average local refractive index $\bar{n}^0$ and optical activity $\frac{1}{\lambda}\frac{\Delta n^0}{2}$, where $\Delta n^0 = n_+^0 - n_-^0$.

B. Transformation of the DJM under rotation of the coordinate system

Equations (37) and (45) give the cholesteric DJM in the local coordinate system attached to the director. Our next step is to find the cholesteric DJM in the rotated coordinate system. Under rotation of the coordinate system by an angle qz, each of the two eigenvectors $\vec{E}_\pm^0$ transform according to the rule given by Eq. (9), such that

$$\vec{E}_\pm = R(qz)\vec{E}_\pm^0. \tag{46}$$

It is worth noticing that the rotation matrix $R(qz)$ can be represented in the form

$$R(qz) = T^{0c}R^d(qz)(T^{0c})^{-1}, \tag{47}$$

where

$$R^d(qz) = \begin{bmatrix} e^{-iqz}, & 0 \\ 0, & e^{iqz} \end{bmatrix} \tag{48}$$

is the diagonal form of the $R(qz)$ matrix, composed of the eigenvalues $e^{\pm iqz}$, and the matrix T^{0c} given by Eqs. (39) and (41) is the transform matrix, composed of the eigenvectors, for the rotation matrix. By substituting Eqs.(46)–(48) in Eq. (23) and performing simple matrix manipulations, one finds, for the electric field vector in the rotated coordinate system,

$$\frac{d\vec{E}}{dz} = T^{0c}N^{0q}(T^{0c})^{-1}\vec{E}, \tag{49}$$

where

$$N^{0q} = -\frac{i}{\lambda}\begin{bmatrix} n_+^0 + q, & 0 \\ 0, & n_-^0 - q \end{bmatrix}. \tag{50}$$

It is seen from the form of Eq. (49) that the matrix factor

$$N^c = T^{0c}N^{0q}(T^{0c})^{-1} \tag{51}$$

in the right-hand side of Eq. (49) is simply the DJM of a cholesteric in the rotated coordinate system and, respectively, N^{0q}, given by Eq. (50), is its diagonal form. Therefore, we are led to conclude that under rotation of the coordinate system by an angle qz, the cholesteric DJM, whose eigenvectors describe circularly polarized waves of opposite handedness, transforms by adding $\pm q$ to its eigenvalues. This conclusion recovers the commonly accepted suggestion [3,9–12] according to which the solution of Maxwell equation (16) in the rotated coordinate system should be of the form $\vec{E}^{\pm} \sim e^{-\frac{i}{\lambda}(n_\pm^0 \pm q)z}$ (see, for example, Eq. (6.22) in Ref. [9]).

The matrix multiplication in Eq. (51) gives the DJM of a cholesteric in the regime of circularly polarized eigenwaves in

the form

$$N^c = \begin{bmatrix} -\frac{i}{\lambda}\bar{n}^0, & -\frac{1}{\lambda}\frac{\Delta n^0}{2} - q \\ \frac{1}{\lambda}\frac{\Delta n^0}{2} + q, & -\frac{i}{\lambda}\bar{n}^0 \end{bmatrix}. \tag{52}$$

The off-diagonal components,

$$N^c_{21} = -N^c_{12} = \frac{1}{\lambda}\frac{\Delta n^0}{2} + q = \mathrm{OA}, \tag{53}$$

are antisymmetric and thus describe only the optical activity (OA) of the cholesteric. It should be noticed that in previous works, the OA for a cholesteric was derived from the form of the eigenwaves. De Vries [2] obtained Eq. (53) using a suggestion that the rotation of the light polarization is an average of two contributions, namely, rotation due to the phase difference for the two eigenwaves in the local coordinate system and rotation of the coordinate system. In Refs. [3,9–12], the Maxwell equations (16) are solved in the rotated coordinate system and thus rotation of the light polarization is calculated simply as a phase difference for the two eigenwaves. In our approach, we obtain the same form of OA directly from the form of the cholesteric DJM, namely, the OA of the cholesteric is the antisymmetric parts of the DJM off-diagonal components. Equation (53) indeed shows that the off-diagonal components $N^c_{21} = -N^c_{12}$ are the result of the sum of the light polarization rotation in the internal coordinate system and the rotation of the coordinate system, thereby confirming the suggestion by de Vries [2], alluded to above.

