Resource theory of non-Gaussian operations

Quntao Zhuang,[1,2,*] Peter W. Shor,[3,4] and Jeffrey H. Shapiro[2]
[1]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[3]*Center For Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[4]*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Received 28 March 2018; published 14 May 2018)

Non-Gaussian states and operations are crucial for various continuous-variable quantum information processing tasks. To quantitatively understand non-Gaussianity beyond states, we establish a resource theory for non-Gaussian operations. In our framework, we consider Gaussian operations as free operations, and non-Gaussian operations as resources. We define entanglement-assisted non-Gaussianity generating power and show that it is a monotone that is nonincreasing under the set of free superoperations, i.e., concatenation and tensoring with Gaussian channels. For conditional unitary maps, this monotone can be analytically calculated. As examples, we show that the non-Gaussianity of ideal photon-number subtraction and photon-number addition equal the non-Gaussianity of the single-photon Fock state. Based on our non-Gaussianity monotone, we divide non-Gaussian operations into two classes: (i) the finite non-Gaussianity class, e.g., photon-number subtraction, photon-number addition, and all Gaussian-dilatable non-Gaussian channels; and (ii) the diverging non-Gaussianity class, e.g., the binary phase-shift channel and the Kerr nonlinearity. This classification also implies that not all non-Gaussian channels are exactly Gaussian dilatable. Our resource theory enables a quantitative characterization and a first classification of non-Gaussian operations, paving the way towards the full understanding of non-Gaussianity.

DOI: 10.1103/PhysRevA.97.052317

I. INTRODUCTION

Bosonic Gaussian states and Gaussian operations are important components in quantum information processing [1]. Despite involving an infinite-dimensional Hilbert space, they are analytically tractable and, more importantly, easy to realize in experiments. Lasers, phase-insensitive optical amplifiers, and phase-sensitive optical amplifiers all produce Gaussian states, viz., coherent states, amplified spontaneous emission (thermal) states, and squeezed states, respectively [2]. In addition, spontaneous parametric down conversion—the most commonly used source of optical entanglement—produces Gaussian states [2]. Important tasks, such as quantum key distribution (QKD), can be performed with only Gaussian sources, Gaussian operations, and Gaussian measurements [3]. Gaussian attacks have also been proven to be optimum for one-way continuous-variable QKD protocols [4] and two-way continuous-variable QKD protocols [5].

However, non-Gaussian states and non-Gaussian operations are necessary for many other quantum information processing tasks, e.g., entanglement distillation [6–9], quantum error correction [10], optimal cloning [11], continuous-variable quantum computation [12,13], and cluster-state quantum computation [14,15]. It has been shown that under a few reasonable assumptions, general quantum resources in the Gaussian domain cannot be distilled with Gaussian free operations [16]. Moreover, non-Gaussian states and non-Gaussian operations can improve the quality of entanglement [17] and the performance of tasks such as teleportation [18–20]. For this reason, non-Gaussian states (e.g., Fock states, N00N states [21], Schrödinger-cat states [22,23]) and non-Gaussian operations (e.g., photon-number addition (PNA) [24–26], photon-number subtraction (PNS) [27–30], the cubic-phase gate [31], the Kerr nonlinearity [32], sum-frequency generation [33], the photon-added Gaussian channels [34], and other examples [36]) are being theoretically analyzed and experimentally realized.

An important task is thus to characterize and quantify the non-Gaussianity (nG) utilized in each task. Quantum resource theory (QRT) [37] answers this type of question. QRT has been established in various areas of physics, e.g., quantum coherence [38,39], superposition [40], athermality [41,42], and asymmetry [43]. The QRT of nG is challenging because the set of Gaussian states is not convex, so the usual framework of QRT [37] does not apply directly, and because of the infinite-dimensional Hilbert space that is involved. Despite these difficulties, the QRT of non-Gaussian states has been developed [44–46]. We explain the basic ingredients of traditional QRT via the example of non-Gaussian states: (i) resource states (non-Gaussian states), (ii) free states (Gaussian states), and (iii) free operations (Gaussian channels). A principal goal of QRT is to quantify the resource with a monotone—a function that maps quantum states or operations to real numbers—that satisfies three conditions: (i) zero for all free states, (ii) nonzero for all resource states, and (iii) nonincreasing under free operations. Indeed, Refs. [44,45] defined such a monotone based on quantum relative entropy [47,48], and evaluated the nG of various non-Gaussian states. However, the above QRT can only characterize the nG of quantum states, the nG of quantum operations is not yet well understood.

*quntao@mit.edu

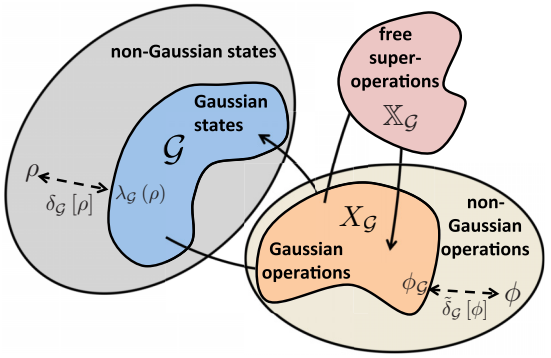

FIG. 1. Schematic of the resource-theory framework for non-Gaussian operations. The set of free states (Gaussian states $\mathcal{G}$) is closed under the set of free operations (Gaussian operations $X_{\mathcal{G}}$). $X_{\mathcal{G}}$ is closed under the set of free superoperations $\mathbb{X}_{\mathcal{G}}$. $\delta_{\mathcal{G}}[\rho]$ is the monotone for nG of a quantum state ρ. It measures the difference between ρ and the Gaussian state $\lambda_{\mathcal{G}}(\rho)$ produced by the resource destroying map $\lambda_{\mathcal{G}}$. $\tilde{\delta}_{\mathcal{G}}[\phi]$ is the monotone for nG of a conditional quantum map ϕ and it also measures the deviation from some Gaussian conditional quantum map $\phi_{\mathcal{G}}$. $\mathcal{G}, X_{\mathcal{G}}$, and $\mathbb{X}_{\mathcal{G}}$ are nonconvex.

In this paper, we establish a resource theory for nG of bosonic quantum operations. In our framework, the main ingredients of QRT for quantum operations are (see the schematic in Fig. 1): (i) resource states (non-Gaussian states), (ii) free states (Gaussian states), (iii) resource operations (non-Gaussian operations), (iv) free operations (Gaussian operations), and (v) free superoperations (concatenation and tensoring with Gaussian channels). To quantify the nG of quantum operations, we propose a monotone—the entanglement-assisted nG generating power—that is zero for all Gaussian operations, nonzero for non-Gaussian operations, and nonincreasing under free superoperations. Note that generating powers for coherence [49–53], entanglement [54,55], and work [56] have been considered in other QRTs. We also derive a lower bound and an upper bound for the monotone. The lower bound—the generating power of nG without entanglement assistance—has been suggested in Refs. [45,46] to be a measure for nG of operations. However, it is challenging to calculate, even for unitary operations. Moreover, it is not nonincreasing under the superoperation of tensoring with Gaussian channels.

Unlike the previous suggestion, our nG monotone is analytically tractable for conditional unitary maps, including all unitary operations. As examples, we evaluate the nG of PNS and PNA. We find that the nG of both maps equals the nG of the single-photon Fock state. Our nG monotone can thus enable a quantitative characterization of nG for conditional unitary maps. Despite the difficulty in the evaluation for general operations, we have identified two classes of operations through our nG monotone—the first class has finite nG while the second class has diverging nG. PNS and PNA are in the first class, while the binary phase-shift (BPS) channel and the Kerr nonlinearity are in the second class. For the first class, nG is finite, thus operations can be directly compared and ordered in terms of nG; for the second class, further classification may be possible by considering the rate of divergence of nG with increasing input or output mean photon number.

By utilizing the nG monotone defined in this paper and its properties, we show that all Gaussian-dilatable non-Gaussian channels defined in Ref. [34,35] are in the finite-nG class. The Gaussian-dilatable non-Gaussian channels are an important class of non-Gaussian channels and a starting point for our understanding of non-Gaussian operations, since their Kraus operators and input-output relations in characteristic-function form are analytically solvable. For example, this class includes the bosonic noise channel defined in Ref. [57], where it has been shown that additivity violation in classical capacity is upper bounded by a constant. It is also conjectured in Ref. [34] that the set of linear bosonic channels and the set of Gaussian-dilatable channels are identical. For general bosonic channels, our result means that going beyond Gaussian-dilatable channels is important for the full understanding of non-Gaussian operations.

This paper is organized as follows. In Sec. II, we introduce Gaussian states, quantum operations, and Gaussian operations, and we review the QRT of nG for non-Gaussian states. In Sec. III, we establish a framework for the QRT of nG for quantum operations and give the monotone, with its lower bound and upper bound. In Sec. IV, we evaluate the nG of two conditional unitary maps—including PNS and PNA. In Sec. V, we propose a classification of non-Gaussian operations. We conclude the main text in Sec. VI with discussions and future research directions. Details and proofs appear in Appendixes A–I.

II. PRELIMINARIES

Here we introduce some preliminary results. In Sec. II A, we introduce Gaussian states; In Sec. II B, we introduce quantum operations; In Sec. II C, we introduce Gaussian operations; In Sec. II D, we summarize the QRT for non-Gaussian states. A complete introduction to Gaussian states and Gaussian channels can be found in Ref. [1].

A. Gaussian states

An n-mode bosonic continuous-variable system is described by annihilation operators $\{a_k, 1 \leqslant k \leqslant n\}$, which satisfy the commutation relation $[a_k, a_j^\dagger] = \delta_{kj}, [a_k, a_j] = 0$. One can also define real quadrature field operators $q_k = a_k + a_k^\dagger, p_k = i(a_k^\dagger - a_k)$ and formally define a real vector $\boldsymbol{x} = (q_1, p_1, \ldots, q_n, p_n)$, which satisfies the canonical commutation relation ($\hbar = 2$) $[x_i, x_j] = 2i\boldsymbol{\Omega}_{ij}$, where $\boldsymbol{\Omega} = i \bigoplus_{k=1}^{n} \boldsymbol{Y}$ and $\boldsymbol{Y}$ is the Pauli matrix. A quantum state ρ can be described by its Wigner characteristic function $\chi(\boldsymbol{\xi}) = \mathrm{Tr}[\rho D(\boldsymbol{\xi})]$, where $\boldsymbol{\xi}$ is a vector of $2n$ real numbers and $D(\boldsymbol{\xi}) = \exp(i\boldsymbol{x}^T \boldsymbol{\Omega} \boldsymbol{\xi})$ is the Weyl operator. A state ρ is Gaussian if its characteristic function has the Gaussian form

$$\chi(\boldsymbol{\xi}) = \exp\left(-\tfrac{1}{2}\boldsymbol{\xi}^T \left(\boldsymbol{\Omega}\boldsymbol{\Lambda}\boldsymbol{\Omega}^T\right)\boldsymbol{\xi} - i(\boldsymbol{\Omega}\bar{\boldsymbol{x}})^T \boldsymbol{\xi}\right). \tag{1}$$

Here the $\bar{\boldsymbol{x}} = \langle \boldsymbol{x} \rangle_\rho$ is the state's mean and

$$\boldsymbol{\Lambda}_{ij} = \tfrac{1}{2} \langle \{x_i - d_i, x_j - d_j\} \rangle_\rho, \tag{2}$$

is its covariance matrix, where $\{,\}$ is the anticommutator and $\langle A \rangle_\rho = \mathrm{Tr}(A\rho)$. We denote the set of normalized (i.e., unity trace) Gaussian states with n modes as $\mathcal{G}[n]$. The set of

Gaussian states $\mathcal{G}$ is the union of all $\mathcal{G}[n]$, with $n \geqslant 1$. Any state with a non-Gaussian characteristic function is non-Gaussian.

As an example of Gaussian state, the two-mode squeezed vacuum (TMSV) state is

$$|\zeta_\lambda\rangle_{AA'} = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n |n\rangle_A |n\rangle_{A'}, \tag{3}$$

where $|n\rangle$ is a Fock state with n photons. The covariance matrix of a TMSV can be obtained as

$$\mathbf{\Lambda}_\zeta = \begin{pmatrix} (2N_S+1)\mathbf{I} & 2C_p\mathbf{Z} \\ 2C_p\mathbf{Z} & (2N_S+1)\mathbf{I} \end{pmatrix}, \tag{4}$$

where $\mathbf{I}, \mathbf{Z}$ are Pauli matrices, $N_S = \lambda^2/(1-\lambda^2)$ is the mean photon number per mode, and $C_p = \sqrt{N_S(N_S+1)}$ is the phase-sensitive cross correlation.

B. Quantum operations

Traditionally, a quantum operation $\mathcal{T}$ is defined as a linear and completely positive (CP) map from density operators to (unnormalized) density operators [47]. It can be expressed in terms of a unitary operator U on the input in state ρ, and an environment E in a pure state $|\psi_E\rangle$, and a projector P onto E [47] as

$$\mathcal{T}(\rho) = \mathrm{Tr}_E[(P\circ U)(\rho\otimes\psi_E)]. \tag{5}$$

For simplicity, we have used the notation $\psi = |\psi\rangle\langle\psi|$ to denote the density operator of a pure state $|\psi\rangle$. We also use the same notation U to denote the unitary channel that applies unitary U on input states, i.e., $U(\rho) = U\rho U^\dagger$, and similarly $P(\rho) = P\rho P$.

When $\mathcal{T}$ is also trace preserving (TP), it is a quantum channel and can be implemented deterministically. $\mathcal{T}$ can also be non-trace-preserving. In that case, $\mathcal{T}$ is implemented probabilistically. The probability of the map $\mathcal{T}$ successfully happening is given by $\mathrm{Tr}[\mathcal{T}(\rho)] \leqslant 1$ and the normalized output state is $\mathcal{T}(\rho)/\mathrm{Tr}[\mathcal{T}(\rho)]$. In various scenarios, we are interested in the enhancement provided only by the successful instances of $\mathcal{T}$, e.g., when operations such as PNA and PNS are used to enhance entanglement [17–20]. In these cases, we care more about the quantum state produced conditioned on success. Thus, we define the following postselected completely positive and trace-preserving (CPTP) maps.

