Laser-modified Coulomb scattering states of an electron in the parabolic quasi-Sturmian-Floquet approach

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Electron scattering states in combined Coulomb and laser fields are investigated with a nonperturbative approach based on the Hermitian Floquet theory. Taking into account the Coulomb-specific asymptotic behavior of the electron wave functions at large distances, a Lippmann-Schwinger-Floquet equation is derived in the Kramers-Henneberger frame. Such a scattering-state equation is solved numerically employing a set of parabolic quasi-Sturmian functions which have the great advantage of possessing, by construction, adequately chosen incoming or outgoing Coulomb asymptotic behaviors. Our quasi-Sturmian-Floquet approach is tested with a calculation of triple differential cross sections for a laser-assisted $(e,2e)$ process on atomic hydrogen within a first-order Born treatment of the projectile-atom interaction. Convergence with respect to the number of Floquet-Fourier expansion terms is numerically demonstrated. The illustration shows that the developed method is very efficient for the computation of light-dressed states of an electron moving in a Coulomb potential in the presence of laser radiation.

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I. INTRODUCTION

The electron scattering states in a Coulomb field are encountered in the theoretical treatments of various phenomena, such as the electron bremsstrahlung in the electric field of an atomic nucleus, atomic photoionization and electron-ion photorecombination, ionization of atoms under charged particle impact, etc. In nonrelativistic quantum mechanics, these states are given by the solutions, with asymptotic outgoing or incoming spherical wave behavior, of the Schrödinger equation with a Coulomb potential. Such solutions are well known (see, e.g., the textbook [1]) and are usually referred to as the Coulomb wave functions or, simply, the Coulomb waves. In the last few decades, the availability of powerful lasers and their application to the study of multiphoton processes have greatly stimulated theoretical investigations of the above-mentioned radiation and ionization phenomena in the presence of intense laser fields. In contrast to the pure Coulomb case, when an electron moves in a Coulomb potential in the presence of a laser electric field, the corresponding Schrödinger equation does not possess closed-form solutions anymore. As a consequence, for laser-modified Coulomb scattering states, one has then to resort either to analytical approximations or to numerical approaches. The most frequently employed approximations are (i) the strong field approximation [2], whereby the Coulomb potential is neglected in the Schrödinger equation, and the electron states are described by the Volkov functions [3]; (ii) the Bunkin-Fedorov approach [4], which treats the Coulomb potential perturbatively; and (iii) the Coulomb-Volkov states [5,6], where both the Coulomb and the laser fields are treated nonperturbatively (see also Ref. [7] for testing the accuracy of the Coulomb-Volkov approximation). Concerning approaches which are purely numerical and free of approximations, we find the treatments of the laser-assisted Coulomb scattering using the R-matrix-Floquet theory [8] and the close-coupling method [9–11] in the accelerated, or Kramers-Henneberger (KH), frame [12].

Our aim is to develop an alternative, nonperturbative approach that is ultimately able to describe efficiently electron scattering states in radiation-assisted processes. In the present contribution, in particular, we present a way of calculating laser-modified Coulomb wave functions within the Hermitian Floquet theory (see, e.g., Ref. [13]). The originality of our approach stands on using recently introduced set of functions [14] (here taken in parabolic coordinates) and on taking advantage of the following observation. When working within the KH frame, a laser field is effectively absent, and an electron moves in a Coulomb potential of a nucleus that oscillates in time (the oscillations are equivalent to those of a classical free electron in a laser field in the laboratory frame). At large distances from the nucleus, the role of the nuclear oscillating motion vanishes: the electron experiences therefore a usual, time-independent, Coulomb-tail force and, accordingly, the leading asymptotic behavior of the electron wave function is represented by a Coulomb-distorted plane wave. We exploit this fact to recast the system of (time-independent) coupled equations formulated for Floquet components [15]; the

original matrix equation for the Floquet state vector becomes a driven equation with an inhomogeneity which incorporates the Coulomb wave. The solution of the driven equation, subject to outgoing or incoming boundary conditions, can be obtained by applying, for example, the standard Sturmian-Floquet method [13,16–19]. In our approach, we propose to solve the equation in parabolic coordinates (with the z axis being parallel to the electron momentum) by expanding the Floquet components in terms of so-called quasi-Sturmian (QS) functions [14] with appropriate outgoing or incoming Coulomb asymptotic behavior. The parabolic QS functions satisfy inhomogeneous Schrödinger-type equations with a Coulomb potential and a driven term consisting of square-integrable Laguerre basis functions; they are constructed in closed form using an integral representation of the Coulomb Green's function in parabolic coordinates [20,21].

In order to validate our approach we needed to examine the convergence of the numerical scattering solutions as both the number of terms in the Floquet-Fourier expansion and the size of the QS basis are increased. Once these tests were performed, we wished to illustrate its efficiency by applying the method to a concrete example. Here we considered the laser-assisted $(e,2e)$ collision on atomic hydrogen:

$$\ell\omega + e^- + \mathrm{H}(1s) \to \mathrm{H}^+ + 2e^-, \tag{1}$$

in which a net number ℓ of photons of frequency ω can be exchanged between the colliding system and the external field while the ionization process takes place. The measurable quantity that characterizes an $(e,2e)$ reaction is the triple differential cross section (TDCS), which is differential in the solid angles of the two emitted electrons and in the energy of one of them. To calculate it one needs to describe both the ejected electron and the initial laser-dressed target wave functions. The latter is often obtained from a perturbation treatment (see, e.g., Ref. [22]). Here, we get it in the KH frame through the diagonalization of the Hamiltonian matrix in the basis of L^2 Laguerre functions. Specifically, we identify this state with an eigenfunction whose zeroth Floquet component is largest in magnitude. Hence, this eigenstate may be regarded as a generalization of the so-called KH state predicted within the high-frequency Floquet theory [23] (see also Ref. [24] and references therein). The main challenge to describing the ionization process (1) is to deal properly with the (slow) ejected electron. As a starting point, it can be described by either Volkov [25] or Coulomb-Volkov states [26,27], or approximated using perturbation theory [28,29]. In the case of intense laser fields, however, one must go beyond perturbative and approximate approaches: this constitutes the main motivation of the present contribution. Within our nonperturbative formulation, we essentially recast the original Floquet matrix method and solve a driven equation using parabolic QS functions. The proposed treatment of the ejected-electron state can indeed be done as illustrated through convergent numerical calculations of a laser-assisted $(e,2e)$ reaction on atomic hydrogen.

The paper is organized as follows. In Sec. II, we present the general matrix equation for the Floquet components of the electron wave function in the KH frame. The scattering-state problem is formulated as a Lippmann-Schwinger-type integral equation. Section III is dedicated to its solutions. We start by looking at the matrix equations corresponding to an expansion on the basis set of square-integrable Laguerre functions in parabolic coordinates. We then propose an alternative approach whereby the Lippmann-Schwinger-type integral equation is recast as a driven equation with purely incoming or outgoing boundary conditions. The scattering-state solution is sought using an expansion in terms of parabolic QS functions possessing adequate Coulomb incoming or outgoing boundary conditions. In Sec. IV, we present the results of some numerical calculations in the context of the laser-assisted ionization process on atomic hydrogen. We study the convergence behavior of the scattering-state solution [representing, e.g., the ejected electron in reaction (1)] as a function of the number both of QS basis functions and of Floquet components. To compute the initial dressed hydrogen state, we first retain only the zeroth component in the Floquet-Fourier expansion and find the KH state, i.e., the lowest state for the time average of the oscillating electron-nucleus interaction in the KH frame. Then we successively increase the number of Floquet components, thereby obtaining a sequence of eigenstates originating from the KH state. Convergence of this sequence of generalized KH states is demonstrated numerically. Finally, in order to illustrate the applicability of the developed approach, laser-assisted $(e,2e)$ triple differential cross sections in the first Born approximation are computed. Section V summarizes this work. Atomic units (a.u., $\hbar = e = m_e = 1$) are used throughout unless otherwise specified.