C. Cholesteric IJM for circularly polarized eigenwaves

The IJM for a cholesteric in the regime of circularly polarized eigenwaves can be obtained from its DJM by substituting Eq. (52) in Eq. (2), with J_0 taken to be the identity matrix,

$$J^c = e^{\int N^c dz}. \tag{54}$$

The DJM given by Eq. (52) is independent of z and thus the IJM given by Eq. (54) can be expressed analytically [19,22]:

$$J^c = e^{\frac{1}{2}(N^c_{11}+N^c_{22})z}\begin{bmatrix} \cosh\frac{D}{2} + \frac{(N^c_{11}-N^c_{22})z}{D}\sinh\frac{D}{2}, & 2\frac{N^c_{12}}{D}\sinh\frac{D}{2} \\ 2\frac{N^c_{21}}{D}\sinh\frac{D}{2}, & \cosh\frac{D}{2} - \frac{(N^c_{11}-N^c_{22})z}{D}\sinh\frac{D}{2} \end{bmatrix}, \tag{55}$$

where $D = z\sqrt{(N^c_{11}-N^c_{22})^2 + 4N^c_{12}N^c_{21}}$.

From Eq. (52), one finds

$$\frac{N^c_{11}+N^c_{22}}{2} = -\frac{i}{\lambda}\bar{n}^0, \quad N^c_{11} - N^c_{22} = 0,$$
$$D = z\sqrt{-4\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)^2} = 2i\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)z, \tag{56}$$

and thus the IJM of a cholesteric in the regime of circularly polarized eigenwaves is of the form

$$J^c = e^{-i\frac{\bar{n}^0}{\lambda}z}\begin{bmatrix} \cos\left\{\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)z\right\}, & -\sin\left\{\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)z\right\} \\ \sin\left\{\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)z\right\}, & \cos\left\{\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)z\right\} \end{bmatrix}$$
$$= e^{-i\frac{\bar{n}^0}{\lambda}z}R\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right). \tag{57}$$

The matrix in Eq. (57) is the rotation matrix $R(\chi)$ with the rotation angle $\chi = (\frac{1}{\lambda}\frac{\Delta n^0}{2} + q)z$, from which the optical activity is of the form

$$\mathrm{OA} = \frac{d\chi}{dz} = \frac{1}{\lambda}\frac{\Delta n^0}{2} + q, \tag{58}$$

which is equivalent to Eq. (53), obtained from the form of the DJM. A linearly polarized wave $\vec{E}^i = [1,0]^{\mathrm{Tr}}$, which is normally incident (superscript i) on a cholesteric with its helical axis along the cell normal, transforms by the matrix J^c [Eq. (57)] into the exiting wave,

$$\vec{E} = J^c\vec{E}^i = \begin{bmatrix} J^c_{11} \\ J^c_{21} \end{bmatrix} = e^{-i\frac{\bar{n}^0}{\lambda}z}\begin{bmatrix} \cos\left\{\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)z\right\} \\ \sin\left\{\left(\frac{1}{\lambda}\frac{\Delta n^0}{2} + q\right)z\right\} \end{bmatrix}. \tag{59}$$

From Eq. (59), one finds that the exiting wave is linearly polarized and its azimuth is defined as $\tan\chi = \frac{J^c_{21}}{J^c_{11}} \Rightarrow \chi = (\frac{1}{\lambda}\frac{\Delta n^0}{2} + q)z$, which leads to the optical activity in the form given by Eq. (58) and which is the same as that given by Eq. (53), obtained from the form of the DJM.