Definition 1. A conditional quantum map ϕ takes input state ρ and yields

$$\phi(\rho) = \frac{1}{\mathrm{Tr}[\mathcal{T}(\rho)]}\mathcal{T}(\rho), \tag{6}$$

where $\mathcal{T}$ is a linear CP map.

Map ϕ can be linear, when $\mathcal{T}$ is TP (so $\mathcal{T}$ is a quantum channel), thus conditional quantum maps include all quantum channels. Map ϕ can also be nonlinear, which occurs when $\mathcal{T}$ is not TP, due to the normalization factor. The complementary map of ϕ is given by $\phi^c(\rho) = \mathcal{T}^c(\rho)/\mathrm{Tr}[\mathcal{T}(\rho)]$, where $\mathcal{T}^c$ is the complementary quantum operation and we note that $\mathrm{Tr}[\mathcal{T}(\rho)] \equiv \mathrm{Tr}[\mathcal{T}^c(\rho)]$. In the rest of the paper, without causing confusion, we refer to conditional quantum maps as quantum operations. Note that the notion of such conditional quantum dynamics has been defined in quantum trajectory theory and quantum control [58–62].

In this paper we are concerned with quantum operations in infinite dimensions. We denote the set of density operators with n modes as $\mathcal{H}[n]$, thus we have $\mathcal{G}[n] = \mathcal{H}[n]\bigcap\mathcal{G}$. Denote the number of input modes to channel ϕ as n_ϕ and the input Hilbert space is thus $\mathcal{H}[n_\phi]$. Denote the identity operation on $\mathcal{H}[n]$ as $\mathcal{I}_n$. In certain cases, we will not explicitly state the dimension for simplicity (e.g., write $\mathcal{I}$ instead of $\mathcal{I}_n$), as long as it does not cause any confusion.

C. Gaussian operations

A quantum operation is Gaussian if it transforms Gaussian states to Gaussian states [7]. Formally, the set of Gaussian operations (conditional maps) $X_\mathcal{G}$ is defined as follows.

Definition 2. A conditional quantum map $\phi \in X_\mathcal{G}$, iff $\forall\rho_\mathcal{G} \in \mathcal{G}[n_\phi+n]$, $n \in \{0,1,\cdots\}$, we have $(\mathcal{I}_n\otimes\phi)(\rho_\mathcal{G}) \in \mathcal{G}$.

Note that if in Eq. (6) $\phi \in X_\mathcal{G}$, then the original linear CP map $\mathcal{T}$ is also Gaussian. Additionally, if ϕ is linear, the requirement in Definition 2 is equivalent to the weaker condition: $\forall\rho_\mathcal{G} \in \mathcal{G}[n_\phi]$, we have $\phi(\rho_\mathcal{G}) \in \mathcal{G}$ [63,64]. Since on Gaussian inputs, Gaussian measurements can also be transformed to TP operations by postprocessing [7], we are particularly interested in the set of Gaussian channels $X_\mathcal{G}^L \subset X_\mathcal{G}$. Any quantum operation outside $X_\mathcal{G}$ is non-Gaussian.

All quantum channels can be extended to unitaries on the input and a vacuum environment (Stinespring dilation) [7], Gaussian unitary operations $X_\mathcal{G}^U$ are therefore essential among $X_\mathcal{G}$. Here we list a few Gaussian unitaries. A trivial Gaussian unitary is the identity operation $\mathcal{I}_n$. Less trivial unitaries include single-mode displacement $D_\alpha = \exp(\alpha a^\dagger - \alpha^\star a)$, single-mode phase rotation $R_\theta = \exp(-i\theta a^\dagger a)$, single-mode squeezing $S_r = \exp[r(a^2 - a^{\dagger 2})/2]$, and two-mode squeezing $S_{2,r} = \exp[-r(ab - a^\dagger b^\dagger)]$. In particular, $S_{2,r}$ generates a TMSV $|\zeta_\lambda\rangle_{AA'}$ from vacuum inputs $|0\rangle_A|0\rangle_{A'}$, i.e., $(\zeta_\lambda)_{AA'} = S_{2,r}(0_{AA'})$, where $\lambda = \tanh(r)$, $(\zeta_\lambda)_{AA'} \equiv |\zeta_\lambda\rangle_{AA'}\langle\zeta_\lambda|_{AA'}$ and $0_{AA'} \equiv |0\rangle_A|0\rangle_{A'}\langle 0|_A\langle 0|_{A'}$.

All Gaussian unitaries can be expressed as affine maps $\boldsymbol{x} \to \boldsymbol{S}\boldsymbol{x} + \Delta\boldsymbol{x}$ in the Heisenberg picture. Commutation relation preservation of $[x_i, x_j] = 2i\mathbf{\Omega}_{ij}$ requires that $\boldsymbol{S}\mathbf{\Omega}\boldsymbol{S}^T = \mathbf{\Omega}$, i.e., $\boldsymbol{S}$ is symplectic. In terms of the mean and covariance matrix, the affine map leads to

$$\bar{\boldsymbol{x}} \to \boldsymbol{S}\bar{\boldsymbol{x}} + \Delta\boldsymbol{x}, \quad \text{and} \quad \mathbf{\Lambda} \to \boldsymbol{S}\mathbf{\Lambda}\boldsymbol{S}^T. \tag{7}$$

Moreover, this is true regardless of whether the input state is Gaussian or not.

An arbitrary n-mode covariance matrix $\mathbf{\Lambda}$ has a symplectic diagonalization, i.e., $\exists\boldsymbol{S}$, such that $\boldsymbol{S}\mathbf{\Omega}\boldsymbol{S}^T = \mathbf{\Omega}$ and $\mathbf{\Lambda} = \boldsymbol{S}(\bigoplus_{k=1}^n \mu_k\boldsymbol{I})\boldsymbol{S}^T$. Here μ_k's are the eigenvalues of $i\mathbf{\Omega}\mathbf{\Lambda}$. Since $\mu_k\boldsymbol{I}$ is the covariance matrix of a thermal state with mean photon number $(\mu_k-1)/2$, this means that an arbitrary Gaussian state can be transformed into a product of thermal states with mean photon numbers $\{(\mu_k-1)/2, 1\leqslant k\leqslant n\}$ by a Gaussian unitary. Thus, the entropy of such a Gaussian state $S(\rho) = \sum_{k=1}^n g((\mu_k-1)/2)$, where $g(N) = (N+1)\log_2(N+1) - N\log_2 N$ is the entropy of a thermal state with mean photon number N.

As an analog to the Schmidt decomposition for finite-dimensional bipartite pure states, we have the following phase-space Schmidt decomposition [65]. Consider an arbitrary bipartite pure Gaussian state ψ_{AB}, with modes $\{A_k, 1\leqslant k\leqslant$

$n_A\}$ and $\{B_k, 1 \leqslant k \leqslant n_B\}$, where $n_A \leqslant n_B$. There are local Gaussian unitaries U_A, U_B that transform ψ_{AB} to a tensor product of n_A TMSV and $n_B - n_A$ vacuum states, i.e.,

$$(U_A \otimes U_B)(\psi_{AB}) = \left[\otimes_{k=1}^{n_A} (\zeta_{\lambda_k})_{A_k B_k}\right] \otimes \left[\otimes_{k=n_A+1}^{n_B} 0_{B_k}\right]. \quad (8)$$

D. Summary of nG resource theory for states

In a QRT, consider the set of free states to be the Gaussian states $\mathcal{G}$. To characterize the nG of a quantum state ρ, a relative-entropy-based monotone, nonincreasing under free operations of Gaussian channels $X_{\mathcal{G}}^L$, has been established [44,45], namely

$$\delta_{\mathcal{G}}[\rho] = \min_{\rho_{\mathcal{G}} \in \mathcal{G}} S(\rho \| \rho_{\mathcal{G}}) = S(\rho \| \lambda_{\mathcal{G}}(\rho)) = S[\lambda_{\mathcal{G}}(\rho)] - S(\rho). \quad (9)$$

Here $S(\rho\|\sigma) \equiv \mathrm{Tr}[\rho(\log_2 \rho - \log_2 \sigma)]$ is the quantum relative entropy; a brief review of its properties is given in Appendix A. The first formula is a natural definition and has been shown to equal to the second formula in Ref. [44]. The second formula is the original proposal from Ref. [45], and equals the third formula, where $\lambda_{\mathcal{G}}$ is the resource-destroying map [66] $\rho \to \tau_\rho$, with $\tau_\rho \in \mathcal{G}$ having the same mean and covariance matrix as ρ. We can obtain the following lemma (proof in Appendix B).

Lemma 1. $\lambda_{\mathcal{G}}$ commutes with any Gaussian channel $\xi_{\mathcal{G}} \in X_{\mathcal{G}}^L$, viz., $\xi_{\mathcal{G}} \circ \lambda_{\mathcal{G}} = \lambda_{\mathcal{G}} \circ \xi_{\mathcal{G}}$.

When Gaussian channels are considered as free operations, this condition guarantees that $S(\rho\|\lambda_{\mathcal{G}}(\rho))$ is a monotone [66]. In general, however, conditional Gaussian maps do not commute with $\lambda_{\mathcal{G}}$. A counterexample is given in Appendix B.

Besides continuity, $\delta_{\mathcal{G}}[\cdot]$ satisfies the following [45].

A1 Non-negativity. $\delta_{\mathcal{G}}[\rho] \geqslant 0$, with equality if and only if (iff) $\rho \in \mathcal{G}$.

A2 $\delta_{\mathcal{G}}[\rho_1 \otimes \rho_2] = \delta_{\mathcal{G}}[\rho_1] + \delta_{\mathcal{G}}[\rho_2]$.

A3 If $\lambda_{\mathcal{G}}(\rho_k)$'s are equal, then $\delta_{\mathcal{G}}[\sum_k p_k \rho_k] \leqslant \sum_k p_k \delta_{\mathcal{G}}[\rho_k]$.

A4 Invariance under a Gaussian unitary. $\delta_{\mathcal{G}}[U_{\mathcal{G}} \rho U_{\mathcal{G}}^\dagger] = \delta_{\mathcal{G}}[\rho], \forall U_{\mathcal{G}} \in X_{\mathcal{G}}^U$.

A5 Monotonically decreasing under a partial trace. $\delta_{\mathcal{G}}[\mathrm{Tr}_2(\rho_{12})] \leqslant \delta_{\mathcal{G}}[\rho_{12}]$.

A6 Monotonically decreasing through Gaussian channels. $\delta_{\mathcal{G}}[\phi_{\mathcal{G}}(\rho)] \leqslant \delta_{\mathcal{G}}[\rho], \forall \phi_{\mathcal{G}} \in X_{\mathcal{G}}^L$.

Note that relative entropy is not superadditive in the traditional sense [67]. The free set of states $\mathcal{G}$ is not convex, thus precluding the results about resource state conversion in Ref. [37] to hold in the resource theory of nG. Property A6 cannot be extended to Gaussian conditional maps, a counterexample in which a Gaussian operation increases the nG of a non-Gaussian state is given in Appendix C. This shows that even Gaussian measurements can be reduced to a Gaussian channel on Gaussian inputs by postprocessing, but on non-Gaussian inputs they need to be treated differently from Gaussian channels.

III. RESOURCE THEORY OF NON-GAUSSIAN OPERATIONS

The goal of this paper is to establish a resource theory for nG of quantum operations. We define the set of free operations to be Gaussian operations $X_{\mathcal{G}}$. To formulate the resource theory of non-Gaussian operations, we need to find a set of superoperations that leave $X_{\mathcal{G}}$ closed (schematic in Fig. 1).

Definition 3. The set of free superoperations $\mathbb{X}_{\mathcal{G}}$ is a set of maps that map each element in $X_{\mathcal{G}}$ to an element in $X_{\mathcal{G}}$. Here we consider

$$\mathbb{X}_{\mathcal{G}} = \{\otimes\phi_{\mathcal{G}}, \circ\, \phi_{\mathcal{G}}, \phi_{\mathcal{G}} \circ\}, \quad (10)$$

which includes tensoring with a Gaussian channel ($\otimes\phi_{\mathcal{G}}$), preconcatenation with a Gaussian channel ($\circ\phi_{\mathcal{G}}$) and postconcatenation with a Gaussian channel ($\phi_{\mathcal{G}}\circ$).

All the above superoperations map a Gaussian operation to another Gaussian operation. However, $\mathbb{X}_{\mathcal{G}}$ does not include general probabilistic mixing, because probabilistic mixing of Gaussian states can be non-Gaussian. We also exclude from $\mathbb{X}_{\mathcal{G}}$ the action of taking the complement. The reason is as follows. If nG is nonincreasing under taking the complement, then it must be invariant under taking the complement, because taking the complement twice gets back to the original map. However, one can construct a channel by swapping the incoming state with a non-Gaussian pure state, the channel is clearly non-Gaussian, but its complementary channel—the identity channel—is Gaussian.

The crucial step in characterizing the nG of quantum operations is to find a monotone. This monotone should be nonincreasing under the set of free superoperations $\mathbb{X}_{\mathcal{G}}$. In Sec. III A, we will propose a monotone $\tilde{\delta}_{\mathcal{G}}[\cdot]$ based on the entanglement-assisted generating power of quantum operations. In Sec. III B, we obtain a lower bound $d_{\mathcal{G}}[\cdot]$ on $\tilde{\delta}_{\mathcal{G}}[\cdot]$. In Sec. III C, we obtain an upper bound $D_{\mathcal{G}}[\cdot]$ on $\tilde{\delta}_{\mathcal{G}}[\cdot]$ based on distance measures between quantum operations. This upper bound is in fact also a monotone. To summarize, we present two monotones, $\tilde{\delta}_{\mathcal{G}}[\cdot]$ and $D_{\mathcal{G}}[\cdot]$, and a lower bound $d_{\mathcal{G}}[\cdot]$, satisfying the following relation.

Theorem 1. For all conditional quantum maps ϕ, $d_{\mathcal{G}}[\phi] \leqslant \tilde{\delta}_{\mathcal{G}}[\phi] \leqslant D_{\mathcal{G}}[\phi]$.

The proof is given after we introduce each quantity. We propose $\tilde{\delta}_{\mathcal{G}}[\cdot]$ instead of $D_{\mathcal{G}}[\cdot]$ to be the measure of nG for quantum operations, since $\tilde{\delta}_{\mathcal{G}}[\cdot]$ is much easier to evaluate, as we will show in Sec. IV. It is open whether the inequalities can be strict.

A. Entanglement-assisted generating power as a monotone

In this section, we propose a monotone for nG of quantum operations based on the entanglement-assisted generating power.