II. FLOQUET THEORY IN THE KRAMERS-HENNEBERGER FRAME

We consider electron scattering states in a Coulomb field in the presence of linearly polarized monochromatic laser radiation with the vector potential

$$\mathbf{A}(t) = \mathbf{A}_0 \cos\omega t. \tag{2}$$

The laser field of frequency ω is assumed to switch on adiabatically at $t = -\infty$.

The electron dynamics is governed by the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[\frac{1}{2}\left(-i\nabla + \frac{1}{c}\mathbf{A}(t)\right)^2 + V(\mathbf{r})\right]\Psi(\mathbf{r},t), \tag{3}$$

with the electron-nucleus Coulomb potential $V(\mathbf{r}) = -Z/r$ (the nucleus of charge $Z > 0$ is assumed to be infinitely heavy as compared to the electron mass). In the accelerated, or space-translated, KH frame [12], Eq. (3) is transformed into

$$i\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left(-\frac{1}{2}\Delta + V[\mathbf{r} + \mathbf{a}(t)]\right)\psi(\mathbf{r},t), \tag{4}$$

where

$$\psi(\mathbf{r},t) = \exp\left[\mathbf{a}(t)\cdot\nabla + \frac{i}{2c^2}\int_{-\infty}^{t} dt' A^2(t')\right]\Psi(\mathbf{r},t), \tag{5}$$

$$\mathbf{a}(t) = \frac{1}{c}\int_{-\infty}^{t} dt'\,\mathbf{A}(t') = \mathbf{a}_0 \sin\omega t \tag{6}$$

is the displacement vector of a classical electron, with $\mathbf{a}_0 = \mathbf{A}_0/\omega c$ called the quiver amplitude.

A. Floquet equations

Within the Hermitian Floquet theory (see, e.g., [13]), one seeks the solution of Eq. (4) in the form

$$\psi(\mathbf{r},t) = e^{-iEt}\sum_{n=-\infty}^{\infty} e^{-in\omega t} F_n(E,\mathbf{r}), \tag{7}$$

where E is a real quantity called the Floquet quasienergy of the state. The harmonic components F_n satisfy an infinite set of time-independent equations:

$$(H_n + \widetilde{V}_0(\mathbf{a}_0,\mathbf{r}) - E)F_n(E,\mathbf{r}) + \sum_{\nu\neq n} V_{n-\nu}(\mathbf{a}_0,\mathbf{r}) \times F_\nu(E,\mathbf{r}) = 0, \quad n = 0, \pm1, \pm2, \ldots \tag{8}$$

Here

$$H_n = H_C - n\omega, \quad H_C = -\frac{1}{2}\Delta - \frac{Z}{r},$$
$$\widetilde{V}_0(\mathbf{a}_0,\mathbf{r}) = V_0(\mathbf{a}_0,\mathbf{r}) + \frac{Z}{r}, \tag{9}$$

and V_n are the Fourier components of the space-translated Coulomb potential $V[\mathbf{r}+\mathbf{a}(t)]$,

$$V_n(\mathbf{a}_0,\mathbf{r}) = \frac{1}{T}\int_0^T dt e^{in\omega t}\, V[\mathbf{r}+\mathbf{a}(t)], \tag{10}$$

with $T = 2\pi/\omega$ being the optical period.

Equation (8) can be written in the matrix form

$$(\mathbb{H} + \mathbb{V} - E\mathbb{I})\mathbb{F} = 0, \tag{11}$$

where $\mathbb{F}$ is the infinite Floquet vector

$$\mathbb{F} = \begin{pmatrix} \vdots \\ F_{-n} \\ \vdots \\ F_{-1} \\ F_0 \\ F_1 \\ \vdots \\ F_n \\ \vdots \end{pmatrix}, \tag{12}$$

$\mathbb{H}$ is the infinite diagonal matrix

$$\mathbb{H} = \begin{pmatrix} \ddots & & & & & & & \\ & H_{-n} & & & & & & \\ & & \ddots & & & & & \\ & & & H_{-1} & & & & \\ & & & & H_0 & & & \\ & & & & & H_1 & & \\ & & & & & & \ddots & \\ & & & & & & & H_n \\ & & & & & & & & \ddots \end{pmatrix}, \tag{13}$$

$\mathbb{I}$ is the infinite identity matrix, and $\mathbb{V}$ is the Hermitian matrix

$$\mathbb{V} = \begin{pmatrix} \ddots & & & & & & & & \\ & \widetilde{V}_0 & V_{-1} & \ldots & V_{-n} & & & & \\ & V_1 & \ddots & \ddots & \ddots & \ddots & & & \\ & \vdots & \ddots & \widetilde{V}_0 & V_{-1} & \ddots & \ddots & & \\ & V_n & \ddots & V_1 & \widetilde{V}_0 & V_{-1} & \ddots & V_{-n} & \\ & & \ddots & \ddots & V_1 & \widetilde{V}_0 & \ddots & \vdots & \\ & & & \ddots & \ddots & \ddots & \ddots & V_{-1} & \\ & & & & V_n & \ldots & V_1 & \widetilde{V}_0 & \\ & & & & & & & & \ddots \end{pmatrix}, \tag{14}$$

whose nth subdiagonal contains V_n.

B. Scattering states

In the field-free case, for an electron with momentum $\mathbf{k}$, the Coulomb scattering states are given by the well-known outgoing (+) or incoming (−) waves [30]:

$$\psi_C^{(\pm)}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}\pi\beta}\Gamma(1\pm i\beta)e^{i\mathbf{kr}}{}_1F_1(\mp i\beta,1;\pm i(kr\mp\mathbf{kr})), \tag{15}$$

where $k=\sqrt{2E}$, and the Sommerfeld parameter $\beta=-Z/k$ measures the strength of the Coulomb field. At asymptotically large distances the Coulomb waves behave as

$$\psi_C^{(\pm)}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \frac{1}{(2\pi)^{3/2}}\left\{\exp\left[i\mathbf{kr}\pm i\beta\ln(kr\mp\mathbf{kr})\right] + f_C^{(\pm)}(E,\hat{\mathbf{r}})\frac{\exp\left[\pm ikr\mp i\beta\ln(2kr)\right]}{r}\right\}, \tag{16}$$

where $\hat{\mathbf{r}}=\mathbf{r}/r$, $f_C^{(+)}$ is the Coulomb scattering amplitude [30], and $f_C^{(-)}=[f_C^{(+)}]^*$.