VII. MAUGUIN REGIME

In the previous section, we have derived the cholesteric DJM and IJM for the extreme case of small optical anisotropy at a high twist corresponding to the condition of small Mauguin number, $\mathrm{Ma} = (P/2\lambda)\Delta n \ll 1$, in which the eigenwaves are circularly polarized waves of opposite handedness. In this section, we consider an opposite extreme case of small twist at high enough linear birefringence, the so-called Mauguin regime (also called the waveguide regime [9]) corresponding to the condition $\mathrm{Ma} \gg 1$. The latter condition is equivalent to $q\lambda \ll \Delta n/2$ and thus at this condition from Eq. (29) one finds $n^0_+ \to n_\parallel$ and $n^0_- \to n_\perp$; for ellipticities from Eq. (32), one has $f_+ \to \infty$ and $f_- \to 0$, from which the eigenwaves corresponding to these eigenvalues and the ellipticities are $\vec{E}^{0+} = [0;1]^{\mathrm{Tr}}$ and $\vec{E}^{0-} = [1;0]^{\mathrm{Tr}}$. Therefore, the eigenwaves in the Mauguin regime are linearly polarized waves as in a nematic. Then, for the diagonal DJM in the Mauguin regime (correspondingly denoted by the superscript M),

$$N^{0M} = -\frac{i}{\lambda}\begin{bmatrix} n_\perp; & 0 \\ 0; & n_\parallel \end{bmatrix}, \tag{60}$$

the corresponding transform matrix T^{0M} in Eq. (34) is the identity matrix and therefore the DJM N^M of a cholesteric in the Mauguin regime is a diagonal matrix, $N^M = N^{0M}$ [Eq. (60)]. Similarly as it was done for the regime of circularly polarized waves, one can find three other normalized transform

matrices, but they appear to be singular and thus are not suitable for substitution in Eq. (34).

The absence of the off-diagonal components in the local DJM [Eq. (60)] indicates that the elementary slice of a cholesteric in the Mauguin regime is not optically active, which is different from that in the regime of circularly polarized waves.

Under rotation of the coordinate system, a diagonal DJM obeys the similarity transformation rule, similar to that for the dielectric permittivity tensor, given by Eq. (3). Then, it is the very case which was considered by Jones [19] for a twisted crystal. The IJM derived by Jones for a twisted crystal whose DJM transforms according to Eq. (3) is given by Eq. (5). Substitution of Eq. (60) in Eq. (5) gives the cholesteric IJM in the Mauguin regime,

$$J^M = R(qz)\exp\left[\left\{N^{0M} - qR\left(\frac{\pi}{2}\right)\right\}z\right]. \tag{61}$$

Since the matrix $N^{0M} - qR(\pi/2)$ in Eq. (61) is independent of z, J^M can be written analytically,

$$J^M = e^{-i\frac{\bar{n}}{\lambda}z}\begin{bmatrix}\cos(qz), & -\sin(qz)\\ \sin(qz), & \cos(qz)\end{bmatrix} \times\begin{bmatrix}\cos(\Gamma z)+\frac{i}{\lambda}\frac{\Delta n}{2\Gamma}\sin(\Gamma z), & \frac{q}{\Gamma}\sin(\Gamma z)\\ -\frac{q}{\Gamma}\sin(\Gamma z), & \cos(\Gamma z)-\frac{i}{\lambda}\frac{\Delta n}{2\Gamma}\sin(\Gamma z)\end{bmatrix}, \tag{62}$$

where $\Gamma = \sqrt{\left(\frac{\Delta n}{4\lambda}\right)^2 + q^2}$. Apart from the notations, Eq. (62) is identical to the form of the IJM given by Yeh and Gu (Eq. (4.3-10) in Ref. [31]) for a twisted nematic cell. It is worth noticing that Yeh and Gu derived it using the approach given by Eq. (1). Therefore, we confirm our statement made in [22], according to which the approach of the DJM developed by Jones for a twisted crystal in Ref. [19] is equivalent to that obtained with IJMs by slicing a cholesteric to elementary nematic plates [10,21,31]. Jones equation (5) is equivalent to Eq. (4.3-10) from Ref. [31], as well as to our Eq. (62), and appears to be applicable for weak twist, corresponding to the Mauguin regime, but it does not work in the spectral region of selective light reflection.