Definition 4. For the input Gaussian state $\rho_{A'} \in \mathcal{G}[n_\phi]$ to conditional quantum map ϕ, consider its purification $\psi_{AA'} \in \mathcal{G}[2n_\phi]$. We define the entanglement-assisted nG generating power as follows:

$$\tilde{\delta}_{\mathcal{G}}[\phi] = \max_{\rho_{A'} \in \mathcal{G}[n_\phi]} \delta_{\mathcal{G}}[(\mathcal{I}_{n_\phi} \otimes \phi)(\psi_{AA'})]. \quad (11)$$

Before proving the properties of $\tilde{\delta}_{\mathcal{G}}[\cdot]$ that allow it to be a monotone for nG, we justify the choice of the number of ancilla modes by the following lemma.

Lemma 2. $\tilde{\delta}_{\mathcal{G}}[\phi]$ is invariant under local isometry on ancilla A and giving ancilla A extra modes.

The proof is based on the phase space Schmidt decomposition, details are in Appendix D. In Definition 4, we have

chosen an ancilla with the minimum number of modes. Also, maximization over $\rho_{A'}$ is equivalent to maximization over the pure state $\psi_{AA'}$ due to this symmetry of purification. This symmetry of purification also guarantees that pure states are optimum, i.e., we have an equivalent definition of $\tilde{\delta}_{\mathcal{G}}[\cdot]$ as follows.

Definition 5. For $\mathcal{H}[n+n_\phi]$ with $n \geqslant n_\phi$ modes,

$$\tilde{\delta}_{\mathcal{G}}[\phi] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}[n+n_\phi]} \delta_{\mathcal{G}}[(\mathcal{I}_n \otimes \phi)(\rho_{\mathcal{G}})]. \tag{12}$$

This means that going to an arbitrary mixed state with an arbitrary number of modes does not increase nG. The proof that Definition 5 and Definition 4 are equivalent is as follows. By Property A5, we have $\delta_{\mathcal{G}}[(\mathcal{I}_n \otimes \phi)(\rho_{\mathcal{G}})] \leqslant \delta_{\mathcal{G}}[(\mathcal{I}_{2n+n_\phi} \otimes \phi)(\psi_{\rho_{\mathcal{G}}})]$, where $\psi_{\rho_{\mathcal{G}}} \in \mathcal{G}[2n+2n_\phi]$ is the purification of $\rho_{\mathcal{G}} \in \mathcal{G}[n+n_\phi]$. Combined with symmetry of purification, we have $\max_{\rho_{\mathcal{G}} \in \mathcal{G}[n+n_\phi]} \delta_{\mathcal{G}}[(\mathcal{I}_n \otimes \phi)(\rho_{\mathcal{G}})] \leqslant \max_{\psi_{\rho_{\mathcal{G}}} \in \mathcal{G}[2n+2n_\phi]} \delta_{\mathcal{G}}[(\mathcal{I}_{2n+n_\phi} \otimes \phi)(\psi_{\rho_{\mathcal{G}}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$. On the other hand, the reverse inequality is trivially satisfied by taking $\rho_{\mathcal{G}}$ to be the product of the pure state in Definition 4 and extra vacuum ancilla.

Now we give properties of $\tilde{\delta}_{\mathcal{G}}[\cdot]$, The proofs are given in Appendix E.

B1 Non-negativity. $\tilde{\delta}_{\mathcal{G}}[\phi] \geqslant 0$, with equality iff $\phi \in X_{\mathcal{G}}$.

B2 Invariance under tensoring with Gaussian channels. $\forall \phi_{\mathcal{G}} \in X^L_{\mathcal{G}}$, we have $\tilde{\delta}_{\mathcal{G}}[\phi \otimes \phi_{\mathcal{G}}] = \tilde{\delta}_{\mathcal{G}}[\phi]$.

B3 Invariance under concatenation with a Gaussian unitary. $\forall U_{\mathcal{G}} \in X^U_{\mathcal{G}}$, $\tilde{\delta}_{\mathcal{G}}[U_{\mathcal{G}} \circ \phi] = \tilde{\delta}_{\mathcal{G}}[\phi \circ U_{\mathcal{G}}] = \tilde{\delta}_{\mathcal{G}}[\phi]$.

B4 Monotonically decreasing under concatenation with partial trace. For ϕ with output AB, we have $\tilde{\delta}_{\mathcal{G}}[\mathrm{Tr}_A \circ \phi] \leqslant \tilde{\delta}_{\mathcal{G}}[\phi]$.

B5 Monotonically increasing under Stinespring dilation with a vacuum environment. Note this property is only for channels, not for general operations. Suppose $\forall \rho, \phi(\rho) = \mathrm{Tr}_E \circ U_\phi(\rho \otimes \mathbf{0}_E)$, we then have $\tilde{\delta}_{\mathcal{G}}[\phi] \leqslant \tilde{\delta}_{\mathcal{G}}[U_\phi]$.

B6 Nonincreasing under concatenation with a Gaussian channel. $\forall \phi_{\mathcal{G}} \in X^L_{\mathcal{G}}$, (i) Postconcatenation: $\tilde{\delta}_{\mathcal{G}}[\phi_{\mathcal{G}} \circ \phi] \leqslant \tilde{\delta}_{\mathcal{G}}[\phi]$. (ii) Preconcatenation: $\tilde{\delta}_{\mathcal{G}}[\phi \circ \phi_{\mathcal{G}}] \leqslant \tilde{\delta}_{\mathcal{G}}[\phi]$.

B7 Superadditivity. $\tilde{\delta}_{\mathcal{G}}[\phi_1 \otimes \phi_2] \geqslant \tilde{\delta}_{\mathcal{G}}[\phi_1] + \tilde{\delta}_{\mathcal{G}}[\phi_2]$.

It is open whether this superadditivity B7 can be strict. Due to superadditivity, if one wants invariance under tensoring with itself, a regularization can be introduced $\tilde{\delta}^\infty_{\mathcal{G}}[\phi] = \lim_{n\to\infty} \tilde{\delta}_{\mathcal{G}}[\phi^{\otimes n}]/n$, such that $\tilde{\delta}^\infty_{\mathcal{G}}[\phi^{\otimes 2}] = \tilde{\delta}^\infty_{\mathcal{G}}[\phi]$. However, unlike the case in communication capacity, where joint encoding between multiple channel uses is natural to consider; here we can simply regard ϕ and $\phi^{\otimes 2}$ as two different quantum operations, thus regularization is not compulsory for our resource theory.

B. Generating power as a lower bound

Suppose we trace out the ancilla in Definition 5, we can define another function as follows.

Definition 6. (nG generating power) $d_{\mathcal{G}}[\phi] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}[n_\phi]} \delta_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})]$.

This has been suggested in Refs. [45,46] to be a measure for the nG of quantum operations. By considering an input in a product state with the ancilla, it is easy to see that $\tilde{\delta}_{\mathcal{G}}[\phi] \equiv d_{\mathcal{G}}[\mathcal{I}_{n_\phi} \otimes \phi] \geqslant d_{\mathcal{G}}[\phi]$, by Property A5. Thus the first part of Theorem 1 is true. If the above inequality can be strict (which seems plausible), because the identity $\mathcal{I}_{n_\phi}$ is a Gaussian channel, we cannot prove invariance nor nonincreasing under tensoring with Gaussian channels. Moreover, $d_{\mathcal{G}}[\phi] = 0$ only implies $\forall \rho_{\mathcal{G}} \in \mathcal{G}, \phi(\rho_{\mathcal{G}}) \in \mathcal{G}$, which does not necessarily mean $\phi \in X_{\mathcal{G}}$ according to Definition 2. Thus, it only satisfies Properties B3–B7 (see Appendix F for details). Additionally, it is difficult to calculate $d_{\mathcal{G}}[\cdot]$ even for unitary operations, since it requires maximization over mixed states and the entropy of a non-Gaussian mixed state is difficult to calculate. In contrast, $\tilde{\delta}_{\mathcal{G}}$ can be analytically evaluated, as we will show in Sec. IV.

C. Upper bound: Distance as a monotone

Another natural definition for the nG of quantum operations can be obtained from a geometric approach. Since the diamond norm [68] is difficult to calculate, here we introduce the following.

Definition 7. Consider conditional quantum maps ϕ_1 and ϕ_2 each with the n input modes. We define a measure for their difference by

$$D_{\mathcal{G}}(\phi_1, \phi_2) = \max_{\psi_{\mathcal{G}} \in \mathcal{G}[2n]} S[(\mathcal{I}_n \otimes \phi_1)(\psi_{\mathcal{G}}) \| (\mathcal{I}_n \otimes \phi_2)(\psi_{\mathcal{G}})], \tag{13}$$

which is equivalent to

$$D_{\mathcal{G}}(\phi_1, \phi_2) = \max_{\rho_{\mathcal{G}} \in \mathcal{G}[n]} S[(\mathcal{I}_n \otimes \phi_1)(\rho_{\mathcal{G}}) \| (\mathcal{I}_n \otimes \phi_2)(\rho_{\mathcal{G}})]. \tag{14}$$

In the first formula, we have restricted the state to be pure and within $\mathcal{G}[2n]$. An argument similar to Lemma 2's proof gives the second formula. Now, one can define a measure of nG by the distance from the closest Gaussian conditional map with the same number of input modes.

Definition 8. (nG distance) $D_{\mathcal{G}}[\phi] \equiv \min_{\phi_{\mathcal{G}} \in X_{\mathcal{G}}} D_{\mathcal{G}}(\phi, \phi_{\mathcal{G}})$.

Now we show that the second part of Theorem 1 is true. We will not explicitly state the dimension in the following proof for simplicity.

$$\begin{aligned} D_{\mathcal{G}}[\phi] &= \min_{\phi_{\mathcal{G}} \in X_{\mathcal{G}}} \max_{\psi_{\mathcal{G}} \in \mathcal{G}} S[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi_{\mathcal{G}})(\psi_{\mathcal{G}})] \\ &\geqslant \max_{\psi_{\mathcal{G}} \in \mathcal{G}} \min_{\phi_{\mathcal{G}} \in X_{\mathcal{G}}} S[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi_{\mathcal{G}})(\psi_{\mathcal{G}})] \\ &\geqslant \max_{\psi_{\mathcal{G}} \in \mathcal{G}} \min_{\rho_{\mathcal{G}} \in \mathcal{G}} S[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}}) \| \rho_{\mathcal{G}}] \\ &= \max_{\psi_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi)(\psi_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]. \end{aligned}$$

The first inequality is due to the max-min inequality [69], the second inequality is due to the fact that $(\mathcal{I} \otimes \phi_{\mathcal{G}})(\psi_{\mathcal{G}}) \in \mathcal{G}$, and the last equality is due to Eq. (9) and Definition 4.

We can show that $D_{\mathcal{G}}[\cdot]$ satisfies Properties B1–B6, which qualifies it to be a measure of nG for quantum operations (see Appendix G for details). Moreover, we can show that it satisfies $D_{\mathcal{G}}[\phi_1 \otimes \phi_2] \geqslant \max(D_{\mathcal{G}}[\phi_1], D_{\mathcal{G}}[\phi_2])$. It is open whether this can be improved to superadditivity.

IV. EXAMPLE: CONDITIONAL UNITARY MAPS

We now introduce conditional unitary maps.

Definition 9. A conditional quantum map is a conditional unitary map if it is one-to-one and maps all pure states to pure states.

Conditional unitary maps include unitary operations, such as the single-mode self-Kerr unitary [46], and operations such as PNA and PNS [17,70]. For a conditional unitary map U, because the output ancilla is jointly pure when the input ancilla is pure, combining Eq. (9) and Definition 4 gives

$$\tilde{\delta}_{\mathcal{G}}[U] = \max_{\rho_{A'} \in \mathcal{G}} S\{\lambda_{\mathcal{G}}[(\mathcal{I} \otimes U)(\psi_{AA'})]\}. \tag{15}$$

For fixed $\rho_{A'}$, $S\{\lambda_{\mathcal{G}}[(\mathcal{I} \otimes U)(\psi_{AA'})]\}$ can be analytically obtained by calculating the entropy of the Gaussian state $\lambda_{\mathcal{G}}[(\mathcal{I} \otimes U)(\psi_{AA'})]$, which can be obtained from its covariance matrix. Moreover, the Gaussian state $\rho_{A'}$ being maximized over can be fully characterized by its mean and covariance matrix. Thus, the overall maximization can be solved analytically without too much difficulty. For example, in the single-mode case, the general input-ancilla state

$$|\psi_{\alpha,\theta,r,\lambda}\rangle_{AA'} = D_\alpha R_\theta S_r |\zeta_\lambda\rangle_{AA'} \tag{16}$$

only depends on four parameters—the displacement α, phase rotation θ, squeezing r, and two-mode squeezing λ. Note here that D_α, R_θ, and S_r act on the input A'.

Below, we consider two specific single-mode conditional maps—the PNS ϕ_{PNS} and PNA ϕ_{PNA}—and evaluate their nG's analytically. For simplicity, we consider the ideal ϕ_{PNS} and ϕ_{PNA}, which are described by the annihilation and creation operators a and $a^\dagger$ [17,70]. Experimental schemes of PNS and PNA can be found in Refs. [24–30]. Both ϕ_{PNS} and ϕ_{PNA} are one-to-one and produce a pure state when the input is pure, thus they are conditional unitary maps.

Photon-number subtraction. When the input and ancilla are in the joint state given by Eq. (16), the joint state of the output and ancilla is $|\psi\rangle_{AB} = N_{\text{PNS}} a_B |\psi_{\alpha,\theta,r,\lambda}\rangle_{AB}$, where the normalization factor is $N_{\text{PNS}} = \{|\alpha|^2 + [(1+2N_S)\cosh(2r) - 1]/2\}^{-1/2}$. Because of Property A4, $|\psi\rangle_{AB}$ has the same nG as

$$\begin{aligned} |\xi\rangle_{AB} &= S_r^\dagger R_\theta^\dagger D_\alpha^\dagger |\psi\rangle_{AB} \\ &= N_{\text{PNS}}(e^{-i\theta}(\cosh(r) a_B - \sinh(r) a_B^\dagger) + \alpha) |\zeta_\lambda\rangle_{AB}, \end{aligned} \tag{17}$$

where $|\xi\rangle_{AB}$ is a superposition of photon-number added TMSV, photon-number subtracted TMSV, and TMSV, so it is non-Gaussian. By changing the global phase properly, we can choose $\alpha > 0$.