In the presence of the laser field [Eq. (2)], the electron scattering states are given by the solutions of the Floquet equations (8) with proper $r\to\infty$ boundary conditions which account for the long-range effect of the potential $V[\mathbf{r}+\mathbf{a}(t)]$. Such conditions can be formulated making use of the properties

$$\psi(\mathbf{r},t) \underset{t\to-\infty}{\longrightarrow} e^{-iEt}\psi_C^{(\pm)}(\mathbf{r}), \quad V_n(\mathbf{a}_0,\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \delta_{n0}V(\mathbf{r}).$$

From the latter it follows that the Floquet channels ($n=0,\pm1,\pm2,\ldots$) asymptotically decouple and, hence, one obtains the outgoing and incoming boundary conditions in the form [15]

$$F_n^{(\pm)}(E,\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \frac{1}{(2\pi)^{3/2}}\left\{\delta_{n0}\exp\left[i\mathbf{kr}\pm i\beta\ln(kr\mp\mathbf{kr})\right] + f_n^{(\pm)}(E,\hat{\mathbf{r}})\frac{\exp\left[\pm ik_nr\mp i\beta_n\ln(2k_nr)\right]}{r}\right\}, \tag{17}$$

where $\beta_n=-Z/k_n$, and

$$k_n=\begin{cases}\sqrt{2(E+n\omega)}, & E+n\omega\geqslant 0,\\ \pm i\sqrt{-2(E+n\omega)}, & E+n\omega<0.\end{cases} \tag{18}$$

In order to satisfy the boundary conditions we recast the matrix equation (11) into a Lippmann-Schwinger equation

$$\mathbb{F}^{(\pm)}=\mathbb{S}^{(\pm)}-\mathbb{G}^{(\pm)}\mathbb{V}\mathbb{F}^{(\pm)}, \tag{19}$$

where

$$\mathbb{S}^{(\pm)}=\begin{pmatrix}\vdots\\0\\\vdots\\0\\\psi_C^{(\pm)}\\0\\\vdots\\0\\\vdots\end{pmatrix}, \tag{20}$$

and $\mathbb{G}^{(\pm)}$ is the diagonal matrix containing the Coulomb Green's functions operators $G_n^{(\pm)}$ whose kernels $G^{(\pm)}(k_n;\mathbf{r},\mathbf{r}')$ satisfy the equation

$$\left(-\frac{1}{2}\Delta-\frac{Z}{r}-\frac{k_n^2}{2}\right)G^{(\pm)}(k_n;\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \tag{21}$$

Since the Fourier components V_n decrease asymptotically as $\sim 1/r^{|n|+1}$, the elements of the matrix $\mathbb{V}$ vanish as fast as $1/r^2$ or faster when $r\to\infty$: the kernel of the integral equation (19) is therefore compact.

Directing the z axis along the electron momentum $\mathbf{k}$ and introducing the parabolic coordinates $(\xi,\ \eta,\ \phi)$ through the relations $x=\sqrt{\xi\eta}\cos\phi$, $y=\sqrt{\xi\eta}\sin\phi$, and $z=(\xi-\eta)/2$, Green's function can be represented by the expansion

$$G^{(\pm)}(k;\mathbf{r},\mathbf{r}')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\mathcal{G}^{|m|(\pm)}(k;\xi,\eta;\xi',\eta'). \tag{22}$$

The partial components $\mathcal{G}^{|m|(\pm)}$ obey the equation

$$\begin{aligned}&\frac{1}{2}\left[-\left(\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\eta\frac{\partial}{\partial\eta}\right)+k\beta\right.\\&\quad\left.+\frac{m^2}{4\xi}+\frac{m^2}{4\eta}-(\xi+\eta)\frac{k^2}{4}\right]\mathcal{G}^{|m|(\pm)}(k;\ \xi,\eta;\ \xi',\eta')\\&\quad=\delta(\xi-\xi')\delta(\eta-\eta'),\end{aligned} \tag{23}$$

and possess the following integral representation [21]

$$\begin{aligned}&\mathcal{G}^{|m|(\pm)}(k;\ \xi,\eta;\ \xi',\eta')\\&\quad=\mp ik\int_0^\infty ds\ \sinh s\left(\coth\frac{s}{2}\right)^{\mp2i\beta}e^{\pm i\frac{k}{2}(\xi+\xi'+\eta+\eta')\cosh s}\\&\qquad\times I_{|m|}(\mp ik\sqrt{\xi\xi'}\sinh s)I_{|m|}(\mp ik\sqrt{\eta\eta'}\sinh s),\end{aligned} \tag{24}$$

where $I_{|m|}$ are the modified Bessel functions [31] [Eq. (24) is slightly different from that in Ref. [20]].

III. QUASI-STURMIAN APPROACH

In this section we present our method for solving the Floquet-Lippmann-Schwinger equation (19). We first outline it by considering square-integrable Coulomb-Sturmian basis functions in parabolic coordinates. Then, we formulate the driven matrix equation to be solved with a basis set of parabolic quasi-Sturmian functions.

A. Square-integrable Coulomb-Sturmian basis functions

Consider the scattering-state Floquet components as the expansions

$$F_n^{(\pm)} = \sum_{m=-\infty}^{\infty} \sum_{n_1,n_2=0}^{\infty} C_{n_1 n_2 m}^{n(\pm)} |n_1 n_2 m\rangle, \tag{25}$$

with

$$|n_1 n_2 m\rangle = \frac{e^{im\phi}}{\sqrt{2\pi}} \varphi_{n_1}^{|m|}(\xi)\, \varphi_{n_2}^{|m|}(\eta), \tag{26}$$

where the square-integrable functions $\varphi_n^{|m|}$ are defined in terms of the associated Laguerre polynomials $L_n^{|m|}$ [31],

$$\varphi_n^{|m|}(\rho) = \sqrt{\frac{2b\Gamma(n+1)}{\Gamma(n+1+|m|)}}\, (2b\rho)^{|m|/2} e^{-b\rho} L_n^{|m|}(2b\rho). \tag{27}$$

Here b is the basis scale parameter. The basis functions $|n_1 n_2 m\rangle$ are orthogonal to

$$\widetilde{|n_1 n_2 m\rangle} = \frac{4}{(\xi+\eta)} |n_1 n_2 m\rangle, \tag{28}$$

i.e., $\langle \widetilde{n_1' n_2' m'} | n_1 n_2 m\rangle = \delta_{n_1' n_1} \delta_{n_2' n_2} \delta_{m' m}$.