For much weaker twist such that $q\lambda \to 0$, while qz is still high enough, the IJM [Eq. (62)] for a twisted nematic cell simplifies to the form

$$J^a = e^{-i\frac{\bar{n}}{\lambda}z}\begin{bmatrix}\cos(qz), & -\sin(qz)\\ \sin(qz), & \cos(qz)\end{bmatrix}\begin{bmatrix}e^{\frac{i}{\lambda}\Delta nz}, & 0\\ 0, & e^{-\frac{i}{\lambda}\Delta nz}\end{bmatrix}. \tag{63}$$

Such a form of the IJM describes the *adiabatic regime* (superscript a in J^a), which is realized in twisted nematic cells, for which the pitch P is comparable to the cell thickness d or larger. Indeed, for a 90°-twisted cell of the thickness $z = d = 10\ \mu$m at the light wavelength $\lambda = 0.5\ \mu$m, one has $qz = \pi/2$, whereas $q\lambda = 0.05$.

VIII. GENERAL CASE

Without any assumption concerning the ellipticity values $f_\pm$ [Eq. (32)], the DJM of a cholesteric is defined by Eq. (37). Taking derivative d/dz from both sides of Eq. (9) and using Eq. (23), one can show that the cholesteric DJM N^{0f} for any values of $f_\pm$ transforms under rotation of the coordinate system as follows:

$$N = R(qz)\left[N^{0f} + qR\left(\frac{\pi}{2}\right)\right]R^{-1}(qz). \tag{64}$$

Equation (64) shows that under rotation of the coordinate system, the matrix N^{0f}, derived in the local coordinate system, does not transform according to the similarity transformation rule, given by Eq. (3). Instead, this law is valid for the matrix $N^{0fq} = N^{0f} + qR(\frac{\pi}{2})$. According to Jones [19], if a DJM (here the matrix N^{0fq}) transforms according to the law, given by Eq. (3), then its IJM transforms according to Eq. (5). Therefore, the IJM of a cholesteric for any $f_\pm$ is of the form

$$J = R(qz)J^{0f}, \tag{65}$$

where

$$J^{0f} = e^{[N^{0f}]z} = e^{-\frac{i}{\lambda}\bar{n}^0 z}\Delta J^{0f}, \tag{66}$$

with

$$\Delta J^{0f} = \begin{bmatrix}\cos(\Gamma z) - i\frac{LB}{\lambda\Gamma}\sin(\Gamma z), & \frac{JD+OA}{\lambda\Gamma}\sin(\Gamma z)\\ \frac{JD-OA}{\lambda\Gamma}\sin(\Gamma z), & \cos(\Gamma z) + i\frac{LB}{\lambda\Gamma}\sin(\Gamma z)\end{bmatrix}. \tag{67}$$

From Eq. (65), one finds the cholesteric IJM in the form

$$J = e^{-i\frac{\bar{n}^0}{\lambda}z}\begin{bmatrix}\Delta J^{0f}_{11}\cos(qz) - \Delta J^{0f}_{21}\sin(qz), & \Delta J^{0f}_{12}\cos(qz) - \Delta J^{0f}_{22}\sin(qz)\\ \Delta J^{0f}_{21}\cos(qz) + \Delta J^{0f}_{11}\sin(qz), & \Delta J^{0f}_{22}\cos(qz) + \Delta J^{0f}_{12}\sin(qz)\end{bmatrix}. \tag{68}$$

