Editors' Suggestion

Mode entanglement of Gaussian fermionic states

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(Received 18 January 2018; published 16 April 2018)

We investigate the entanglement of n-mode n-partite Gaussian fermionic states (GFS). First, we identify a reasonable definition of separability for GFS and derive a standard form for mixed states, to which any state can be mapped via *Gaussian local unitaries* (GLU). As the standard form is unique, two GFS are equivalent under GLU if and only if their standard forms coincide. Then, we investigate the important class of *local operations assisted by classical communication* (LOCC). These are central in entanglement theory as they allow one to partially order the entanglement contained in states. We show, however, that there are no nontrivial Gaussian LOCC (GLOCC) among pure n-partite (fully entangled) states. That is, any such GLOCC transformation can also be accomplished via GLU. To obtain further insight into the entanglement properties of such GFS, we investigate the richer class of Gaussian *stochastic local operations assisted by classical communication* (SLOCC). We characterize Gaussian SLOCC classes of pure n-mode n-partite states and derive them explicitly for few-mode states. Furthermore, we consider certain fermionic LOCC and show how to identify the maximally entangled set of pure n-mode n-partite GFS, i.e., the minimal set of states having the property that any other state can be obtained from one state inside this set via fermionic LOCC. We generalize these findings also to the pure m-mode n-partite (for $m > n$) case.

DOI: 10.1103/PhysRevA.97.042325

I. INTRODUCTION

Entanglement [1] plays a crucial role in understanding the quantum physics of systems composed of many subsystems or many particles. It is the primary resource of many applications in quantum computation and communication and is the basis of many of the intriguing effects of quantum many-body physics.

In multipartite systems, there are various qualitatively different kinds of entanglement. Relating them to physical properties [2] or to performable tasks [3,4] contributes to elucidating the role of entanglement in nature and as a resource for quantum technologies [5].

One very successful approach to identify different classes of entanglement is to consider whether states can be converted into each other using some naturally restricted set of quantum operations, defining states for which such conversion is mutually impossible to belong to distinct classes. This has lead to the discovery of inequivalent kinds of entanglement [6,7] and to their classification [8,9]. Furthermore, the maximally entangled states and sets, which are the most relevant states regarding local state transformations, have been identified [1,10–14].

Most of these notions have been developed considering systems of distinguishable particles, and with system Hilbert spaces that have a natural tensor-product structure imposed by the spatial separation of subsystems. When applying them to systems of *indistinguishable* particles, central notions of entanglement theory have to be adapted to account for (anti)commutation relations and superselection rules that restrict the set of allowed operations and modify the structure of "local" operations.

In the present article, we investigate the entanglement properties of multipartite *fermionic* states. There are both fundamental and practical reasons to do so: On the one hand, fermions are the fundamental constituents of matter, and hence to understand the entanglement properties of quantum many-body systems the fermionic perspective is indispensable. This has motivated a broad effort to study fermionic entanglement and work out the differences with qubit systems; see, e.g., Refs. [15–24].

Even in quantum information, where bosonic or effectively distinguishable particles play the major important role, genuinely fermionic systems such as single semiconductor electrons or holes in quantum dots [25], ballistic electrons in quantum wires or edge channels [26–28], or Majorana fermions in quantum wires [29] are of increasing interest. On the other hand, this analysis gives insights into the nature of entanglement in general and the comparison of fermionic, bosonic, and distinguishable systems affords a clearer picture of the role of statistics.

Here we apply this state-conversion-based entanglement classification to multipartite *Gaussian* fermionic states. This important family of states contains the eigen- and thermal states of quadratic Hamiltonians, i.e., those describing quasifree single-particle dynamics. Despite their simplicity, these states comprise a large range of different kinds of entangled states, including GHZ-like states, spin-squeezed states, paired states [30], and topological states [31], thus serving as a convenient test bed for entanglement studies, and they can be used for basic quantum information processing tasks such as probabilistic teleportation [32], entanglement distillation [33], or metrology [30,34,35], while for universal quantum computation, the

Gaussian states and operations have to be augmented by a non-Gaussian measurement [36]. In the present work, we focus first on pure n-partite states with a single mode per party and investigate their transformation properties under different kinds of local fermionic operations. Then we generalize some of the results to m-mode n-partite (for $m > n$) states.

The outline of the remainder of the paper is the following. In Sec. II, we recall the definition and some properties of fermionic states (FS), Gaussian fermionic states (GFS), and Gaussian operations. Moreover, we recall the mapping between GFS and spin states using the Jordan-Wigner transformation. In Sec. III, we consider mixed GFS and first recall the various definitions of separability for FS [16]. We identify a reasonable definition of separability of GFS. Then, we introduce a standard form for the covariance matrix (CM), which is invariant under Gaussian local unitaries (GLU). In the last two sections, Secs. IV and V, we investigate the entanglement properties of pure GFS considering Gaussian local operations assisted by classical communication (GLOCC). As this class of operations turns out to be trivial for n-mode n-partite as well as multipartite multimode pure fully entangled GFS, we study also Gaussian stochastic local operations assisted by classical communication (GSLOCC) and fermionic local operations assisted by classical communication (FLOCC) to still obtain insights into the entanglement of GFS. In particular, the following results are presented: (i) We characterize the separable Gaussian fermionic states and different kinds of local Gaussian fermionic operations (GLOCC, GSLOCC, Gaussian separable operations (GSEP)); (ii) we derive a standard form for n-mode, n-partite GFS into which any such state can be transformed by GLU; as this standard form is unique, two GFS are GLU equivalent if and only if (iff) their standard forms coincide; (iii) we show that there are no nontrivial Gaussian fermionic LOCC transformations between fully entangled pure n-partite GFS; (iv) we characterize the Gaussian SLOCC classes for pure n-mode, n-partite GFS; (v) we consider fermionic LOCC between Gaussian states and identify the corresponding maximally entangled set (MES), and, finally, (vi) we generalize some of these findings to the m-mode n-partite ($m > n$) case.