To calculate the covariance matrix of ξ_{AB}, we consider the expectation values of operators $X \in \{a_A, a_B, a_A^2, a_B^2, a_A^\dagger a_A, a_B^\dagger a_B, a_A^\dagger a_B, a_A a_B\}$, which can be found from

$$\begin{aligned} \langle X\rangle_{\xi_{AB}} \equiv \langle \xi|_{AB} X |\xi\rangle_{AB} = N^2 \big\{ & \alpha^2 \langle X\rangle_{\zeta_\lambda} \\ & + \alpha e^{-i\theta} \cosh(r) \langle X a_B\rangle_{\zeta_\lambda} - \alpha e^{-i\theta} \sinh(r) \langle X a_B^\dagger\rangle_{\zeta_\lambda} \\ & + \alpha e^{i\theta} \cosh(r) \langle a_B^\dagger X\rangle_{\zeta_\lambda} - \alpha e^{i\theta} \sinh(r) \langle a_B X\rangle_{\zeta_\lambda} \\ & + \cosh^2(r) \langle a_B^\dagger X a_B\rangle_{\zeta_\lambda} + \sinh(r)^2 \langle a_B X a_B^\dagger\rangle_{\zeta_\lambda} \\ & - \tfrac{1}{2} \sinh(2r) \big(\langle a_B X a_B\rangle_{\zeta_\lambda} + \langle a_B^\dagger X a_B^\dagger\rangle_{\zeta_\lambda} \big) \big\}. \end{aligned} \tag{18}$$

Since TMSV ζ_λ has zero mean, each term can be solved by Gaussian moment factoring. The covariance matrix can be obtained by the method in Appendix H, however, the expression is too lengthy to display here. With the covariance matrix in hand, the entropy can be obtained easily by the method in Sec. II C.

After the maximization over r, α, θ, N_S, we find that

$$\tilde{\delta}_{\mathcal{G}}[\phi_{\text{PNS}}] = \delta_{\mathcal{G}}[|1\rangle\langle 1|] = 2, \tag{19}$$

which is achieved by $\alpha = 0$ and arbitrary r, θ, N_S. This result equals the lower bound $d'_{\mathcal{G}}$ obtained in Ref. [71] for the special case of $N_S = 0, \alpha = 0$.

Photon-number addition. The nG analysis for PNA parallels what we have done for PNS. The joint state of the output and ancilla is $|\psi\rangle_{AB} = N_{\text{PNA}} a_B^\dagger |\psi_{\alpha,\theta,r,\lambda}\rangle_{AB}$, where $N_{\text{PNA}} = (|\alpha|^2 + ((1+2N_S)\cosh(2r) + 1)/2)^{-1/2}$. Because of Property A4, $|\psi\rangle_{AB}$ has the same nG as

$$\begin{aligned} |\xi\rangle_{AB} &= S_r^\dagger R_\theta^\dagger D_\alpha^\dagger |\psi\rangle_{AB} \\ &= N_{\text{PNA}}(e^{i\theta}(\cosh(r) a_B^\dagger - \sinh(r) a_B) + \alpha^\star) |\zeta_\lambda\rangle_{AB}. \end{aligned} \tag{20}$$

Intuitively, since it is again a superposition of photon-number added TMSV, photon-number subtracted TMSV, and TMSV, the maximum nG should be the same as that of ϕ_{PNS}. However, because $\cosh(r) \geqslant \sinh(r)$, the parameter space here is slightly different. This difference can be dealt with by realizing that the new expectation values can be obtained by exchanging $-\sinh(r)$ with $\cosh(r)$ and θ with $-\theta$ in Eq. (18) (fixing $\alpha > 0$), and using the new normalization factor.

After the maximization over r, α, θ, N_S, we find that

$$\tilde{\delta}_{\mathcal{G}}[\phi_{\text{PNA}}] = \tilde{\delta}_{\mathcal{G}}[\phi_{\text{PNS}}] = \delta_{\mathcal{G}}[|1\rangle\langle 1|] = 2, \tag{21}$$

which is achieved by $\alpha = 0$ and arbitrary r, θ, N_S.

V. CLASSIFICATION: FINITE nG AND DIVERGING nG

In the above examples, nG is finite. However, for other quantum operations there is a potential divergence caused by the infinite dimensionality of states in $\mathcal{G}$. Consider Definition 4. If the overall output energy is bounded by N_S, then $\tilde{\delta}_{\mathcal{G}}[\phi] \leqslant \max_{\rho_{A'} \in \mathcal{G}[n_\phi]} S[\lambda_{\mathcal{G}}(\rho_{AB})] \leqslant n g(N_S/n)$. The factor n is the total number of modes in the output and ancilla. Since $g(N_S) \sim \log_2 N_S$, when $N_S \gg 1$, the growth rate of $\tilde{\delta}_{\mathcal{G}}[\phi]$ with the allowed output energy is at most logarithmic. It may be tempting to constrain the input or output energy in Definition 4 to define an energy-constrained version of generating power. However, because concatenation of Gaussian channels can change the energy constraint on the input or output of the original conditional quantum map, such constraints will invalidate Properties B3 and B6. So an energy-constrained generating power is not a meaningful monotone for nG.

Based on the above observation, we classify non-Gaussian operations into two classes [schematic in Fig. 2(b)]. The first class of operations has finite $\tilde{\delta}_{\mathcal{G}}$ despite allowing the input to have infinite energy. We denote this class of operations Φ_F.

Definition 10. Finite-nG class.

$$\Phi_F = \{\text{conditional quantum map } \phi \mid 0 < \tilde{\delta}_{\mathcal{G}}[\phi] < \infty\}. \tag{22}$$

As we have shown in Sec. IV, PNA and PNS both belong to this class, i.e.,

$$\phi_{\text{PNS}} \in \Phi_F, \phi_{\text{PNA}} \in \Phi_F. \tag{23}$$

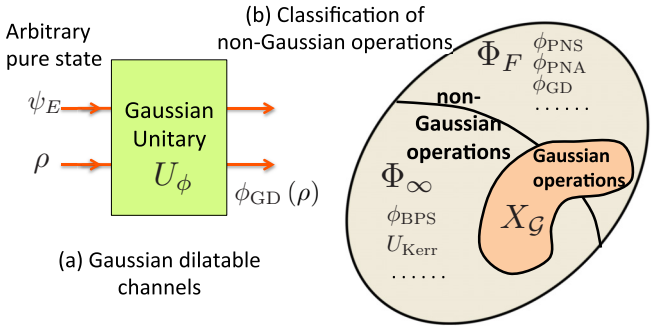

FIG. 2. (a) Schematic of a Gaussian-dilatable channel $\phi_{\rm GD}$, ψ_E is the environment in an arbitrary pure state. (b) Schematic of the classification of non-Gaussian operations into: (i) finite-nG class Φ_F, including $\phi_{\rm PNS}$, $\phi_{\rm PNA}$ and Gaussian-dilatable non-Gaussian channels $\phi_{\rm GD}$, and (ii) diverging-nG class Φ_∞, including the binary phase-shift channel $\phi_{\rm BPS}$ and the self-Kerr unitary $U_{\rm Kerr}$.

For operations in Φ_F, we can compare and rank their nG based on the $\tilde{\delta}_\mathcal{G}[\phi]$ value.

The second class of operations has diverging $\tilde{\delta}_\mathcal{G}$, when the output energy increases. We denote this class of operations as Φ_∞.

Definition 11. Diverging-nG class.

$$\Phi_\infty = \{\text{conditional quantum map } \phi \mid \tilde{\delta}_\mathcal{G}[\phi] = \infty\}. \quad (24)$$

To identify the diverging-nG class, it is often useful to consider the lower bound

$$\tilde{\delta}_\mathcal{G}[\phi] \geqslant d_\mathcal{G}[\phi] \geqslant \delta_\mathcal{G}[\phi(|\alpha\rangle\langle\alpha|)], \quad (25)$$

where the coherent state $|\alpha\rangle$ is the input to the map. If we can show that $\delta_\mathcal{G}[\phi(|\alpha\rangle\langle\alpha|)]$ diverges to ∞ as $|\alpha|^2$ increases, then we can conclude that $\phi \in \Phi_\infty$. In the following, we give more examples of operations in Φ_F and Φ_∞.

Gaussian-dilatable channels. In Ref. [34], a class of non-Gaussian channels called Gaussian-dilatable non-Gaussian channels is introduced. A channel is Gaussian dilatable if it has a Stinespring dilation composed of a Gaussian unitary $U_\phi \in X_\mathcal{G}^U$ and an ancilla in a fixed pure state ψ_E with finite energy [schematic in Fig. 2(a)]. A Gaussian-dilatable channel $\phi_{\rm GD}$'s output on arbitrary input ρ can be written as

$$\phi_{\rm GD}(\rho) = {\rm Tr}_E[U_\phi(\rho \otimes \psi_E)]. \quad (26)$$

All Gaussian channels are trivially Gaussian dilatable. $\phi_{\rm GD}$ is non-Gaussian when ψ_E is non-Gaussian. For Gaussian-dilatable channels, the output's characteristic function can be analytically obtained from the input's characteristic function and the Kraus operators are also analytically attainable. Thus, Gaussian-dilatable channels are an important starting point for the study of non-Gaussian channels and operations. For example, it includes the bosonic noise channel defined in Ref. [57], where it has been shown that its additivity violation in classical capacity is upper bounded by a constant. It is also conjectured in Ref. [34] (see Conjecture 1 in the reference) that the set of linear bosonic channels and the set of Gaussian-dilatable channels are identical.

The nG of a Gaussian-dilatable channel satisfies

$$\begin{aligned}\tilde{\delta}_\mathcal{G}[\phi_{\rm GD}] &= \max_{\psi_\mathcal{G}} \delta_\mathcal{G}[(\mathcal{I}_{n_\phi} \otimes ({\rm Tr}_E \circ U_\phi))(\psi_\mathcal{G} \otimes \psi_E)] \\ &\leqslant \max_{\psi_\mathcal{G}} \delta_\mathcal{G}[(\mathcal{I}_{n_\phi} \otimes U_\phi)(\psi_\mathcal{G} \otimes \psi_E)] \\ &= \max_{\psi_\mathcal{G}} \delta_\mathcal{G}(\psi_\mathcal{G} \otimes \psi_E) = \delta_\mathcal{G}[\psi_E], \end{aligned} \quad (27)$$

where the first inequality is from Property A5, the second equality is from Property A4 and the last equality is from Property A2. Because the nG of the state ψ_E is finite and does not depend on the input or output, we immediately have the following theorem.

Theorem 2. Every Gaussian-dilatable non-Gaussian channel is in the finite-nG class, i.e.,

$$\phi_{\rm GD} \in \Phi_F. \quad (28)$$

The fact that $\tilde{\delta}_\mathcal{G}[\phi_{\rm GD}] \leqslant \delta_\mathcal{G}[\psi_E]$ is intuitive, since all nG of this channel comes from the non-Gaussian environment and all other operations are Gaussian. Here we have considered an ancilla with finite energy. An ancilla with infinite energy is only meaningful when one considers a sequence of ancilla with increasing finite energy. However, the ancilla of a fixed channel cannot depend on the energy of the input state, thus in terms of the growth with the input energy, the amount of nG is bounded for Gaussian-dilatable channels.[1] Note that our argument does not rule out the possibility that all channels might be approximately Gaussian dilatable. The formulation of approximate Gaussian-dilatable channels still requires more work.

Binary phase-shift channel. Consider a single-mode channel that applies a phase shift R_π with probability 1/2, i.e.,

$$\phi_{\rm BPS}(\rho) = \tfrac{1}{2}\rho + \tfrac{1}{2}R_\pi \rho R_\pi^\dagger. \quad (29)$$

Let the input be a coherent state $|\alpha\rangle$ ($\alpha > 0$), so that the mean and covariance matrix of the output $\phi_{\rm BPS}(|\alpha\rangle\langle\alpha|) = \frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|-\alpha\rangle\langle-\alpha|$ are (0,0) and Diag($4\alpha^2+1$,1). The entropy of the Gaussian state with the same mean and covariance matrix is $g[(\sqrt{4\alpha^2+1}-1)/2]$, while $S[\phi_{\rm BPS}(|\alpha\rangle\langle\alpha|)] \leqslant 1$. Thus we have $\delta_\mathcal{G}[\phi_{\rm BPS}(|\alpha\rangle\langle\alpha|)] \geqslant g[(\sqrt{4\alpha^2+1}-1)/2] - 1$, and equality is achieved as $\alpha \to \infty$. It is diverging as α increases. Thus $\delta_\mathcal{G}[\phi_{\rm BPS}(|\alpha\rangle\langle\alpha|)]$ diverges as α increases, so

$$\phi_{\rm BPS} \in \Phi_\infty. \quad (30)$$

In fact, if one considers the input and ancilla to be in a TMSV, it is straightforward to show (details in Appendix I) that $\tilde{\delta}_\mathcal{G}[\phi_{\rm BPS}] \geqslant 2g(N_S/2) - 1$, when the output and ancilla have total energy constraint N_S. Thus the rate of divergence is $\log_2(N_S)$, which is the maximum rate of divergence.

Self-Kerr unitary. Consider now the single-mode self-Kerr unitary

$$U_{\rm Kerr} = \exp(-i\gamma(a^\dagger a)^2). \quad (31)$$

The lower bound $\delta_\mathcal{G}[U_{\rm Kerr}(|\alpha\rangle\langle\alpha|)]$ has been found to diverge maximally, as $\log_2(N_S)$, where $N_S = |\alpha|^2$ [46]. So we have

$$U_{\rm Kerr} \in \Phi_\infty. \quad (32)$$

[1] In principle, one can encode all possible output states into an ancilla with infinite energy, thus considering an infinite-energy ancilla is not meaningful.

We have classified non-Gaussian operations into two classes Φ_F and Φ_∞. Within the class Φ_F, nG is finite and thus comparing and ordering different operations is straightforward. Within the class Φ_∞, even though all nG are infinite, they can have different rates of divergence. So classification based on those rates is possible.