If a linearly polarized light wave normally incidences on a crystalline slab, the azimuth χ of the polarization ellipse long axis for the exiting light wave is determined by the following expression [32]:

$$\tan 2\chi = \frac{r + r^*}{1 - |r|^2}, \tag{69}$$

where

$$r = \frac{J_{21}}{J_{11}} = \frac{\Delta J^{0f}_{21}\cos(qz) + \Delta J^{0f}_{11}\sin(qz)}{\Delta J^{0f}_{11}\cos(qz) - \Delta J^{0f}_{21}\sin(qz)}, \tag{70}$$

and r^* is the complex conjugate of r. Substituting Eq. (70) in Eq. (69), after some trigonometric transformations, we find

$$\chi = \theta + qz, \tag{71}$$

where θ is defined through the relation

$$\tan 2\theta = \frac{\mathrm{Re}\left\{J_{11}^{0f}\right\}\mathrm{Re}\left\{J_{21}^{0f}\right\} + \mathrm{Im}\left\{J_{11}^{0f}\right\}\mathrm{Im}\left\{J_{21}^{0f}\right\}}{\left|J_{11}^{0f}\right|^2 - \left|J_{21}^{0f}\right|^2}. \tag{72}$$

From Eq. (71), one finds optical activity in the form

$$\mathrm{OA} = \frac{d\chi}{dz} = \frac{d\theta}{dz} + q. \tag{73}$$

The analytic expression for $\frac{d\theta}{dz}$, obtained from Eq. (72), appears to be lengthy and we do not present it here. There are no principal difficulties to derive it, though one has to keep in mind that the forms of the real and imaginary parts of the matrix components J_{11}^{0f}, J_{21}^{0f} as well as their moduli $|J_{11}^{0f}| = \sqrt{(\mathrm{Re}\{J_{ij}\})^2 + (\mathrm{Im}\{J_{ij}\})^2}$ are different outside and inside the spectral region $Pn_\perp < \lambda < Pn_\parallel$ of selective light reflection, where the refractive index n_-^0 [Eq. (29)] for one of the eigenwaves becomes imaginary.

IX. DISCUSSION

Although the two different approaches of, respectively, integral and differential matrices were proposed by Jones in the same cycle of papers [13–20], written in a relatively close time period, they have received different attention in the literature. The integral Jones matrices (IJMs) originally developed for systems with discrete optical elements are intensively and successfully exploited for these purposes (see [33] for review). IJMs are also used for the modeling of spatially modulated optical media and appear to be even more popular than the differential Jones matrices (DJMs), especially introduced by Jones for this purpose. Reports on successful application of the DJM method are scarce [22,34]. Below we argue that there are some problems in basing assumptions of IJM and DJM approaches, which might explain why both approaches [19,21] do not work in the spectral regime of selective reflection for cholesterics.

Two key hypotheses as a rule are used in Jones' calculus when applied to nonuniform optical media. First, in the IJM approach, it is assumed that a nonuniform optical medium can be modeled by a stack of thin enough slices such that each slice can be considered as being optically equivalent to the medium in its undistorted (parent) state, which is further referred to as the *assumption of parent state*. However, from the point of view of crystallographic symmetry, the assumption of parent state for an elementary slice of a distorted medium is questionable. Indeed, once the medium is distorted, its crystallographic symmetry (described by the point group) lowers and becomes different from that of its parent state, even locally. The difference in symmetry implies differences in physical properties. Intuitively, one expects that for weak distortions, these differences might be vanishing, whereas for stronger distortions, they cannot be neglected even for a reasonably thin elementary slice. Strong enough distortions might lead to the situation when the response of the medium to the electric field of the light wave depends not only on the electric field in a given point of the medium, but also on the space derivatives of the field. These are effects of spatial dispersion [35,36].