A method for solving a Lippmann-Schwinger equation in the context of square-integrable functions can be found elsewhere (see, e.g., [32]). Specifically, in order to write out a matrix equation for the coefficients $C_{n_1 n_2 m}^{n(\pm)}$ of expansion (25), one uses the formal basis set (26) representations of the potentials V_n and Green's operators $G_n^{(\pm)}$:

$$\begin{aligned} V_n &= \sum_{n_1,n_2,m} \sum_{n_1',n_2',m'} \widetilde{|n_1 n_2 m\rangle} \langle n_1 n_2 m | V_n | n_1' n_2' m' \rangle \widetilde{\langle n_1' n_2' m'|} \\ &= \sum_{n_1,n_2,m} \sum_{n_1',n_2',m'} [V]^n_{n_1 n_2 m; n_1' n_2' m'} \widetilde{|n_1 n_2 m\rangle} \widetilde{\langle n_1' n_2' m'|}, \end{aligned} \tag{29}$$

$$\begin{aligned} G_n^{(\pm)} &= \sum_{n_1,n_2,m} \sum_{n_1',n_2',m'} |n_1 n_2 m\rangle \widetilde{\langle n_1 n_2 m|} G_n^{(\pm)} \widetilde{|n_1' n_2' m'\rangle} \langle n_1' n_2' m'| \\ &= \sum_{n_1,n_2,m} \sum_{n_1',n_2',m'} [G^{(\pm)}]^n_{n_1 n_2 m; n_1' n_2' m'} |n_1 n_2 m\rangle \langle n_1' n_2' m'|. \end{aligned} \tag{30}$$

Using the two equivalent forms of the unit operator

$$\mathbb{I}_1 = \sum_{n_1,n_2,m} |n_1 n_2 m\rangle \widetilde{\langle n_1 n_2 m|}, \quad \mathbb{I}_2 = \sum_{n_1,n_2,m} \widetilde{|n_1 n_2 m\rangle} \langle n_1 n_2 m|, \tag{31}$$

one can rewrite Eq. (19) as

$$\mathbb{F}^{(\pm)} = \mathbb{S}^{(\pm)} - \mathbb{I}_1 \mathbb{G}^{(\pm)} \mathbb{I}_2 \mathbb{V} \mathbb{I}_1 \mathbb{F}^{(\pm)}. \tag{32}$$

The $n = 0$ components of $\mathbb{S}^{(\pm)}$, which are given by $\psi_C^{(\pm)}$, are expanded as

$$S_0^{(\pm)}(E,\mathbf{r}) = \sum_{n_1,n_2,m} \mathcal{S}_{n_1 n_2 m}^{(\pm)} |n_1 n_2 m\rangle, \tag{33}$$

where the coefficients $\mathcal{S}_{n_1 n_2 m}^{(\pm)} \equiv \widetilde{\langle n_1 n_2 m|} \psi_C^{(\pm)}\rangle$ can be readily derived in terms of the Gauss hypergeometric function ${}_2F_1$ [31]:

$$\mathcal{S}_{n_1 n_2 m}^{(+)} = \frac{\delta_{m0}}{2\pi} e^{-\frac{1}{2}\pi\beta} \zeta^{-i\beta} \Gamma(1+i\beta) \frac{(1+\frac{1}{\zeta})(1+\zeta)}{2b} (-\zeta)^{n_1-n_2} {}_2F_1(-n_2,-i\beta;1;1-\zeta^2), \tag{34}$$

$$\mathcal{S}_{n_1 n_2 m}^{(-)} = \frac{\delta_{m0}}{2\pi} e^{-\frac{1}{2}\pi\beta} \zeta^{-i\beta} \Gamma(1-i\beta) \frac{(1+\frac{1}{\zeta})(1+\zeta)}{2b} (-\zeta)^{n_1-n_2} {}_2F_1(-n_1,i\beta;1;1-\zeta^{-2}). \tag{35}$$

The parameter ζ is defined by

$$\zeta = \frac{b + i\frac{k}{2}}{b - i\frac{k}{2}}. \tag{36}$$

The matrix elements of Green's function,

$$[G^{(\pm)}]^n_{n_1 n_2 m; n_1' n_2' m'} \equiv \delta_{m' m} \mathcal{G}^{|m|(\pm)}_{n_1' n_2'; n_1 n_2}(\beta_n, k_n), \tag{37}$$

where

$$\mathcal{G}^{|m|(\pm)}_{n_1' n_2'; n_1 n_2}(\beta_n, k_n) = \int_0^\infty d\xi \int_0^\infty d\eta \int_0^\infty d\xi' \int_0^\infty d\eta' \varphi_{n_1'}^{|m|}(\xi) \varphi_{n_2'}^{|m|}(\eta)\, \mathcal{G}^{|m|(\pm)}(k_n;\xi,\eta;\xi',\eta') \varphi_{n_1}^{|m|}(\xi') \varphi_{n_2}^{|m|}(\eta') \tag{38}$$

can be expressed as the finite sum

$$\begin{aligned} \mathcal{G}^{|m|(\pm)}_{n_1' n_2'; n_1 n_2}(\beta,k) = \pm\frac{2i}{k}(1-\zeta^{\pm 2})(-\zeta)^{\pm K} \sqrt{\binom{n_1'+|m|}{n_1'}\binom{n_2'+|m|}{n_2'}\binom{n_1+|m|}{n_1}\binom{n_2+|m|}{n_2}} \sum_{\ell=0}^{u+v} c_\ell\, \zeta^{\mp 2\ell} \\ \times \frac{\Gamma(|m|+\ell+1\pm i\beta)\Gamma(K+1-2\ell)}{\Gamma(K+|m|+2-\ell\pm i\beta)} {}_2F_1\big(K+1-2\ell,-|m|-\ell\pm i\beta;K+|m|+2-\ell\pm i\beta;\zeta^{\pm 2}\big), \end{aligned} \tag{39}$$

where $K = n_1' + n_2' + n_1 + n_2$; the coefficients c_ℓ are given by

$$c_\ell = \sum_{j=\max(\ell-v,0)}^{\min(\ell,u)} \frac{\binom{n_1}{j}\binom{n_1'}{j}\binom{n_2}{\ell-j}\binom{n_2'}{\ell-j}}{\binom{j+|m|}{j}\binom{\ell-j+|m|}{\ell-j}},$$

with $u = \min(n_1, n_1')$ and $v = \min(n_2, n_2')$.

Substituting the expansions (29), (30), and (33) into Eq. (32), one obtains for the expansion coefficients of Eq. (25) the following equations:

$$C^{n(\pm)}_{n_1 n_2 m} = \delta_{n0} \mathcal{S}^{(\pm)}_{n_1 n_2 m} - \sum_{n'} \sum_{n_1', n_2', m'} \sum_{n_1'', n_2'', m''} [G^{(\pm)}]^{n}_{n_1 n_2 m; n_1' n_2' m'} [V]^{n-n'}_{n_1' n_2' m'; n_1'' n_2'' m''} C^{n'(\pm)}_{n_1'' n_2'' m''}. \tag{40}$$

B. Parabolic quasi-Sturmian basis functions

An alternative and, in principle, more efficient way consists of representing the solutions of Eq. (19) as expansions on basis functions possessing the expected asymptotic behavior rather than on square-integrable ones (26). Expressing the Floquet components $F_n^{(\pm)}$ in the form

$$F_n^{(\pm)}(E, \mathbf{r}) = \widetilde{F}_n^{(\pm)}(E, \mathbf{r}) + \delta_{n0} \psi_C^{(\pm)}(\mathbf{r}), \tag{41}$$

Eq. (19) can be recast as follows:

$$(\mathbb{H} + \mathbb{V} - E\,\mathbb{I})\widetilde{\mathbb{F}}^{(\pm)} = -\mathbb{V}\mathbb{S}^{(\pm)}, \tag{42}$$

whose solutions exhibit, for $r \to \infty$, outgoing or incoming spherical wave behavior. Thus, they can be expanded in terms of Sturmian functions [13].