It is an open question whether all linear maps (quantum channels) in Φ_F are Gaussian dilatable. If it is true, then because of Theorem 2, it would imply that the class of Gaussian-dilatable non-Gaussian channels and the class of finite-nG channels are equal. It is also open whether there is a minimum set of operations in Φ_F, such that any other operations in Φ_F can be simulated by this set of operations and Gaussian operations in $X_\mathcal{G}$, in terms of the generation of non-Gaussian states from Gaussian inputs.

VI. CONCLUSIONS

Gaussian states and Gaussian operations are inadequate for various tasks, such as universal quantum computing, entanglement distillation, and quantum error correction. So non-Gaussian states and operations are naturally considered as resources for these tasks. A quantum resource theory for nG in states and operations is a starting point for understanding the utility of nG.

In this paper, we extended the resource theory of non-Gaussian states in Refs. [44–46] to non-Gaussian operations and established a monotone to quantify the amount of nG. This monotone can be analytically calculated for conditional unitary maps such as PNS and PNA. We also provided a lower bound and an upper bound for this monotone to assist in the calculation and analysis of nG.

More importantly, our monotone enables us to classify non-Gaussian operations into (1) the finite-nG class, and (2) the diverging-nG class. Within the first class, nG is finite, thus direct comparison and ordering of operations is straightforward. Within the second class, nG diverges as the output energy increases. Further classification may be possible through comparing rates of divergence.

We gave several examples of quantum operations in each class. In particular, we showed that all Gaussian-dilatable non-Gaussian channels are in the finite-nG class. Thus, not all non-Gaussian channels are Gaussian dilatable. Gaussian-dilatable channels are important because their properties, such as their Kraus operators, are relatively easy to obtain, making them a starting point for studying of non-Gaussian channels and operations. For example, recent results [57] show that the nonadditivity violation in a bosonic noise channel, which is Gaussian dilatable, is mild. However, our results suggest that focusing on Gaussian-dilatable channels is not enough for the full understanding of non-Gaussian channels.

An important future research direction is the operational resource theory of non-Gaussian operations, such as the one for coherence [39]. For example, how to quantify the power of different non-Gaussian operations for specific tasks, such as quantum computation and entanglement distillation, is worthy of investigation. This problem is also related to channel simulation in terms of production of non-Gaussian states. One can also ask whether there is a finite set of universal non-Gaussian operations, such that all non-Gaussian states can be produced by this set of non-Gaussian operations and arbitrary Gaussian operations starting from Gaussian states. The answer is yes, because universal quantum computation is possible with Gaussian operations plus one single non-Gaussian operation [12]. However, it is not clear whether the class of finite-nG operations can enable universal quantum computing or it is necessary to have operations from the diverging-nG class.

Another important future task is the further classification of non-Gaussian operations. As an analog, there are bound entanglement states [72] that have zero distillable entanglement, and cannot be directly used to enhance teleportation. Similarly, a mixture of Gaussian channels, e.g., the BPS channel, seems less useful than the Kerr nonlinearity for many tasks such as universal computation, while they are both in the diverging-nG class with the same rate of divergence. A more delicate classification, based on the convex resource theory of non-Gaussianity [73,74], which distinguishes these two types of non-Gaussian operations is an important step towards the full classification of non-Gaussian operations.

ACKNOWLEDGMENTS

Q.Z. thanks Zi-Wen Liu and Ryuji Takagi for discussions. Q.Z. and J.H.S. are supported by the Air Force Office of Scientific Research Grant No. FA9550-14-1-0052. Q.Z. also acknowledges the support of the Claude E. Shannon Research Assistantship. P.W.S. is supported by the National Science Foundation under Grant No. CCF-1525130 and National Science Foundation through the Science and Technology Centers for Science of Information under Grant No. CCF0-939370.

APPENDIX A: PROPERTIES OF QUANTUM RELATIVE ENTROPY

The relative entropy of two quantum states ρ and σ is defined as $S(\rho\|\sigma) \equiv \mathrm{Tr}[\rho(\log_2\rho - \log_2\sigma)]$. Besides continuity, it has the following properties [47,48].

O1 Non-negativity (Klein's inequality). $S(\rho\|\sigma) \geqslant 0$.

O2 Joint convexity.

$$S[p\rho_1 + (1-p)\rho_2 \| p\sigma_1 + (1-p)\sigma_2] \leqslant pS(\rho_1\|\sigma_1) + (1-p)S(\rho_2\|\sigma_2).$$

O3 Monotonically decreasing under partial trace.

$$S(\mathrm{Tr}_2\rho_{12}\|\mathrm{Tr}_2\sigma_{12}) \leqslant S(\rho_{12}\|\sigma_{12}).$$

O4 Monotonically decreasing under quantum operation. $S[\varepsilon(\rho)\|\varepsilon(\sigma)] \leqslant S(\rho\|\sigma)$. Equal when ε is an isometry.

O5 Additivity of product states. $S(\rho_1\otimes\rho_2\|\sigma_1\otimes\sigma_2) = S(\rho_1\|\sigma_1) + S(\rho_2\|\sigma_2)$.

O6 $2S(\rho_{12}\|\sigma_{12}) \geqslant S(\rho_1\|\sigma_1) + S(\rho_2\|\sigma_2)$. Superadditivity can be established by a better multiplicative constant [75].

APPENDIX B: PROOF OF LEMMA 1

Proof. A Gaussian channel $\xi_\mathcal{G}$ can be extended to a Gaussian unitary on its input and an environment [1,7], which can be expressed as a linear transform on the mean and covariance matrix in Eq. (7). The output can be obtained by tracing out part of the joint output of this Gaussian unitary. Thus $\xi_\mathcal{G}$ produces a state (not necessarily Gaussian) with mean

and covariance matrix $(\bar{\boldsymbol{x}}', \boldsymbol{\Lambda}')$ as function of the mean and covariance matrix $(\bar{\boldsymbol{x}}, \boldsymbol{\Lambda})$ of the input ρ. So $(\lambda_{\mathcal{G}} \circ \xi_{\mathcal{G}})(\rho)$ is a Gaussian state with mean and covariance matrix $(\bar{\boldsymbol{x}}', \boldsymbol{\Lambda}')$. On the other hand, $(\xi_{\mathcal{G}} \circ \lambda_{\mathcal{G}})(\rho)$ is also a Gaussian state with mean and covariance matrix $(\bar{\boldsymbol{x}}', \boldsymbol{\Lambda}')$. Since a Gaussian state is uniquely specified by its mean and covariance matrix, we have $(\xi_{\mathcal{G}} \circ \lambda_{\mathcal{G}})(\rho) = (\lambda_{\mathcal{G}} \circ \xi_{\mathcal{G}})(\rho), \forall \rho$.

A counterexample for the generalization to conditional Gaussian maps is constructed here. Consider the conditional map

$$\mathcal{T}_\alpha(\rho_{AA'}) = \frac{\langle\alpha|_{A'} \rho_{AA'} |\alpha\rangle_{A'}}{\mathrm{Tr}_A \langle\alpha|_{A'} \rho_{AA'} |\alpha\rangle_{A'}}, \tag{B1}$$

which projects on A' and outputs A, where $|\alpha\rangle_{A'}$ is the coherent state with amplitude $\alpha > 0$. Consider the input $\sigma_{AA'} = (|\alpha\rangle_A \langle\alpha| \otimes |\alpha\rangle_{A'} \langle\alpha| + |-\alpha\rangle_A \langle-\alpha| \otimes |-\alpha\rangle_{A'} \langle-\alpha|)/2$. In the following, we show that $(\lambda_{\mathcal{G}} \circ \mathcal{T}_\alpha)(\sigma_{AA'})$ and $(\mathcal{T}_\alpha \circ \lambda_{\mathcal{G}})(\sigma_{AA'})$ have different means and are thus different Gaussian states. We have that

$$\mathcal{T}_\alpha(\sigma_{AA'}) = \frac{1}{1 + e^{-4\alpha^2}} (|\alpha\rangle_A \langle\alpha| + e^{-4\alpha^2} |-\alpha\rangle_A \langle-\alpha|), \tag{B2}$$

where expectation value is

$$\langle a \rangle_{\mathcal{T}_\alpha(\sigma_{AA'})} = \frac{1 - e^{-4\alpha^2}}{1 + e^{-4\alpha^2}} \alpha. \tag{B3}$$

From results in Appendix H, the mean of $\sigma_{AA'}$ is $(0,0,0,0)$, and its covariance matrix is

$$\Lambda_\sigma = \begin{pmatrix} 4\alpha^2 + 1 & 0 & 4\alpha^2 & 0 \\ 0 & 1 & 0 & 0 \\ 4\alpha^2 & 0 & 4\alpha^2 + 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{B4}$$

The density matrix of $\lambda_{\mathcal{G}}(\sigma_{AA'})$ can be obtained through the P function [1] as

$$\lambda_{\mathcal{G}}(\sigma_{AA'}) = \int_{-\infty}^{\infty} d\alpha_1 \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{\alpha_1^2}{2\alpha^2}} |\alpha_1\rangle_A \langle\alpha_1| \otimes |\alpha_1\rangle_{A'} \langle\alpha_1|. \tag{B5}$$

The output of the map is $(\mathcal{T}_\alpha \circ \lambda_{\mathcal{G}})(\sigma_{AA'}) \propto \int_{-\infty}^{\infty} d\alpha_1 \exp(-\frac{\alpha_1^2}{2\alpha^2} - (\alpha_1^2 + \alpha^2 - 2\alpha_1\alpha)) |\alpha_1\rangle_A \langle\alpha_1|$. It is then straightforward to see that

$$\langle a \rangle_{\mathcal{T}_\alpha \circ \lambda_{\mathcal{G}}(\sigma_{AA'})} = \frac{2\alpha^3}{1 + \alpha^2}, \tag{B6}$$

which is not equal to $\langle a \rangle_{\lambda_{\mathcal{G}} \circ \mathcal{T}_\alpha(\sigma_{AA'})} = \langle a \rangle_{\mathcal{T}_\alpha(\sigma_{AA'})}$ given in Eq. (B3) for finite $\alpha > 0$.

APPENDIX C: COUNTEREXAMPLE

Consider a non-Gaussian state $\rho_{AA'} \simeq \sqrt{\epsilon} |\alpha\rangle_{A'} \langle\alpha| \otimes |n\rangle_A \langle n| + \sqrt{1-\epsilon} |-\alpha\rangle_{A'} \langle-\alpha| \otimes \rho_A$, where $\epsilon \ll 1, \alpha \gg 1$ and $\rho_A \in \mathcal{G}$. We have $\delta_{\mathcal{G}}[\rho_{AA'}] \ll 1$ from continuity. For the Gaussian conditional map in Eq. (B1), we have $\mathcal{T}_\alpha(\rho_{AA'}) \simeq |n\rangle_A \langle n|$. This means that $\delta_{\mathcal{G}}[\mathcal{T}_\alpha(\rho_{AA'})] \simeq \delta_{\mathcal{G}}[|n\rangle_A \langle n|] \gg \delta_{\mathcal{G}}[\rho_{AA'}]$, i.e., nG can increase under a Gaussian conditional map.

APPENDIX D: PROOF OF LEMMA 2

Proof. We use methods similar to those in Ref. [76]. Any pure Gaussian state $\psi_{AA'}$, with A having $n \geqslant n_\phi$ modes and A' having n_ϕ modes, has phase-space Schmidt decomposition [Eq. (8) in main text]

$$U_A(\psi_{AA'}) = \left[\otimes_{k=n_\phi+1}^{n} 0_{A_k}\right] \otimes \left[U_{A'}^\dagger (\otimes_{k=1}^{n_\phi} (\zeta_{\lambda_k})_{A_k A_k'})\right].$$

Thus, U_A can allow a Gaussian isometry u_A from $\mathcal{H}[n]$ to $\mathcal{H}[n_\phi]$ such that $\psi_{AA'} = [(u_A^{-1} \circ u_A) \otimes \mathcal{I}_{n_\phi}](\psi_{AA'})$.

Now let $\psi_{A_\phi A'} = (u_A \otimes \mathcal{I}_{n_\phi})(\psi_{AA'}) \in \mathcal{G}[2n_\phi]$. Due to relative entropy's invariance under isometries, $u_A \in X_{\mathcal{G}}^L$, and Lemma 1, we get

$$\begin{aligned} &\delta_{\mathcal{G}}[(\mathcal{I}_n \otimes \phi)(\psi_{AA'})] \\ &= S[(\mathcal{I}_n \otimes \phi)(\psi_{AA'}) \| \lambda_{\mathcal{G}}((\mathcal{I}_n \otimes \phi)(\psi_{AA'}))] \\ &= S[(u_A \otimes \phi)(\psi_{AA'}) \| u_A \circ \lambda_{\mathcal{G}}((\mathcal{I}_n \otimes \phi)(\psi_{AA'}))] \\ &= S[(\mathcal{I}_{n_\phi} \otimes \phi)(\psi_{A_\phi A'}) \| \lambda_{\mathcal{G}}((u_A \otimes \phi)(\psi_{AA'}))] \\ &= \delta_{\mathcal{G}}[(\mathcal{I}_{n_\phi} \otimes \phi)(\psi_{A_\phi A'})]. \end{aligned} \tag{D1}$$

APPENDIX E: PROOFS OF PROPERTIES B1–B7

In most proofs we use Definition 5 as a starting point, and we will simplify the notation for the domain of maximization, e.g., writing $\rho_{\mathcal{G}} \in \mathcal{G}[n + n_\phi]$ as $\rho_{\mathcal{G}} \in \mathcal{G}$. Also, we will not explicitly state the dimension of the identity operator $\mathcal{I}$ when it's not necessary.

B1 *Proof.* Non-negativity follows directly from Property A1. If $\phi \in X_{\mathcal{G}}$, it is easy to see that $\tilde{\delta}_{\mathcal{G}}[\phi] = 0$ since $\mathcal{I} \otimes \phi \in X_{\mathcal{G}}$. Now we prove the reverse part. Suppose $\tilde{\delta}_{\mathcal{G}}[\phi] = 0$, then by Definition 5, $\forall \rho_{\mathcal{G}} \in \mathcal{G}$, we have $\delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi)(\rho_{\mathcal{G}})] = 0$. By Definition 2 and Property A1, we get $\phi \in X_{\mathcal{G}}$.

B2 *Proof.* (i) $\tilde{\delta}_{\mathcal{G}}[\phi \otimes \phi_{\mathcal{G}}] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi \otimes \phi_{\mathcal{G}})(\rho_{\mathcal{G}})] \geqslant \max_{\rho'_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi)(\rho'_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$. The inequality is obtained by taking trace over the output of $\phi_{\mathcal{G}}$ and using Property A5.