In application to a cholesteric, the assumption of parent state implies that the elementary slices, to which the medium is split in the IJM approach [21] and the local DJM (in the DJM approach [19]), are assumed to be those of a uniform nematic. Point group symmetry of the cholesteric is $\infty 2$, whereas for the undistorted nematic it is ∞/mm. Once the nematic director field is twisted, the symmetry of the sample lowers to that of the cholesteric, which leads to optical properties similar to those of the cholesteric, including optical activity. The forms of the DJM and IJM, derived above for different regimes, indicate that differences in optical properties of the elementary slice of the twisted nematic or cholesteric in comparison with those of the undistorted nematic are negligible for weak twist but show up when the twist becomes strong enough. The assumption of parent state appears to be valid only for long-pitch cholesterics in the Mauguin regime. For short-pitch cholesterics out of the Mauguin regime, the elementary IJM and the local DJM appear to be different from those for a uniform nematic. We find that in a general case, the local DJM and, consequently, the elementary IJM correspond to an elementary plate possessing optical activity (also called gyrotropy), linear birefringence, and Jones dichroism;, see Eq. (38). In the extreme approximation of circularly polarized eigenwaves corresponding to high values of the ratio $\lambda/P \gg 1$ (high twist), linear birefringence and Jones dichroism vanish such that the cholesteric in this regime becomes an optically isotropic gyrotropic medium. In the opposite extreme approximation $\lambda/P \ll 1$ (low twist), the eigenwaves are linearly polarized and the elementary slice is equivalent to a uniform nematic layer. With the assumption of parent state in the IJM and DJM approaches, a specific optical effect in the spectral range of selective light reflection appears to be excluded from consideration. We argue below that by applying the assumption of parent state to the cholesteric, one neglects the contribution of spatial dispersion to the local refractive index.

It is worth noticing that the DJM of the cholesteric in the Mauguin regime is a diagonal matrix N^{0M} [Eq. (60)], with zero antisymmetric off-diagonal components responsible for optical activity. Optical activity *per se* is an effect of spatial dispersion [35,36] and, thus, is expected to be strong for strong enough twist of the director field and should vanish for weak twist. This is what we find for the DJMs of short- and long-pitch cholesterics. The absence of the antisymmetric off-diagonal components, responsible for optical activity in the DJM [Eq. (60)], for a long-pitch cholesteric (or mechanically twisted nematics) in the Mauguin regime indicates that the spatial dispersion in this regime is weak. Though the antisymmetric DJM components are zero in the Mauguin regime, there is a giant rotation of the light polarization coming from the form of the corresponding IJM [Eq. (61)], which appears to be a matrix product of the rotation matrix $R(qz)$ by the matrix exponent of $N^{0M} - qR(\pi/2)$ and, thus, implies rotation of the light polarization. The absence of the off-diagonal components, responsible for optical activity in the local DJM N^{0M}, but the presence of antisymmetric components in its corresponding IJM indicates that there are two different sources of the rotation

of the light polarization. Namely, it can be a result either of nonzero antisymmetric off-diagonal DJM components, which by their physical sense describe OA, or of the nonzero off-diagonal IJM components, at zero off-diagonal antisymmetric DJM components. The latter statement is even better illustrated for the adiabatic regime, in which both the DJM and the exponent matrix in Eq. (63) are those of a uniform nematic with zero off-diagonal components. However, in the corresponding IJM [Eq. (63)], the diagonal exponent matrix is multiplied by the rotation matrix $R(qz)$, which is the origin of the rotation of the light polarization. To distinguish between these two origins of the light polarization rotation, we propose to call the rotation of light polarization coming from antisymmetric off-diagonal components of the local DJM as *differential optical activity*, whereas the light polarization rotation coming from the off-diagonal components of the IJM at zero differential optical activity should be classified as the *integral optical activity*. In the presence of both differential and integral optical activity contributions, it can be termed the *mixed optical activity*.

Therefore, the optical activity of the cholesteric (or twisted nematic) in the Mauguin regime is purely of integral origin. Optical activity of the elementary slice in the regime of circularly polarized eigenwaves in the local coordinate system is of the differential origin. In a general case, the optical activity of the cholesteric is of mixed character.