Here, our proposal consists of expanding the components $\widetilde{F}_n^{(\pm)}$,

$$\widetilde{F}_n^{(\pm)}(E, \mathbf{r}) = \sum_{m=-\infty}^{\infty} \sum_{n_1, n_2=0}^{\infty} C^{n(\pm)}_{n_1 n_2 m} Q^{(\pm)}_{n_1 n_2 m}(k_n; \xi, \eta, \phi), \tag{43}$$

over parabolic QS functions subject to the appropriate boundary conditions as defined in Ref. [14] through Green's operator $G_n^{(\pm)}$, namely,

$$Q^{(\pm)}_{n_1 n_2 m}(k_n; \xi, \eta, \phi) = G_n^{(\pm)} |\widetilde{n_1 n_2 m}\rangle. \tag{44}$$

One possible representation of these QS functions is in terms of the basis functions (26):

$$Q^{(\pm)}_{n_1 n_2 m}(k; \xi, \eta, \phi) = \sum_{n_1', n_2'=0}^{\infty} \mathcal{G}^{|m|(\pm)}_{n_1' n_2';\, n_1 n_2}(\beta, k) |n_1' n_2' m\rangle. \tag{45}$$

Here, we separate out the ϕ variable

$$Q^{(\pm)}_{n_1 n_2 m}(k; \xi, \eta, \phi) = \frac{e^{im\phi}}{\sqrt{2\pi}} \mathcal{P}^{|m|(\pm)}_{n_1 n_2}(k; \xi, \eta), \tag{46}$$

and use the integral representation (24) to write the "radial" part as

$$\begin{aligned} \mathcal{P}^{|m|(\pm)}_{n_1 n_2}(k; \xi, \eta) = {} & 2(1 + \zeta^{\pm 1}) \sqrt{\frac{n_1! n_2!}{(n_1 + |m|)!(n_2 + |m|)!}} (2b\sqrt{\xi\eta})^{|m|} e^{-b(\xi+\eta)} \\ & \times \int_0^1 ds\, (1-s)^{\pm i\beta + |m|} \left(1 - \zeta^{\pm 1} s\right)^{\mp i\beta + |m|} \left(1 - s - \zeta^{\pm 1} s\right)^{n_1 + n_2} \exp\left[2b(\xi + \eta) \frac{\zeta^{\pm 1} s}{(1 + \zeta^{\pm 1})}\right] \\ & \times L_{n_1}^{|m|}\left(2b\xi \frac{(1-s)(1-\zeta^{\pm 1} s)}{(1 - s - \zeta^{\pm 1} s)}\right) L_{n_2}^{|m|}\left(2b\eta \frac{(1-s)(1-\zeta^{\pm 1} s)}{(1 - s - \zeta^{\pm 1} s)}\right). \end{aligned} \tag{47}$$

This representation allows one to show analytically that the leading asymptotic behavior is given by

$$\mathcal{P}^{|m|(\pm)}_{n_1 n_2}(k; \xi, \eta) \underset{r \to \infty}{\longrightarrow} \mathcal{A}^{|m|(\pm)}_{n_1 n_2}(\theta) \frac{\exp\{\pm i[kr - \beta \ln(2kr)]\}}{kr}, \tag{48}$$

where the amplitudes $\mathcal{A}^{|m|(\pm)}_{n_1 n_2}$ are expressed as

$$\begin{aligned} \mathcal{A}^{|m|(\pm)}_{n_1 n_2}(\theta) = {} & \pm i \sqrt{\frac{n_1! n_2!}{(n_1 + |m|)!(n_2 + |m|)!}}\, \zeta^{-i\beta} e^{-\frac{1}{2}\pi\beta} \left[\pm\left(\frac{1}{\zeta} - \zeta\right)\right]^{|m|+1} \left(\frac{\sin\theta}{2}\right)^{|m|} \\ & \times (-\zeta^{\pm 1})^{n_1 + n_2} \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} c^{(n_1, |m|)}_{\nu_1} c^{(n_2, |m|)}_{\nu_2}\, \Gamma(\pm i\beta + |m| + \nu_1 + \nu_2 + 1)(1 - \zeta^{\mp 2})^{\nu_1 + \nu_2} \left(\cos\frac{\theta}{2}\right)^{2\nu_1} \left(\sin\frac{\theta}{2}\right)^{2\nu_2}, \end{aligned} \tag{49}$$

with

$$c_{\nu}^{(n,\kappa)} = (-1)^{\nu} \frac{(n+\kappa)!}{(n-\nu)!(\nu+\kappa)!\nu!} \tag{50}$$

being the coefficients of x^{ν} in the Laguerre polynomial $L_n^{\kappa}(x)$ [31].

From Eqs. (43) and (44) we deduce that

$$\widetilde{F}_n^{(\pm)}(E,\mathbf{r}) = G_n^{(\pm)} \bar{F}_n^{(\pm)}(\mathbf{r}), \tag{51}$$

where

$$\bar{F}_n^{(\pm)}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} \sum_{n_1,n_2=0}^{\infty} C_{n_1 n_2 m}^{n(\pm)} |\widetilde{n_1 n_2 m}\rangle. \tag{52}$$

Inserting Eq. (51) into Eq. (42) and accounting for Eq. (21), we thus obtain a driven matrix equation

$$(\mathbb{I} + \mathbb{V}\mathbb{G}^{(\pm)})\bar{\mathbb{F}}^{(\pm)} = -\mathbb{V}\mathbb{S}^{(\pm)}. \tag{53}$$

Substituting expansions (29), (30), and (33) into Eq. (53) and projecting onto the basis functions (26), we obtain the equations for the expansion coefficients

$$C_{n_1 n_2 m}^{n(\pm)} + \sum_{n'} \sum_{n_1',n_2',m'} \left[VG^{(\pm)}\right]_{n_1 n_2 m; n_1' n_2' m'}^{n;n'} C_{n_1' n_2' m'}^{n'(\pm)} = -\delta_{n0} D_{n_1 n_2 m}, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{54}$$

where

$$\left[VG^{(\pm)}\right]_{n_1 n_2 m; n_1' n_2' m'}^{n;n'} = \sum_{n_1'',n_2'',m''} [V]_{n_1 n_2 m; n_1'' n_2'' m''}^{n-n'} [G^{(\pm)}]_{n_1'' n_2'' m''; n_1' n_2' m'}^{n'}, \quad D_{n_1 n_2 m} = \sum_{n_1',n_2',m'} [V]_{n_1 n_2 m; n_1' n_2' m'}^{0} \mathcal{S}_{n_1' n_2' m'}^{(\pm)}.$$

IV. ILLUSTRATION

As indicated in the Introduction, we wish to apply the presented Floquet formulation to study laser-assisted processes. Here we shall consider the $(e,2e)$ process (1) in asymmetric kinematics, where the transferred momentum $\mathbf{q} = \mathbf{p}_0 - \mathbf{p}_s$ is very small compared to the momenta $\mathbf{p}_0$ and $\mathbf{p}_s$ of, respectively, the incident and scattered electrons. With these momenta being much larger than the momentum of the ejected electron $\mathbf{k}_f$, exchange effects between the two electrons can be safely neglected. The energy of the projectile electron ($E_0 = p_0^2/2$ and $E_s = p_s^2/2$) is taken high enough to treat the S matrix of the ionization process in the first Born approximation

$$S_{fi} = -i \int_{-\infty}^{\infty} dt \, \langle \chi_{\mathbf{p}_s} \psi_f | W | \chi_{\mathbf{p}_0} \psi_i \rangle. \tag{55}$$