(ii) $\tilde{\delta}_{\mathcal{G}}[\phi \otimes \phi_{\mathcal{G}}] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi \otimes \phi_{\mathcal{G}})(\rho_{\mathcal{G}})] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi \otimes \mathcal{I}_{\phi_{\mathcal{G}}})(\mathcal{I} \otimes \mathcal{I}_{n_\phi} \otimes \phi_{\mathcal{G}})(\rho_{\mathcal{G}})] \leqslant \max_{\rho'_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi \otimes \mathcal{I}_{n_{\phi_{\mathcal{G}}}})(\rho'_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$, where the inequality follows since $\mathcal{I} \otimes \mathcal{I}_{n_\phi} \otimes \phi_{\mathcal{G}}(\rho_{\mathcal{G}}) \in \mathcal{G}$ and in the last equality we have used the symmetry of purification in Lemma 2.

B3 *Proof.* (i) From Property A4, $\tilde{\delta}_{\mathcal{G}}[U_{\mathcal{G}} \circ \phi] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes U_{\mathcal{G}}) \circ (\mathcal{I} \otimes \phi)(\rho_{\mathcal{G}})] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi)(\rho_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$.

(ii) $\tilde{\delta}_{\mathcal{G}}[\phi \circ U_{\mathcal{G}}] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi) \circ (\mathcal{I} \otimes U_{\mathcal{G}})(\rho_{\mathcal{G}})] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi)(\rho_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$, where we have used $(\mathcal{I} \otimes U_{\mathcal{G}})(\mathcal{G}) = \mathcal{G}$.

B4 *Proof.* $\tilde{\delta}_{\mathcal{G}}[\mathrm{Tr}_A \circ \phi] = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}\{[\mathcal{I} \otimes (\mathrm{Tr}_A \circ \phi)](\rho_{\mathcal{G}})\} = \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}\{\mathrm{Tr}_A[(\mathcal{I} \otimes \phi)(\rho_{\mathcal{G}})]\} \leqslant \max_{\rho_{\mathcal{G}} \in \mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I} \otimes \phi)(\rho_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$. The inequality follows from Property A5.

B5 *Proof.* $\tilde{\delta}_{\mathcal{G}}[\phi] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}\{[\mathcal{I}\otimes(\mathrm{Tr}_E\circ U_\phi)](\rho_{\mathcal{G}}\otimes\mathbf{0}_E)\} \leqslant \max_{\rho'_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}\{[\mathcal{I}\otimes(\mathrm{Tr}_E\circ U_\phi)](\rho'_{\mathcal{G}})\} = \tilde{\delta}_{\mathcal{G}}[\mathrm{Tr}_E\circ U_\phi] \leqslant \tilde{\delta}_{\mathcal{G}}[U_\phi]$. The first inequality is due to expanding the set of states over which the maximization is performed. The second inequality is because of Property B4.

B6 *Proof.* (i) From Property A6, $\tilde{\delta}_{\mathcal{G}}[\phi_{\mathcal{G}}\circ\phi] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I}\otimes\phi_{\mathcal{G}})\circ(\mathcal{I}\otimes\phi)(\rho_{\mathcal{G}})] \leqslant \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I}\otimes\phi)(\rho_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$.

(ii) $\tilde{\delta}_{\mathcal{G}}[\phi\circ\phi_{\mathcal{G}}] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I}\otimes\phi)\circ(\mathcal{I}\otimes\phi_{\mathcal{G}})(\rho_{\mathcal{G}})] \leqslant \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[(\mathcal{I}\otimes\phi)(\rho_{\mathcal{G}})] = \tilde{\delta}_{\mathcal{G}}[\phi]$. The inequality uses the fact that $(\mathcal{I}\otimes\phi_{\mathcal{G}})(\rho_{\mathcal{G}})\in\mathcal{G}$.

B7 *Proof.* In Definition 5, choose the ancilla to be in $\mathcal{H}[n_{\phi_1}]\otimes\mathcal{H}[n_{\phi_2}]$, so we can write $\mathcal{I}=\mathcal{I}_1\otimes\mathcal{I}_2$, where $\mathcal{I}_k$ is the identity operator on $\mathcal{H}[n_{\phi_k}]$, thus $\tilde{\delta}_{\mathcal{G}}[\phi_1\otimes\phi_2] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}[2n_{\phi_1}+2n_{\phi_2}]} \delta_{\mathcal{G}}[(\mathcal{I}_1\otimes\phi_1\otimes\mathcal{I}_2\otimes\phi_2)(\rho_{\mathcal{G}})] \geqslant \max_{\rho_1\in\mathcal{G}[2n_{\phi_1}]}\max_{\rho_2\in\mathcal{G}[2n_{\phi_2}]} \delta_{\mathcal{G}}[(\mathcal{I}_1\otimes\phi_1\otimes\mathcal{I}_2\otimes\phi_2)(\rho_1\otimes\rho_2)] = \tilde{\delta}_{\mathcal{G}}[\phi_1]+\tilde{\delta}_{\mathcal{G}}[\phi_2]$, where in the last step we used Property A2.

APPENDIX F: PROPERTIES OF $d_{\mathcal{G}}$

C1 Invariance under concatenation with a Gaussian unitary. $\forall U_{\mathcal{G}}\in X^U_{\mathcal{G}}$, we have $d_{\mathcal{G}}[U_{\mathcal{G}}\circ\phi] = d_{\mathcal{G}}[\phi\circ U_{\mathcal{G}}] = d_{\mathcal{G}}[\phi]$.

Proof. (i) $d_{\mathcal{G}}[U_{\mathcal{G}}\circ\phi] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}\{U_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})]\} = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})] = d_{\mathcal{G}}[\phi]$, where we used Property A4.

(ii) $d_{\mathcal{G}}[\phi\circ U_{\mathcal{G}}] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\phi\circ U_{\mathcal{G}}(\rho_{\mathcal{G}})] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})] = d_{\mathcal{G}}[\phi]$, where we have used $U_{\mathcal{G}}(\mathcal{G})=\mathcal{G}$.

C2 Monotonically decreasing under the concatenation with partial trace. For ϕ with output AB, we have $d_{\mathcal{G}}[\mathrm{Tr}_A\circ\phi]\leqslant d_{\mathcal{G}}[\phi]$.

Proof. $d_{\mathcal{G}}[\mathrm{Tr}_A\circ\phi] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\mathrm{Tr}_A\circ\phi(\rho_{\mathcal{G}})] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}\{\mathrm{Tr}_A[\phi(\rho_{\mathcal{G}})]\} \leqslant \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})] = d_{\mathcal{G}}[\phi]$. The inequality follows from Property A5.

C3 Monotonically increasing under Stinespring dilation with a vacuum environment. Note this property is only for channels, not for general operations. Suppose $\forall\rho,\phi(\rho)=\mathrm{Tr}_E\circ U_\phi(\rho\otimes\mathbf{0}_E)$, we have $d_{\mathcal{G}}[\phi]\leqslant d_{\mathcal{G}}[U_\phi]$.

Proof. $d_{\mathcal{G}}[\phi] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\mathrm{Tr}_E\circ U_\phi(\rho_{\mathcal{G}}\otimes\mathbf{0}_E)] \leqslant \max_{\rho'_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\mathrm{Tr}_E\circ U_\phi(\rho'_{\mathcal{G}})] = d_{\mathcal{G}}[\mathrm{Tr}_E\circ U_\phi] \leqslant d_{\mathcal{G}}[U_\phi]$ The first inequality is due to expanding the set of states over which the maximization is performed. The second inequality is from Property C2.

C4 Nonincreasing under concatenation with a Gaussian channel. $\forall\phi_{\mathcal{G}}\in X^L_{\mathcal{G}}$, (i) Postconcatenation: $d_{\mathcal{G}}[\phi_{\mathcal{G}}\circ\phi]\leqslant d_{\mathcal{G}}[\phi]$. (ii) Preconcatenation: $d_{\mathcal{G}}[\phi\circ\phi_{\mathcal{G}}]\leqslant d_{\mathcal{G}}[\phi]$.

Proof. (i) $d_{\mathcal{G}}[\phi_{\mathcal{G}}\circ\phi] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}\{\phi_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})]\} \leqslant \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})] = d_{\mathcal{G}}[\phi]$, where we used Property A6.

(ii) $d_{\mathcal{G}}[\phi\circ\phi_{\mathcal{G}}] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}} \delta_{\mathcal{G}}[\phi\circ\phi_{\mathcal{G}}(\rho_{\mathcal{G}})] \leqslant \max_{\rho_{\mathcal{G}}\in\phi_{\mathcal{G}}(\mathcal{G})} \delta_{\mathcal{G}}[\phi(\rho_{\mathcal{G}})] \leqslant d_{\mathcal{G}}[\phi]$, where we have used $\phi_{\mathcal{G}}(\mathcal{G})\subset\mathcal{G}$.

C5 Superadditivity. $d_{\mathcal{G}}[\phi_1\otimes\phi_2]\geqslant d_{\mathcal{G}}[\phi_1]+d_{\mathcal{G}}[\phi_2]$.

Proof. $d_{\mathcal{G}}[\phi_1\otimes\phi_2] = \max_{\rho_{\mathcal{G}}\in\mathcal{G}[n_{\phi_1}+n_{\phi_2}]} \delta_{\mathcal{G}}[(\phi_1\otimes\phi_2)(\rho_{\mathcal{G}})] \geqslant \max_{\rho_1\in\mathcal{G}[n_{\phi_1}]}\max_{\rho_2\in\mathcal{G}[n_{\phi_2}]} \delta_{\mathcal{G}}[(\phi_1\otimes\phi_2)(\rho_1\otimes\rho_2)] = d_{\mathcal{G}}[\phi_1]+d_{\mathcal{G}}[\phi_2]$, where in the last step we used Property A2.

APPENDIX G: PROPERTIES OF $D_{\mathcal{G}}$

D1 Non-negativity. $D_{\mathcal{G}}[\phi]\geqslant 0$, with equality iff $\phi\in X_{\mathcal{G}}$.

Proof. This follows from Property B1 of $\tilde{\delta}_{\mathcal{G}}[\phi]$ and Theorem 1. Alternatively, this result can be obtained from the non-negativity of quantum relative entropy.

D2 Invariance under tensoring with a Gaussian channel. $\forall\xi_{\mathcal{G}}\in X^L_{\mathcal{G}}$, we have $D_{\mathcal{G}}[\phi\otimes\xi_{\mathcal{G}}] = D_{\mathcal{G}}[\phi]$.

Proof. (i) First we prove $D_{\mathcal{G}}[\phi\otimes\xi_{\mathcal{G}}]\geqslant D_{\mathcal{G}}[\phi]$.

$$\begin{aligned} D_{\mathcal{G}}[\phi\otimes\xi_{\mathcal{G}}] &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi\otimes\xi_{\mathcal{G}})(\rho_{\mathcal{G}})\|(\mathcal{I}\otimes\phi_{\mathcal{G}})(\rho_{\mathcal{G}})] \geqslant \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho'_{\mathcal{G}}\in\mathcal{G}} S[\mathcal{I}\otimes\phi\otimes\xi_{\mathcal{G}}(\rho'_{\mathcal{G}}\otimes\rho_1)\|\mathcal{I}\otimes\phi_{\mathcal{G}}(\rho'_{\mathcal{G}}\otimes\rho_1)] \\ &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho'_{\mathcal{G}}\in\mathcal{G}} S[\mathcal{I}\otimes\phi(\rho'_{\mathcal{G}})\otimes\xi_{\mathcal{G}}(\rho_1)\|\mathcal{I}\otimes\phi_{\mathcal{G}}(\rho'_{\mathcal{G}}\otimes\rho_1)] \geqslant \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho'_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi)(\rho'_{\mathcal{G}})\|\mathcal{I}\otimes(\mathrm{Tr}_{\xi_{\mathcal{G}}}\circ\phi_{\mathcal{G}})(\rho'_{\mathcal{G}}\otimes\rho_1)] \\ &= \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho'_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi)(\rho'_{\mathcal{G}})\|(\mathcal{I}\otimes\phi'_{\mathcal{G}})(\rho'_{\mathcal{G}})] = D_{\mathcal{G}}[\phi]. \end{aligned} \tag{G1}$$

The first inequality is from limiting the maximization to states of the form $\rho'_{\mathcal{G}}\otimes\rho_1$. The second inequality is from relative entropy's monotonically decreasing under partial trace. The last equality is because $\forall\phi_{\mathcal{G}}\in X_{\mathcal{G}}$, $(\mathrm{Tr}_{\xi_{\mathcal{G}}}\circ\phi_{\mathcal{G}})$ is a Gaussian operation that takes input σ and outputs to $\mathcal{H}[n_\phi]$, and every Gaussian operation with the same input and output dimension with ϕ can be extended to another Gaussian operation by trivially tensoring with the identity.

(ii) Now we prove $D_{\mathcal{G}}[\phi\otimes\xi_{\mathcal{G}}]\leqslant D_{\mathcal{G}}[\phi]$.

$$\begin{aligned} D_{\mathcal{G}}[\phi\otimes\xi_{\mathcal{G}}] &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi\otimes\xi_{\mathcal{G}})(\rho_{\mathcal{G}})\|(\mathcal{I}\otimes\phi_{\mathcal{G}})(\rho_{\mathcal{G}})] \leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi\otimes\xi_{\mathcal{G}})(\rho_{\mathcal{G}})\|(\mathcal{I}\otimes\phi'_{\mathcal{G}}\otimes\xi_{\mathcal{G}})(\rho_{\mathcal{G}})] \\ &\leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}'\otimes\phi)(\rho_{\mathcal{G}})\|(\mathcal{I}'\otimes\phi'_{\mathcal{G}})(\rho_{\mathcal{G}})] = D_{\mathcal{G}}[\phi]. \end{aligned} \tag{G2}$$

The first inequality is due to limiting the minimization to operations of the form $\phi'_{\mathcal{G}}\otimes\xi_{\mathcal{G}}$. The last inequality is due to relative entropy's monotonically decreasing under quantum operations and symmetry in the ancilla.

D3 Invariance under concatenation with a Gaussian unitary. $\forall U_{\mathcal{G}}\in X^U_{\mathcal{G}}$, we have $D_{\mathcal{G}}[U_{\mathcal{G}}\circ\phi] = D_{\mathcal{G}}[\phi\circ U_{\mathcal{G}}] = D_{\mathcal{G}}[\phi]$.