By its origin, the differential optical activity is the effect of spatial dispersion. Applying the assumption of parent state to the elementary slice in the IJM approach or to the DJM, one neglects the effects of spatial dispersion, which are important for short-pitch cholesterics. A parameter controlling the importance of spatial dispersion for the cholesteric is the ratio λ/P. At $\lambda \ll P$, from Eq. (29) one finds that the refractive indices of the eigenwaves, propagating in the cholesteric, approach the values of the refractive indices of the parent nematic, namely, $n^0_{\pm}|_{\frac{\lambda}{P}\to 0} \to n_{\parallel/\perp}$. At short enough cholesteric pitch, the refractive indices of the eigenwaves depend on the pitch, indicating that the spatial dispersion contribution becomes important.

One might argue that the assumption of parent state, according to which the cholesteric is sliced to nematic elementary slabs, is successfully used in Ref. [23], which correctly describes selective light reflection in the IJM approach. However, to achieve this, one accounts for multiple reflection between adjacent slices. The electric field of the transmitted (reflected) light appears to be dependent on the field of reflected (transmitted) light and on the field in the adjacent slab. This is simply the account for spatial dispersion of the medium. The resulting IJM of an elementary slab contains four nonequal components. The presence of nonequal off-diagonal components, which are complex numbers, indicates that such a Jones matrix describes a slab, which is not a simple nematic.

The second basic assumption, used in both IJM and DJM approaches, concerns the transformation rule for the rotation of Jones matrices. In the traditional Jones matrix calculus with the applied assumption of parent state for an elementary slice or local DJM, the modulation of optical properties of a cholesteric is modeled by the z dependence of the rotation matrix, $R(qz)$. Namely, in previous works [19,21], it was assumed that under rotation of the coordinate system, respectively, the DJM and IJM transform by the similarity rule [Eqs. (3) and (13)], which was applied in Refs. [1,2] to the tensor of dielectric permittivity [Eq. (10)]. We have shown in the previous section that this second assumption is also of limited applicability for a cholesteric. The local DJM and, consequently, the IJM obey the transformation rule, given by Eqs. (3) and (13) only in the Mauguin regime. In a general case, under rotation of the coordinate system, the DJM transforms according to Eq. (64) and its corresponding IJM transforms by Eq. (65).

Analysis performed in previous sections shows that in a general case, the IJM of the elementary slice and the DJM cannot be assumed to be those of the undistorted nematic. Their forms were derived from Maxwell equations. We have shown that the Mauguin solutions of Maxwell equations for the refractive indices n^0_+ and n^0_- [Eq. (29)] of the eigenwaves give the eigenvalues of the DJM and the electric field vectors of the eigenwaves are its eigenvectors [Eq. (33)]. The eigenvalues of the DJM give the two nonzero components $N^{0d}_{11} = -\frac{i}{\lambda}n^0_+$ and $N^{0d}_{22} = -\frac{i}{\lambda}n^0_-$ of the diagonal form of the DJM [Eq (35)]. The full form N^{0f} [Eqs. (37) and (38)] of the DJM is reconstructed from its diagonal form N^{0d} [Eq. (35)] using the transform matrix T^{0f} [Eq. (36)] formed of its eigenvectors [Eq. (31)]. As a result, in a general case, the DJM for a cholesteric appears to not be a matrix of a nematic plate.

It should be noticed that the Jones matrix calculus deals with the polarization of the light wave propagating in a medium. The direction of the light propagation is not considered with this method, but can be described in a framework of the so-called ray tracing matrix (RTM) approach [37,38].

X. CONCLUSION

The IJM and DJM approaches to the modeling of cholesteric liquid crystals, available in the literature [19,21], lead to results which are of limited applicability. They do not describe the optical properties of the cholesteric in the selective reflection spectral region. We find that this is a result of limited applicability of the basic assumptions used in these approaches. First, it is traditionally assumed that the IJM J^0 of an elementary slice and the local DJM N^0 of the cholesteric correspond to a uniform (parent) nematic plate (assumption of parent state). According to the second assumption, the IJM or DJM in the rotated coordinate system is obtained simply by rotation of the nematic matrices J^0 or N^0. We have shown that both of these assumptions are valid only for weak twist in the Mauguin regime and argue that the assumption of parent state excludes from consideration the spatial dispersion contribution to the refractive indices.