Here $\psi_{i(f)}(\mathbf{r},t)$ is the initial (final) state of the field-dressed hydrogen atom ($E_i < 0$ and $E_f = k_f^2/2 > 0$). Setting $\mathbf{r}_0$ to indicate the position vector of the projectile,

$$\chi_{\mathbf{p}_0}(\mathbf{r}_0,t) = e^{-iE_0 t} e^{i\mathbf{p}_0 \mathbf{r}_0}, \tag{56a}$$

$$\chi_{\mathbf{p}_s}(\mathbf{r}_0,t) = e^{-iE_s t} e^{i\mathbf{p}_s \mathbf{r}_0} \tag{56b}$$

are Gordon-Volkov wave functions describing, in the KH representation, the incident and scattered electrons in a laser field. The perturbation is given by the projectile-target interaction

$$W = \frac{1}{|\mathbf{r} - \mathbf{r}_0|} - \frac{Z}{|\mathbf{r}_0 + \mathbf{a}(t)|}, \tag{57}$$

where $Z = 1$ for atomic hydrogen. Using Eqs. (56a) and (56b) and the Floquet expansion (7), the time integration yields for the S matrix:

$$S_{fi} = -2\pi i \sum_{\ell} T_{fi}^{(\ell)} \, \delta(E_s + E_f - E_0 - E_i - \ell\omega), \tag{58}$$

where the ℓ-photon transition amplitudes are

$$T_{fi}^{(\ell)} = \frac{4\pi}{q^2} \sum_n \left[\int d\mathbf{r} \, F_{n-\ell}^{(f)*}(E_f,\mathbf{r}) e^{i\mathbf{q}\mathbf{r}} F_n^{(i)}(E_i,\mathbf{r}) - Z \sum_{n'} J_{n'}(-\mathbf{q}\mathbf{a}_0) \int d\mathbf{r} \, F_{n-n'-\ell}^{(f)*}(E_f,\mathbf{r}) F_n^{(i)}(E_i,\mathbf{r}) \right], \tag{59}$$

with $J_{n'}$ being the Bessel functions of integer order [31].

The triple differential cross section of the laser-assisted $(e,2e)$ process (1) involving ℓ photons is given by

$$\frac{d\sigma^{(\ell)}}{dE_f \, d\Omega_s \, d\Omega_f} = \frac{1}{(2\pi)^2} \frac{p_s k_f}{p_0} |T_{fi}^{(\ell)}|^2. \tag{60}$$

A. Calculation of dressed atomic states

Both the initial and final dressed atomic states are obtained within the Hermitian Floquet formulation, with real quasienergies E_i and E_f. Specifically, in order to test our parabolic quasi-Sturmian-Floquet approach, the final state $\mathbb{F}^{(f)}$ of the ejected electron is derived by solving equations (54) with incoming boundary conditions. In the calculations we truncate expansion (52), such that $n_1, n_2 < N$ and $|m| \leqslant M$, and the potential matrix (14), such that $|n| \leqslant \mathfrak{N}_f$. For illustrative purposes and for the sake of convenience, the initial state $\mathbb{F}^{(i)}$ is also constructed in the KH frame, namely, as an eigensolution

of Eq. (11) satisfying zero boundary conditions. Its Floquet components are approximated by the truncated expansions

$$F_n^{(i)} = \sum_{m=-M}^{M} \sum_{n_1,n_2=0}^{N-1} C_{n_1 n_2 m}^{n(i)} |n_1 n_2 m\rangle \tag{61}$$

in terms of $\mathcal{N} = (2M+1) \times N^2$ Laguerre basis functions (26). Then, the expansion coefficients $C_{n_1 n_2 m}^{n(i)}$ and eigenenergy E_i represent a solution of the generalized eigenvalue problem

$$(\mathbb{H} + \mathbb{V})\mathbb{C}^{(i)} = E_i \mathbb{B}\mathbb{C}^{(i)}, \tag{62}$$

where $\mathbb{H}$ is the block-diagonal matrix obtained from Eq. (13) by replacing each Hamiltonian H_n with its matrix representation, whose size is determined by $\mathcal{N}$. Specifically, the matrix elements of H_n read

$$\langle n_1' n_2' m' | H_n | n_1 n_2 m\rangle = \langle n_1' n_2' m' | H_C | n_1 n_2 m\rangle - n\omega \langle n_1' n_2' m' | n_1 n_2 m\rangle, \tag{63}$$

with

$$\langle n_1' n_2' m' | H_C | n_1 n_2 m\rangle = \tfrac{1}{4}\,\delta_{m'm}(\delta_{n_2' n_2} h_{n_1' n_1}^m + \delta_{n_1' n_1} h_{n_2' n_2}^m - \delta_{n_1' n_1}\delta_{n_2' n_2} 2Z), \tag{64}$$

where the nonvanishing elements $h_{n'n}^m$ are given by

$$h_{nn}^m = b(2n + |m| + 1),$$
$$h_{nn+1}^m = h_{n+1n}^m = b\sqrt{(n+1)(n+1+|m|)}. \tag{65}$$

The overlap integrals read

$$\langle n_1' n_2' m' | n_1 n_2 m\rangle = \tfrac{1}{4}\,\delta_{m'm}(\delta_{n_2' n_2} q_{n_1' n_1}^m + \delta_{n_1' n_1} q_{n_2' n_2}^m), \tag{66}$$

with nonvanishing $q_{n'n}^m$ given by

$$q_{nn}^m = \frac{1}{2b}(2n + |m| + 1),$$
$$q_{nn+1}^m = q_{n+1n}^m = -\frac{1}{2b}\sqrt{(n+1)(n+1+|m|)}. \tag{67}$$

By analogy with $\mathbb{H}$, the matrix $\mathbb{V}$ is constructed from the blocks of order $\mathcal{N}$ for each V_n. The matrix $\mathbb{B}$ has a block-diagonal structure with $\mathcal{N} \times \mathcal{N}$ matrices of the overlap integrals (66) along the main diagonal.

Below a method based on Eqs. (61) and (62) is applied for constructing the solution $\mathbb{F}^{(i)}$ associated with the lowest KH state [23].

B. Numerical results

We inspect the case of the laser-assisted $(e,2e)$ process (1) in coplanar kinematics, i.e., when the incident electron momentum $\mathbf{p}_0$ and the two outgoing electron momenta $\mathbf{p}_s$ and $\mathbf{k}_f$ lie in the same plane. For the numerical results presented below the ejected electron momentum is set to $k_f = 0.93$, and the Laguerre basis (26) scale parameter to $b = 0.6$. The laser field is taken with the polarization vector perpendicular to the scattering plane, a quiver amplitude $a_0 = 5$ and a frequency $\omega = 0.05 = 1.36$ eV (the values are close to those in the experiment of Höhr *et al.* [33], where $I = 4 \times 10^{12}$ W/cm^2 and $\omega = 1.17$ eV).