Proof. (i) $U_{\mathcal{G}}$ has inverse $U_{\mathcal{G}}^{-1}$. So

$$\begin{aligned} D_{\mathcal{G}}[U_{\mathcal{G}}\circ\phi] &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{[\mathcal{I}\otimes(U_{\mathcal{G}}\circ\phi)](\psi_{\mathcal{G}})\|(\mathcal{I}\otimes\phi_{\mathcal{G}})(\psi_{\mathcal{G}})\} \\ &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{(\mathcal{I}\otimes\phi)(\psi_{\mathcal{G}})\|[\mathcal{I}\otimes(U_{\mathcal{G}}^{-1}\circ\phi_{\mathcal{G}})](\psi_{\mathcal{G}})\} = \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi)(\psi_{\mathcal{G}})\|(\mathcal{I}\otimes\phi'_{\mathcal{G}})(\psi_{\mathcal{G}})] = D_{\mathcal{G}}[\phi]. \end{aligned} \tag{G3}$$

We have used the invariance of relative entropy under isometries.

(ii) $\forall\phi_{\mathcal{G}}\in X_{\mathcal{G}}$, let $\phi'_{\mathcal{G}}=\phi_{\mathcal{G}}\circ U_{\mathcal{G}}^{-1}\in X_{\mathcal{G}}$.

$$\begin{aligned} D_{\mathcal{G}}[\phi\circ U_{\mathcal{G}}] &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{[\mathcal{I}\otimes(\phi\circ U_{\mathcal{G}})](\psi_{\mathcal{G}})\|(\mathcal{I}\otimes\phi_{\mathcal{G}})(\psi_{\mathcal{G}})\} \\ &= \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S[\mathcal{I}\otimes(\phi\circ U_{\mathcal{G}})(\psi_{\mathcal{G}})\|\mathcal{I}\otimes(\phi'_{\mathcal{G}}\circ U_{\mathcal{G}})(\psi_{\mathcal{G}})] = \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi'_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi)(\psi'_{\mathcal{G}})\|(\mathcal{I}\otimes\phi'_{\mathcal{G}})(\psi'_{\mathcal{G}})] = D_{\mathcal{G}}[\phi]. \end{aligned} \tag{G4}$$

We have used $U_{\mathcal{G}}(\mathcal{G})=\mathcal{G}$.

D4 Monotonically decreasing under concatenation with a partial trace. For ϕ with output AB, we have $D_{\mathcal{G}}[\mathrm{Tr}_A\circ\phi]\leqslant D_{\mathcal{G}}[\phi]$.

Proof. $D_{\mathcal{G}}[\mathrm{Tr}_A\circ\phi]$

$$\begin{aligned} &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{[\mathcal{I}\otimes(\mathrm{Tr}_A\circ\phi)](\psi_{\mathcal{G}})\|(\mathcal{I}\otimes\phi_{\mathcal{G}})(\psi_{\mathcal{G}})\} \leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S[\mathcal{I}\otimes(\mathrm{Tr}_A\circ\phi)(\psi_{\mathcal{G}})\|\mathcal{I}\otimes(\mathrm{Tr}_A\circ\phi'_{\mathcal{G}})(\psi_{\mathcal{G}})] \\ &\leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi)(\psi_{\mathcal{G}})\|(\mathcal{I}\otimes\phi'_{\mathcal{G}})(\psi_{\mathcal{G}})] = D_{\mathcal{G}}[\phi]. \end{aligned} \tag{G5}$$

The first inequality is due to limiting to minimization over $\phi_{\mathcal{G}}$ that can be written as $\mathrm{Tr}_A\circ\phi'_{\mathcal{G}}$. The second inequality is due to relative entropy's monotonically decreasing under a partial trace.

D5 Monotonically increasing under Stinespring dilation with a vacuum environment. Note this property is only for channels, not for general operations. Suppose $\forall\rho,\phi(\rho)=\mathrm{Tr}_E\circ U_\phi(\rho\otimes\mathbf{0}_E)$, then $D_{\mathcal{G}}[\phi]\leqslant D_{\mathcal{G}}[U_\phi]$.

Proof. We have

$$\begin{aligned} D_{\mathcal{G}}[\phi] &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{[\mathcal{I}\otimes(\mathrm{Tr}_E\circ U_\phi)](\psi_{\mathcal{G}}\otimes\mathbf{0}_E)\|(\mathcal{I}\otimes\phi_{\mathcal{G}})(\psi_{\mathcal{G}})\} \\ &\leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{[\mathcal{I}\otimes(\mathrm{Tr}_E\circ U_\phi)](\psi_{\mathcal{G}}\otimes\mathbf{0}_E)\|[\mathcal{I}\otimes(\mathrm{Tr}_E\circ\phi'_{\mathcal{G}})](\psi_{\mathcal{G}}\otimes\mathbf{0}_E)\} \\ &\leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes U_\phi)(\psi_{\mathcal{G}}\otimes\mathbf{0}_E)\|(\mathcal{I}\otimes\phi'_{\mathcal{G}})(\psi_{\mathcal{G}}\otimes\mathbf{0}_E)] \leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi'_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes U_\phi)(\psi'_{\mathcal{G}})\|(\mathcal{I}\otimes\phi'_{\mathcal{G}})(\psi'_{\mathcal{G}})] = D_{\mathcal{G}}[U_\phi]. \end{aligned} \tag{G6}$$

The first inequality is from limiting the set of operations $\phi_{\mathcal{G}}$ over which the minimization is performed; the second inequality is from relative entropy's monotonically decreasing under a partial trace; and the third inequality is from expanding the set of states over which the maximization is performed.

D6 Nonincreasing under concatenation with a Gaussian channel. $\forall\xi_{\mathcal{G}}\in X_{\mathcal{G}}^L$, (i) Postconcatenation: $D_{\mathcal{G}}[\xi_{\mathcal{G}}\circ\phi]\leqslant D_{\mathcal{G}}[\phi]$. (ii) Preconcatenation: $D_{\mathcal{G}}[\phi\circ\xi_{\mathcal{G}}]\leqslant D_{\mathcal{G}}[\phi]$.

Proof. (i) $D_{\mathcal{G}}[\xi_{\mathcal{G}}\circ\phi]$

$$\begin{aligned} &= \min_{\phi_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{[\mathcal{I}\otimes(\xi_{\mathcal{G}}\circ\phi)](\psi_{\mathcal{G}})\|(\mathcal{I}\otimes\phi_{\mathcal{G}})(\psi_{\mathcal{G}})\} \leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\psi_{\mathcal{G}}\in\mathcal{G}} S\{[\mathcal{I}\otimes(\xi_{\mathcal{G}}\circ\phi)](\psi_{\mathcal{G}})\|[\mathcal{I}\otimes(\xi_{\mathcal{G}}\circ\phi'_{\mathcal{G}})](\psi_{\mathcal{G}})\} \\ &\leqslant \min_{\phi'_{\mathcal{G}}\in X_{\mathcal{G}}}\max_{\rho_{\mathcal{G}}\in\mathcal{G}} S[(\mathcal{I}\otimes\phi)(\rho_{\mathcal{G}})\|(\mathcal{I}\otimes\phi'_{\mathcal{G}})(\rho_{\mathcal{G}})] = D_{\mathcal{G}}[\phi]. \end{aligned} \tag{G7}$$

The first inequality is due to limiting to minimization over $\phi_{\mathcal{G}}$ that can be written as $\xi_{\mathcal{G}}\circ\phi'_{\mathcal{G}}$; and the second inequality is due to relative entropy's monotonically decreasing under a quantum operation.

(ii) $D_{\mathcal{G}}[\phi \circ \xi_{\mathcal{G}}]$

$$= \min_{\phi_{\mathcal{G}} \in X_{\mathcal{G}}} \max_{\psi_{\mathcal{G}} \in \mathcal{G}} S\{[\mathcal{I} \otimes (\phi \circ \xi_{\mathcal{G}})](\psi_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi_{\mathcal{G}})(\psi_{\mathcal{G}})\}$$

$$\leqslant \min_{\phi'_{\mathcal{G}} \in X_{\mathcal{G}}} \max_{\psi_{\mathcal{G}} \in \mathcal{G}} S[\mathcal{I} \otimes (\phi \circ \xi_{\mathcal{G}})(\psi_{\mathcal{G}}) \| \mathcal{I} \otimes (\phi'_{\mathcal{G}} \circ \xi_{\mathcal{G}})(\psi_{\mathcal{G}})] \leqslant \min_{\phi'_{\mathcal{G}} \in X_{\mathcal{G}}} \max_{\rho_{\mathcal{G}} \in \mathcal{G}} S[(\mathcal{I} \otimes \phi)(\rho_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi'_{\mathcal{G}})(\rho_{\mathcal{G}})] = D_{\mathcal{G}}[\phi]. \tag{G8}$$

The first inequality is due to limiting to minimization over $\phi_{\mathcal{G}}$ that can be written as $\phi'_{\mathcal{G}} \circ \xi_{\mathcal{G}}$; and the second inequality is due to $(\mathcal{I} \otimes \xi_{\mathcal{G}})(\psi_{\mathcal{G}}) \in \mathcal{G}$.

D7 $D_{\mathcal{G}}[\phi_1 \otimes \phi_2] \geqslant \max(D_{\mathcal{G}}[\phi_1], D_{\mathcal{G}}[\phi_2])$.

Proof. $D_{\mathcal{G}}[\phi_1 \otimes \phi_2]$

$$= \min_{\phi_{\mathcal{G}} \in X_{\mathcal{G}}} \max_{\rho_{\mathcal{G}} \in \mathcal{G}} S[(\mathcal{I} \otimes \phi_1 \otimes \phi_2)(\rho_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi_{\mathcal{G}})(\rho_{\mathcal{G}})] \geqslant \min_{\phi_{\mathcal{G}} \in X_{\mathcal{G}}} \max_{\rho'_{\mathcal{G}} \in \mathcal{G}} S[\mathcal{I} \otimes \phi_1 \otimes \phi_2(\rho'_{\mathcal{G}} \otimes \sigma) \| \mathcal{I} \otimes \phi_{\mathcal{G}}(\rho'_{\mathcal{G}} \otimes \sigma)]$$

$$\geqslant \min_{\phi'_{\mathcal{G}} \in X_{\mathcal{G}}} \max_{\rho'_{\mathcal{G}} \in \mathcal{G}} S[(\mathcal{I} \otimes \phi_1)(\rho'_{\mathcal{G}}) \| (\mathcal{I} \otimes \phi'_{\mathcal{G}})(\rho'_{\mathcal{G}})] = D_{\mathcal{G}}[\phi_1]. \tag{G9}$$

The first inequality is due to limiting to maximization over $\rho_{\mathcal{G}}$ that has a product form $\rho_{\mathcal{G}} \otimes \sigma$, where $\sigma \in \mathcal{G}[n_{\phi_2}]$ is fixed. The second inequality is by taking a trace over the input to ϕ_2 and that $\phi'_{\mathcal{G}} \equiv \mathrm{Tr}_2 \circ \phi_{\mathcal{G}}$ is a Gaussian channel. Similarly, one can prove $D_{\mathcal{G}}[\phi_1 \otimes \phi_2] \geqslant D_{\mathcal{G}}[\phi_2]$.

APPENDIX H: COVARIANCE MATRIX AND CORRELATIONS

The 4×4 covariance matrix Λ of a two-mode (denote them as A and B) quantum state ρ can be obtained as follows. Note that $\Lambda = \Lambda^T$. The first diagonal block is given by

$$\Lambda(1,1) = 2\mathrm{Re}\left\langle a_A^2 \right\rangle_\rho + 2\left\langle a_A^\dagger a_A \right\rangle_\rho + 1 - (2\mathrm{Re}\left\langle a_A \right\rangle_\rho)^2,$$

$$\Lambda(2,2) = -2\mathrm{Re}\left\langle a_A^2 \right\rangle_\rho + 2\left\langle a_A^\dagger a_A \right\rangle_\rho + 1 - (2\mathrm{Im}\left\langle a_A \right\rangle_\rho)^2,$$

$$\Lambda(1,2) = 2\mathrm{Im}\left\langle a_A^2 \right\rangle_\rho - 4\mathrm{Re}\left\langle a_A \right\rangle_\rho \mathrm{Im}\left\langle a_A \right\rangle_\rho.$$

The second diagonal block is given by replacing A with B and $\Lambda(i,j)$ with $\Lambda(i+2, j+2)$ in the above equations.

The cross terms are given as follows:

$$\Lambda(1,3) = 2\mathrm{Re}(\langle a_A a_B \rangle_\rho + \langle a_A^\dagger a_B \rangle_\rho) - 4\mathrm{Re}\left\langle a_A \right\rangle_\rho \mathrm{Re}\left\langle a_B \right\rangle_\rho,$$

$$\Lambda(2,4) = 2\mathrm{Re}(\langle a_A^\dagger a_B \rangle_\rho - \langle a_A a_B \rangle_\rho) - 4\mathrm{Im}\left\langle a_A \right\rangle_\rho \mathrm{Im}\left\langle a_B \right\rangle_\rho,$$

$$\Lambda(1,4) = 2\mathrm{Im}(\langle a_A^\dagger a_B \rangle_\rho + \langle a_A a_B \rangle_\rho) - 4\mathrm{Re}\left\langle a_A \right\rangle_\rho \mathrm{Im}\left\langle a_B \right\rangle_\rho,$$

$$\Lambda(2,3) = 2\mathrm{Im}(\langle a_A a_B \rangle_\rho - \langle a_A^\dagger a_B \rangle_\rho) - 4\mathrm{Im}\left\langle a_A \right\rangle_\rho \mathrm{Re}\left\langle a_B \right\rangle_\rho.$$

APPENDIX I: MIXED UNITARY CHANNELS

The binary phase-shift channel ϕ_{BPS} is a probabilistic mixture of Gaussian unitaries. We begin our analysis of it by considering the general case of probabilistic mixing of K Gaussian unitaries $\{U_k, 1 \leqslant k \leqslant K\}$, with probabilities $\{p_k, 1 \leqslant k \leqslant K\}$, i.e.,

$$\phi_{\mathrm{mix}}(\rho) = \sum_{k=1}^{K} U_k \rho U_k^\dagger. \tag{I1}$$

From Definition 4 and Eq. (9), with $\rho_{AB} = \mathcal{I}_{\phi_{\mathrm{mix}}} \otimes \phi_{\mathrm{mix}}(\psi_{AA'}) = \sum_{k=1}^{K} p_k U_k \psi_{AA'} U_k^\dagger$, we have $0 \leqslant S(\rho_{AB}) \leqslant h(\{p_k\}) \equiv -\sum_{k=1}^{K} p_k \log_2 p_k$. Let $S_{\mathcal{G}}^{\max} = \max_{\rho_{A'} \in \mathcal{G}} S[\lambda_{\mathcal{G}}(\rho_{AB})]$. We have,

$$\tilde{\delta}_{\mathcal{G}}[\phi_{\mathrm{mix}}] = \max_{\rho_{A'} \in \mathcal{G}} S[\lambda_{\mathcal{G}}(\rho_{AB})] - S(\rho_{AB}) \tag{I2}$$

$$\in [S_{\mathcal{G}}^{\max} - h(\{p_k\}), S_{\mathcal{G}}^{\max}]. \tag{I3}$$

Because $h(\{p_k\})$ is finite, if one can show that either $S_{\mathcal{G}}^{\max}$ or $\tilde{\delta}_{\mathcal{G}}[\phi_{\mathrm{mix}}]$ diverges, then the rate of divergence of $\tilde{\delta}_{\mathcal{G}}[\phi_{\mathrm{mix}}]$ is the same with $S_{\mathcal{G}}^{\max}$.