We have derived the DJM and IJM for a cholesteric liquid crystal without these assumptions. Our DJM and IJM, derived for the general case of any ellipticity value $f_\pm$ of the eigenwaves, correspond to an optically anisotropic plate possessing gyrotropy, linear birefringence, and Jones dichroism. In the limiting approximations of circularly polarized waves and that corresponding to the Mauguin regime, the DJM and IJM reduce to those known from the literature. In the approximation of the Mauguin regime, the eigenwaves become linearly polarized; the transform matrix T^{0f} reduces to the identity matrix; Jones

dichroism and optical activity vanish such that the local DJM becomes diagonal and corresponds to a nondistorted nematic plate. In the adiabatic regime, both the DJM and the exponent matrix are diagonal and correspond to those of the uniform nematic.

To derive the cholesteric DJM, we established the relation between the diagonal form N^{0d} in the local coordinate system to the Mauguin solutions of Maxwell equations for refractive indices n_+^0 and n_-^0 [Eq. (29)]. We have shown that the eigenvalues N_{11}^{0d} and N_{22}^{0d} of the local DJM are the wave numbers $-\frac{i}{\lambda}n_+^0$ and $-\frac{i}{\lambda}n_-^0$ for the two eigenwaves propagating in the sample. Then the full form N^{0f} of the local cholesteric DJM reconstructs from its diagonal form N^{0d}. We found that the DJM in the rotated coordinate system is not simply the matrix, obtained by rotation of the local DJM N^{0f}.

The cholesteric IJM is derived from its corresponding DJM for a general case and for two extreme approximations of the circularly polarized waves and the Mauguin regime. Rotation of the light polarization derived from the IJM for the two approximation regimes is in agreement with those obtained in this paper from the forms of corresponding DJMs as well as with those known from the literature. Our results show that the IJM of an elementary slice, to which the cholesteric is split in the IJM approach in a general case, is a gyrotropic plate, possessing linear birefringence and Jones dichroism. It reduces to the IJM of an undistorted nematic plate only in the approximation of the Mauguin regime corresponding to weak twist of the director field. Though the antisymmetric components in the DJM [Eq. (60)], corresponding to the Mauguin regime, including adiabatic regime, are zero, rotation of the light polarization comes from the IJM, in which the diagonal exponent matrix is multiplied by the rotation matrix $R(qz)$ [Eq. (63)].

In the opposite extreme approximation of circularly polarized eigenwaves propagating in short-pitch cholesterics with low linear birefringence of the parent nematic, the IJM reduces to the rotation matrix $J^c \sim R(\frac{1}{2}\Delta n^0 z + qz)$ [Eq. (57)], which gives rotation of the light polarization in the same form as that obtained from the DJM and those known from the literature.

The main finding of this paper stating that the IJM of the elementary slice in the IJM approach and the local DJM differ from those of the undistorted nematic is explained based on the symmetry arguments. Namely, the crystallographic symmetry of the cholesteric (point group $\infty 2$) or mechanically twisted nematic is different from the symmetry of the undistorted nematic (point group ∞/mm). The difference in the symmetry implies differences in physical properties. For this reason, the IJM of the elementary slice and the DJM for a cholesteric in a general case are different from those for the undistorted nematic, but reduce to the elementary IJM and local DJM of a nematic slice at weak twist in the Mauguin and adiabatic regimes.

We believe that the DJM and IJM derived in this paper, which account for the spatial dispersion, can be used for modeling of the optical properties of distorted cholesterics, subjected to an external field. We predict that our elementary IJM and local DJM are suitable for modeling of the twist grain boundary (TGB) [39–44] and nematic twist-bend (N_{tb}) [45–48] phases as well as for the heliconical state [49–53] of cholesterics with low K_{33} elastic modulus under an external field.

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