Let us first comment on the advantage of using a basis set of QS functions instead of the customary square-integrable

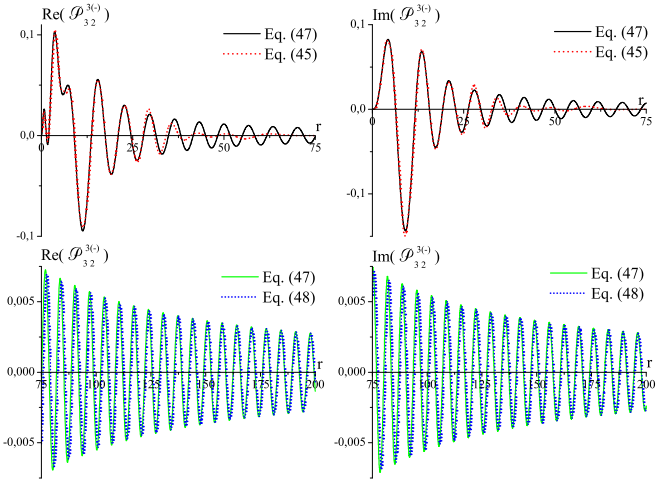

FIG. 1. Real and imaginary parts of the "radial" QS function $\mathcal{P}_{32}^{3(-)}$ (46) calculated along the $\theta = \frac{\pi}{3}$ direction, where θ is the angle between the $\mathbf{k}_f$ and $\mathbf{r}$ vectors in the scattering plane, using the integral representation (47). In the top panels, the same functions calculated with expansion (45) in the L^2 basis (26), truncated at an upper limit of 25, are shown with dotted lines. In the bottom panels, the leading asymptotic behavior (48) is also plotted with dotted lines. All values are given in a.u.

Coulomb-Sturmian functions. For both of them one uses a matrix representation in a finite set of L^2 Laguerre basis functions (26), and solving the corresponding matrix equations gives the basis representations of the wave function. In the QS case, owing to the asymptotic behavior (48), the truncated expansion (with the number of terms being equal to the size of the potential matrix $\mathbb{V}$) allows one to compute the scattering wave function in the entire configuration space. In turn, with square-integrable Coulomb-Sturmian functions, the domain that can be properly described is critically determined by the number of L^2 functions (26): in order to adequately describe the continuum wave function at large distances one should take a sufficiently large number. Moreover, the efficiency of a QS versus Coulomb-Sturmian basis was illustrated, for example, in Ref. [14] through the representation of the scattering solution for a Coulomb plus Yukawa-type potential (see also Ref. [34]).

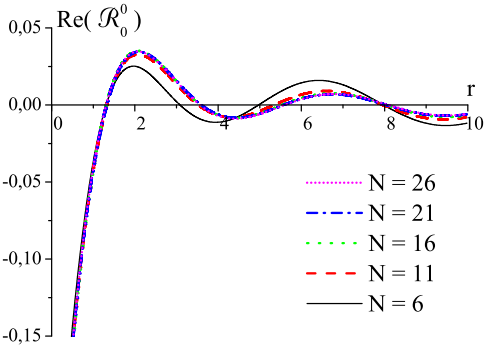

FIG. 2. Convergence behavior of the QS expansion (43) when $\mathfrak{N}_f = 0$. The real parts of the radial functions $\mathcal{R}_0^0$ (69) calculated along the direction $\theta = \frac{\pi}{6}$ are presented for several values of N radial basis functions. All values are given in a.u.

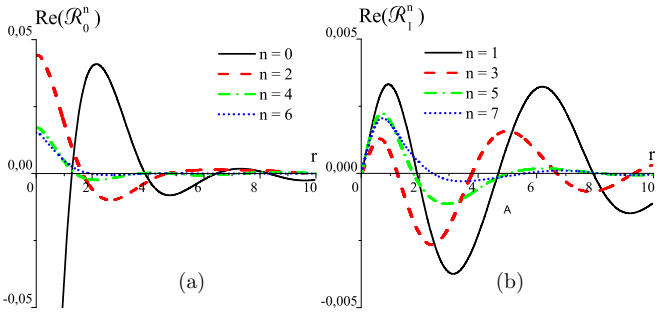

FIG. 3. Comparison of the Floquet components of the solution to the driven Eq. (53), obtained with upper limit $\mathfrak{N}_f = 7$ in the Floquet-Fourier expansion (7) of the final dressed hydrogen state. The calculation is performed along the direction $\theta = \frac{\pi}{6}$. (a) Real part of radial functions $\mathcal{R}^n_0$ (69) of $\widetilde{F}^{(-)}_{n=0,2,4,6}$. (b) Real part of radial functions $\mathcal{R}^n_1$ (69) of $\widetilde{F}^{(-)}_{n=1,3,5,7}$. All values are given in a.u.

In Fig. 1 we show the real and imaginary parts of a radial QS function $\mathcal{P}^{|m|(\pm)}_{n_1 n_2}$ as defined by Eq. (46). On one hand, we compute them using the (exact) integral representation (47) and, on the other hand, with an L^2 (approximate) representation, i.e., a truncated expansion (45) in the L^2 basis (26). The comparison of the two calculations is shown in the top panels. While a good match is observed at short distances from the nucleus, clearly the truncated expansion is not sufficient to provide a satisfactory description of the QS function at large distances. In the bottom panels, we compare the numerical radial function with its asymptotic form (48) in a region well off the origin: a very good match is observed. This comparison illustrates that expanding a continuum wave function in terms of the QS basis functions satisfying the asymptotic boundary conditions (48) and (49) is very appropriate, and is expected to be efficient in dealing with scattering problems.

Let us now investigate some issues related to the convergence of the proposed expansion for the final dressed atomic state. We have essentially two truncations. One in the number $2\mathfrak{N}_f + 1$ of final-state Floquet components taken in the Floquet-Fourier expansion (7). The other is the number of QS basis functions, specifically the number N of radial basis functions for both coordinates ξ and η.

We first solved the driven matrix equation (53) in the $\mathfrak{N}_f = 0$ case. In the matrix representation $\widetilde{\mathbb{V}}_0$ of the potential $\widetilde{V}_0$ (9) we restricted the azimuthal quantum number m to range from -7 to 7. Figure 2 shows the results for the real part of the radial functions $\mathcal{R}^n_m$ (for $n = m = 0$) defined as

$$\widetilde{F}^{(-)}_n(E_f, \mathbf{r}) = \sum_{m=-M}^{M} \frac{e^{im\phi}}{\sqrt{2\pi}} \mathcal{R}^n_m(\xi, \eta), \tag{68}$$

$$\mathcal{R}^n_m(\xi, \eta) = \sum_{n_1, n_2 = 0}^{N-1} C^n_{n_1 n_2 m} \mathcal{P}^{|m|}_{n_1 n_2}(k_n; \xi, \eta). \tag{69}$$

We progressively increased the upper value N up to $N_{\max} = 26$, and clearly observed that convergence of the QS expansion is fast and is achieved for $N \simeq 15$.

Next, we set $\mathfrak{N}_f = 7$ and compute the different Floquet components $\widetilde{F}^{(-)}_n$ of the solution to the driven equation (53), for a fixed basis number $N = 16$ and $M = 7$ meaning $\mathcal{N} = 57\,600$. Figure 3 shows the expected feature that as n grows, the components decrease in magnitude appreciably.