For the case of ϕ_{BPS}, we have $h(\{p_k\}) = 1$ and when the output and ancilla have total energy N_S, $S_{\mathcal{G}}^{\max} = 2g(N_S/2)$ is achieved by input-ancilla in a TMSV.

[1] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[2] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer Science & Business Media, Berlin, 2007).

[3] F. Grosshans and P. Grangier, Continuous Variable Quantum Cryptography Using Coherent States, Phys. Rev. Lett. **88**, 057902 (2002).

[4] R. García-Patrón and N. J. Cerf, Unconditional Optimality of Gaussian Attacks Against Continuous-Variable Quantum Key Distribution, Phys. Rev. Lett. **97**, 190503 (2006).

[5] Q. Zhuang, E. Y. Zhu, and P. W. Shor, Additive Classical Capacity of Quantum Channels Assisted by Noisy Entanglement, Phys. Rev. Lett. **118**, 200503 (2017).

[6] J. Eisert, S. Scheel, and M. B. Plenio, Distilling Gaussian States with Gaussian Operations is Impossible, Phys. Rev. Lett. **89**, 137903 (2002).

[7] G. Giedke and J. I. Cirac, Characterization of Gaussian operations and distillation of Gaussian states, Phys. Rev. A **66**, 032316 (2002).

[8] J. Fiurášek, Gaussian Transformations and Distillation of Entangled Gaussian States, Phys. Rev. Lett. **89**, 137904 (2002).

[9] S. L. Zhang and P. van Loock, Distillation of mixed-state continuous-variable entanglement by photon subtraction, Phys. Rev. A **82**, 062316 (2010).

[10] J. Niset, J. Fiurášek, and N. J. Cerf, No-Go Theorem for Gaussian Quantum Error Correction, Phys. Rev. Lett. **102**, 120501 (2009).

[11] N. J. Cerf, O. Krüger, P. Navez, R. F. Werner, and M. M. Wolf, Non-Gaussian Cloning of Quantum Coherent States is Optimal, Phys. Rev. Lett. **95**, 070501 (2005).

[12] S. Lloyd and S. L. Braunstein, Quantum Computation Over Continuous Variables, Phys. Rev. Lett. **82**, 1784 (1999).

[13] S. D. Bartlett and B. C. Sanders, Universal continuous-variable quantum computation: Requirement of optical nonlinearity for photon counting, Phys. Rev. A **65**, 042304 (2002).

[14] M. Ohliger, K. Kieling, and J. Eisert, Limitations of quantum computing with Gaussian cluster states, Phys. Rev. A **82**, 042336 (2010).

[15] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Universal Quantum Computation with Continuous-Variable Cluster States, Phys. Rev. Lett. **97**, 110501 (2006).

[16] L. Lami, B. Regula, X. Wang, R. Nichols, A. Winter, and G. Adesso, Gaussian quantum resource theories, arXiv:1801.05450.

[17] C. Navarrete-Benlloch, R. García-Patrón, J. H Shapiro, and N. J. Cerf, Enhancing quantum entanglement by photon addition and subtraction, Phys. Rev. A **86**, 012328 (2012).

[18] T. Opatrný, G. Kurizki, and D.-G. Welsch, Improvement on teleportation of continuous variables by photon subtraction via conditional measurement, Phys. Rev. A **61**, 032302 (2000).

[19] P. T. Cochrane, T. C. Ralph, and G. J. Milburn, Teleportation improvement by conditional measurements on the two-mode squeezed vacuum, Phys. Rev. A **65**, 062306 (2002).

[20] S. Olivares, M. G. A. Paris, and R. Bonifacio, Teleportation improvement by inconclusive photon subtraction, Phys. Rev. A **67**, 032314 (2003).

[21] B. C. Sanders, Quantum dynamics of the nonlinear rotator and the effects of continual spin measurement, Phys. Rev. A **40**, 2417 (1989).

[22] A. Ourjoumtsev, R. Tualle-Brouri, J. Laurat, and P. Grangier, Generating optical schrödinger kittens for quantum inf. process. Science **312**, 83 (2006).

[23] A. Ourjoumtsev, H. Jeong, R. Tualle-Brouri, and P. Grangier, Generation of optical 'schrödinger cats' from photon number states, Nature (London) **448**, 784 (2007).

[24] V. Parigi, A. Zavatta, M. Kim, and M. Bellini, Probing quantum commutation rules by addition and subtraction of single photons to/from a light field, Science **317**, 1890 (2007).

[25] J. Fiurášek, Engineering quantum operations on traveling light beams by multiple photon addition and subtraction, Phys. Rev. A **80**, 053822 (2009).

[26] P. Marek, H. Jeong, and M. S. Kim, Generating "squeezed" superpositions of coherent states using photon addition and subtraction, Phys. Rev. A **78**, 063811 (2008).

[27] A. Kitagawa, M. Takeoka, M. Sasaki, and A. Chefles, Entanglement evaluation of non-Gaussian states generated by photon subtraction from squeezed states, Phys. Rev. A **73**, 042310 (2006).

[28] N. Namekata, Y. Takahashi, G. Fujii, D. Fukuda, S. Kurimura, and S. Inoue, Non-Gaussian operation based on photon subtraction using a photon-number-resolving detector at a telecommunications wavelength, Nature Photonics **4**, 655 (2010).

[29] J. Fiurášek, R. García-Patrón, and N. J. Cerf, Conditional generation of arbitrary single-mode quantum states of light by repeated photon subtractions, Phys. Rev. A **72**, 033822 (2005).

[30] K. Wakui, H. Takahashi, A. Furusawa, and M. Sasaki, Photon subtracted squeezed states generated with periodically poled ktiopo 4, Opt. Express **15**, 3568 (2007).

[31] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Phys. Rev. A **64**, 012310 (2001).

[32] K. Nemoto and W. J. Munro, A Near Deterministic Linear Optical Cnot Gate, Phys. Rev. Lett. **93**, 250502 (2004).

[33] Q. Zhuang, Z. Zhang, and J. H Shapiro, Optimum Mixed-State Discrimination for Noisy Entanglement-Enhanced Sensing, Phys. Rev. Lett. **118**, 040801 (2017).

[34] K. K. Sabapathy and A. Winter, Non-gaussian operations on bosonic modes of light: Photon-added Gaussian channels, Phys. Rev. A **95**, 062309 (2017).

[35] T. J. Volkoff, Linear bosonic quantum channels defined by superpositions of maximally distinguishable Gaussian environments, Quantum Inf. Comput. **18**, 0481 (2018).

[36] J. Wenger, R. Tualle-Brouri, and P. Grangier, Non-Gaussian Statistics from Individual Pulses of Squeezed Light, Phys. Rev. Lett. **92**, 153601 (2004).

[37] F. G. S. L. Brandão and G. Gour, Reversible Framework for Quantum Resource Theories, Phys. Rev. Lett. **115**, 070503 (2015).

[38] A. Streltsov, G. Adesso, and M. B. Plenio, Colloquium: Quantum coherence as a resource, Rev. Mod. Phys. **89**, 041003 (2017).

[39] A. Winter and D. Yang, Operational Resource Theory of Coherence, Phys. Rev. Lett. **116**, 120404 (2016).

[40] T. Theurer, N. Killoran, D. Egloff, and M. B. Plenio, Resource Theory of Superposition, Phys. Rev. Lett. **119**, 230401 (2017).

[41] F. G. S. L. Brandao, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource Theory of Quantum States Out of Thermal Equilibrium, Phys. Rev. Lett. **111**, 250404 (2013).

[42] M. Horodecki and J. Oppenheim, Fundamental limitations for quantum and nanoscale thermodynamics, Nature Commun. **4**, 2059 (2013).

[43] G. Gour and R. W. Spekkens, The resource theory of quantum reference frames: Manipulations and monotones, New J. Phys. **10**, 033023 (2008).
[44] P. Marian and T. A. Marian, Relative entropy is an exact measure of non-Gaussianity, Phys. Rev. A **88**, 012322 (2013).
[45] M. G. Genoni, M. G. A. Paris, and K. Banaszek, Quantifying the non-Gaussian character of a quantum state by quantum relative entropy, Phys. Rev. A **78**, 060303 (2008).
[46] M. G. Genoni and M. G. A. Paris, Quantifying non-Gaussianity for quantum information, Phys. Rev. A **82**, 052341 (2010).
[47] M. A. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
[48] M. B. Ruskai, Inequalities for quantum entropy: A review with conditions for equality, J. Math. Phys. **43**, 4358 (2002).
[49] A. Mani and V. Karimipour, Cohering and decohering power of quantum channels, Phys. Rev. A **92**, 032331 (2015).
[50] P. Zanardi, G. Styliaris, and L. C. Venuti, Measures of coherence-generating power for quantum unital operations, Phys. Rev. A **95**, 052307 (2017).
[51] P. Zanardi, G. Styliaris, and L. C. Venuti, Coherence-generating power of quantum unitary maps and beyond, Phys. Rev. A **95**, 052306 (2017).
[52] K. Bu, A. Kumar, L. Zhang, and J. Wu, Cohering power of quantum operations, Phys. Lett. A **381**, 1670 (2017).
[53] K. B. Dana, M. G. Díaz, M. Mejatty, and A. Winter, Resource theory of coherence: Beyond states, Phys. Rev. A **95**, 062327 (2017).
[54] C. H. Bennett, A. W. Harrow, D. W. Leung, and J. A. Smolin, On the capacities of bipartite hamiltonians and unitary gates, IEEE Trans. Inf. Theory **49**, 1895 (2003).
[55] M. S. Leifer, L. Henderson, and N. Linden, Optimal entanglement generation from quantum operations, Phys. Rev. A **67**, 012306 (2003).
[56] M. Navascués and L. P. García-Pintos, Nonthermal Quantum Channels as a Thermodynamical Resource, Phys. Rev. Lett. **115**, 010405 (2015).
[57] S. Huber and R. König, Coherent state coding approaches the capacity of non-Gaussian bosonic channels, J. Phys. A: Math. Theor. **51**, 184001 (2018).
[58] A Barchielli, Stochastic differential equations anda posteriori states in quantum mechanics, Int. J. Theor. Phys. **32**, 2221 (1993).
[59] H. M. Wiseman and G. J. Milburn, Quantum Theory of Optical Feedback via Homodyne Detection, Phys. Rev. Lett. **70**, 548 (1993).
[60] D. A. R. Dalvit, J. Dziarmaga, and W. H. Zurek, Unconditional Pointer States from Conditional Master Equations, Phys. Rev. Lett. **86**, 373 (2001).
[61] J. Dziarmaga, D. A. R. Dalvit, and W. H. Zurek, Conditional quantum dynamics with several observers, Phys. Rev. A **69**, 022109 (2004).
[62] M. G. Genoni, L. Lami, and A. Serafini, Conditional and unconditional Gaussian quantum dynamics, Contemp. Phys. **57**, 331 (2016).
[63] G. De Palma, A. Mari, V. Giovannetti, and A. S. Holevo, Normal form decomposition for Gaussian-to-Gaussian superoperators, J. Math. Phys. **56**, 052202 (2015).
[64] G. De Palma, Gaussian optimizers and other topics in quantum information, arXiv:1710.09395.
[65] G. Adesso, Entanglement of Gaussian states, arXiv:quant-ph/0702069.
[66] Z.-W. Liu, X. Hu, and S. Lloyd, Resource Destroying Maps, Phys. Rev. Lett. **118**, 060502 (2017).
[67] L. Zhang, J. Wu, and S.-M. Fei, Comment on "convergence of macrostates under reproducible processes", Phys. Lett. A **377**, 1794 (2013); **374**, 3715 (2010).
[68] A. Y. Kitaev, A. Shen, and M. N. Vyalyi, *Classical and Quantum Computation* (American Mathematical Society, Providence, 2002), Vol. 47.
[69] S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, 2004).
[70] M. S. Kim, H. Jeong, A. Zavatta, V. Parigi, and M. Bellini, Scheme for Proving the Bosonic Commutation Relation Using Single-Photon Interference, Phys. Rev. Lett. **101**, 260401 (2008).
[71] M. G. Genoni, M. G. A. Paris, and K. Banaszek, Measure of the non-Gaussian character of a quantum state, Phys. Rev. A **76**, 042327 (2007).
[72] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature? Phys. Rev. Lett. **80**, 5239 (1998).
[73] R. Takagi and Q. Zhuang, Convex resource theory of non-Gaussianity, arXiv:1804.04669v1.
[74] F. Albarelli, M. G. Genoni, M. G. A. Paris, and A. Ferraro, Resource theory of quantum non-Gaussianity and Wigner negativity, arXiv:1804.05763v1.
[75] Á. Capel, A. Lucia, and D. Pérez-García, Superadditivity of quantum relative entropy for general states, arXiv:1705.03521.
[76] J. Watrous, Advanced Topics in Quantum Information Processing. Lecture Notes, Chap. 20 (https://cs.uwaterloo.ca/watrous/LectureNotes.html, 2011).