Let us turn to the initial dressed atomic state. We also start from $\mathfrak{N}_i = 0$, and in this case we find for the lowest eigenvalue $E_i^{\mathfrak{N}_i=0} = -0.20182$ which agrees well with the KH state energy $E_{\rm KH} = -0.20196$ obtained in Ref. [23]. Further, we successively increase the number $2\mathfrak{N}_i + 1$ of Floquet components involved in the computation up to 15 ($\mathfrak{N}_i = 7$). For each $\mathfrak{N}_i$ we choose the eigenstate whose zeroth component $F^{(i)}_0$ is largest, i.e., contains the largest (in modulus) coefficients $C^{0(i)}_{n_1 n_2 m}$ in expansion (61). The convergence behavior of the thereby obtained sequence of such generalized KH states can be appreciated in Table I, where the eigenvalues $E_i^{\mathfrak{N}_i}$ along with the corresponding largest coefficients $C^{0(i)}_{n_1 n_2 m}$ are presented for $\mathfrak{N}_i = 0, 1, \ldots, 7$. While the $E_i^{\mathfrak{N}_i}$ sequence exhibits clear convergence, it should be mentioned that calculating the initial dressed hydrogen state within the Floquet theory in the KH frame is supposed to be efficient in the high-frequency regime, which is far from the present choice of laser parameters. Therefore, since our analysis is primarily focused on the efficiency of calculating the final dressed hydrogen state rather than the initial one, in the $(e,2e)$ calculations discussed below we restrict ourselves to the $\mathfrak{N}_i = 7$ case in Table I as far as the initial state is concerned.

As the convergence of expansions is under control, we wish to test our approach by applying it to the laser-assisted $(e,2e)$ process (1); we compute TDCS when $\ell = 0$, that is, when the net number of photons exchanged between the colliding system and laser field equals zero. We consider increasing

TABLE I. Calculated eigenvalues $E_i^{\mathfrak{N}_i}$ and the corresponding largest coefficients $C^{0(i)}_{n_1 n_2 0}$ of the zeroth Floquet component ($n = 0$) of the generalized KH states are given for increasing values of $\mathfrak{N}_i$. A number in square brackets stands for the power of 10 by which the preceding value is to be multiplied.

$\mathfrak{N}_i$	$E_i^{\mathfrak{N}_i}$	$C^{0(i)}_{000}$	$C^{0(i)}_{100}$	$C^{0(i)}_{110}$
0	−0.201819	0.551647 −i0.646912[−14]	−0.355924 +i0.417389[−14]	0.235527 −i0.276201[−14]
1	−0.326097	0.264437 +i0.254086[−14]	−0.199877 −i0.192053[−14]	0.209358 +i0.201163[−14]
2	−0.330410	0.405213 −i0.213726[−12]	−0.196557 +i0.103595[−12]	0.112545 −i0.593281[−13]
3	−0.389163	0.350215 +i0.221954[−14]	−0.195399 −i0.123909[−14]	0.156212 +i0.993447[−15]
4	−0.419223	0.371691 −i0.890313[−10]	−0.184586 +i0.442137[−10]	0.150770 −i0.361136[−10]
5	−0.438350	0.360758 +i0.122863[−14]	−0.184227 −i0.629620[−15]	0.159845 +i0.547415[−15]
6	−0.450722	0.360771 +i0.103656[−6]	−0.179809 −i0.516623[−7]	0.159936 +i0.459524[−7]
7	−0.459299	0.360064 −i0.483796[−16]	−0.176820 +i0.244482[−16]	0.158099 −i0.197072[−16]

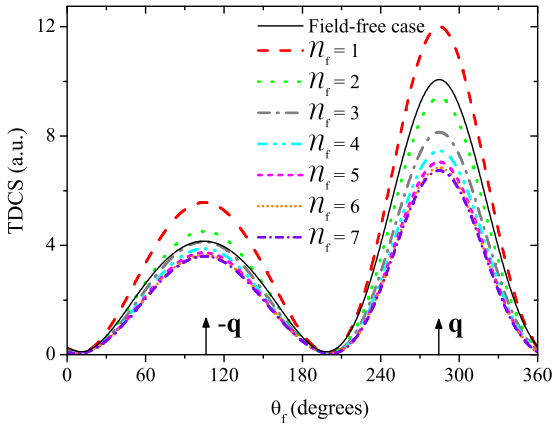

FIG. 4. Convergence behavior of the laser-assisted TDCS in the $\ell = 0$ case when the $\mathfrak{N}_f$ number in the Floquet-Fourier expansion of the final hydrogen state is varied from 0 to 7 (N is fixed here at 16). The field-free TDCS is also plotted (solid line). The quiver amplitude is $a_0 = 5$ and the laser frequency is $\omega = 0.05$. In the chosen coplanar geometry, the momenta of the incident and scattered electrons are $p_0 = 19.21$ and $p_s = 19.17$, respectively, and the ejected electron momentum is $k_f = 0.93$. The scattered electron is observed at $\theta_s = 0.43^\circ$, the electron angles being measured counterclockwise with respect to the incident electron beam direction. This results in a momentum transfer of modulus $q = 0.15$ and pointing into the direction indicated by an arrow.

values of the $2\mathfrak{N}_f + 1$ terms used in the Floquet-Fourier expansion (7) of the final atomic state, while this number for the initial atomic state is fixed as remarked above (i.e., $\mathfrak{N}_i = 7$). With such dressed atomic states we calculate the TDCS defined by Eq. (60) and study its convergence behavior as the number $2\mathfrak{N}_f + 1$ of the final-state Floquet components is increased. We choose the kinematical situation where the fast scattered electron energy is $E_s = 5$ keV and the scattering angle is $\theta_s = 0.43^\circ$ (the momentum transfer is $q = 0.15$). The first Born approximation in the projectile-hydrogen interaction is supposed to be well applicable in such kinematics, thus validating the use of Eq. (55) for the S matrix. The TDCS results presented in Fig. 4 exhibit a typical pattern of the ejected-electron angular distribution in the scattering plane, namely, a two-peak structure with the binary peak in the direction of the momentum transfer ($\mathbf{q}$) and the recoil peak in the opposite direction ($-\mathbf{q}$). From comparison with the field-free cross section (solid line) one can see how significantly the ionization process is affected by the presence of the laser field. What matters most here is that one observes a fast numerical convergence with increasing number of terms in the Floquet-Fourier expansion (7) of the final atomic state. This demonstrates that our quasi-Sturmian-Floquet methodology is able to generate light-dressed states of an electron moving in a Coulomb field in the presence of laser radiation, and that it can be applied to study laser-assisted ionization processes.

V. SUMMARY

We have applied the Floquet-Fourier expansion method to the solution of the time-dependent Schrödinger equation describing the electron states in combined Coulomb and laser fields. The system of coupled Floquet equations for the electron states subject to outgoing or incoming asymptotic boundary conditions has been formulated in the Kramers-Henneberger frame. The Floquet components of the electron scattering states are expanded on parabolic quasi-Sturmian basis functions and the driven matrix equation is derived in this representation. The efficiency of the developed approach has been illustrated in the case of the laser-assisted $(e,2e)$ collision on atomic hydrogen. Numerical calculations have exhibited fast convergence while increasing both the number of the components in the Floquet-Fourier expansion and the size of the parabolic quasi-Sturmian basis. Thus, one might expect the present approach to be efficient in the theoretical treatment of electron states in various laser-assisted radiation and ionization processes. The formalism was presented in this work for the case of a pure Coulomb potential and applied here to atomic hydrogen as a showcase. It can be readily used to treat a general potential with a Coulomb tail, which typically mimics nonhydrogenic atomic systems with a single active electron. The only difference between the two cases will consist in the potential matrix $\mathbb{V}$ involved in the driven matrix equation (53).

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