II. PRELIMINARIES

We summarize here some results concerning GFS and introduce our notation. We consider systems composed of n fermionic modes. To each mode $k = 1,\dots,n$ belongs a creation and an annihilation operator $b_k, b_k^\dagger$, obeying the anticommutation relations $\{b_k^\dagger, b_l^\dagger\} = \{b_k, b_l\} = 0, \{b_k, b_l^\dagger\} = \delta_{kl}$. The antisymmetric Fock space over n modes is spanned by the Fock basis defined as

$$|k_1,\dots,k_n\rangle = (b_1^\dagger)^{k_1}\dots(b_n^\dagger)^{k_n}|0\rangle, \tag{1}$$

where $k_i \in \{0,1\}$ for all $i \in \{1,\dots,n\}$ and the vacuum state $|0\rangle$ obeys $b_i|0\rangle = 0\ \forall i$. Note that $|k_1,\dots,k_n\rangle$ is an eigenstate of all number operators $n_i = b_i^\dagger b_i$ to eigenvalue k_i.

It is sometimes more convenient to consider the $2n$ Hermitian fermionic Majorana operators,

$$\tilde{c}_{2k-1} = b_k + b_k^\dagger, \quad \tilde{c}_{2k} = -i(b_k - b_k^\dagger), \tag{2}$$

instead of the creation and annihilation operators. The anticommutation relations are then equivalent to

$$\{\tilde{c}_k, \tilde{c}_l\} = 2\delta_{kl}. \tag{3}$$

For any Clifford algebra satisfying the relation above, the operators $b_k = \frac{1}{2}(\tilde{c}_{2k-1} + i\tilde{c}_{2k})$ obey the anticommutation relation and vice versa.

A linear transformation of the fermionic operators $\{\tilde{c}_k\}$, i.e., $\tilde{c}_k \to \tilde{c}'_k = \sum_l O_{kl}\tilde{c}_l$, preserves the canonical anticommutation relations iff $O \in \mathcal{O}(2n,\mathbb{R})$, i.e., iff O is a real orthogonal matrix. These are called canonical transformations or Bogoliubov transformations. They realize a basis change in the fermionic phase space and can be implemented by Gaussian operations (see below).

A. Gaussian states

A GFS of n modes is defined as the thermal (Gibbs) state of a quadratic Hamiltonian, $H = \frac{i}{4}\tilde{c}^T G\tilde{c}$ with G a real antisymmetric $2n \times 2n$ matrix and $\tilde{c} = (\tilde{c}_1,\dots,\tilde{c}_{2n})$, i.e.,

$$\rho = Ke^{-\frac{i}{4}\tilde{c}^T G\tilde{c}}, \tag{4}$$

where K denotes a normalization constant (or, to include states of nonmaximal rank, can be expressed as a limit of such expressions). Equivalently, they can be characterized as those states satisfying Wick's theorem, i.e., for which all cumulants vanish [37,38].

It is well known that any real antisymmetric $2n \times 2n$ matrix can be transformed into a normal form via a real special orthogonal matrix [39]. More precisely, there exists a matrix $O \in \mathcal{SO}(2n,\mathbb{R})$ such that

$$OGO^T = \oplus_{k=1}^n \beta_k J_2, \text{where } J_2 = \begin{pmatrix} 0 & 1, \\ -1 & 0 \end{pmatrix}, \beta_k \in \mathbb{R}. \tag{5}$$

Hence, a GFS is a state of the form

$$\rho = \tilde{\otimes}_{k=1}^n \rho'_k, \tag{6}$$

where $\rho'_k = \frac{1}{2}(\mathbb{1} - \mu_k[b'^\dagger_k, b'_k])$ for $\mu_k = \tanh(\beta_k/2)$. Note that here and in the following $\tilde{\otimes}$ denotes a product of operators which are acting only on distinct sets of modes. However, we only use this notation if the operators fulfill a commutation relation. Here, the operators $b'_k = \sum_l u_{lk} b_l$ obey again the anticommutation relations; i.e., they are fermionic annihilation operators [39]. Thus, ρ can be written as

$$\rho = \frac{1}{N}e^{-\sum_k \beta_k b'^\dagger_k b'_k}, \tag{7}$$

where $N = \prod_k(1 + e^{-\beta_k})$ denotes a normalization constant. It is evident that a Gaussian state is completely determined by its second moments, which are usually collected in the CM. In terms of the Majorana operators the CM of a GFS, ρ, which we denote by γ, is defined as

$$\gamma_{kl} = -\frac{1}{2i}\mathrm{tr}(\rho[\tilde{c}_k, \tilde{c}_l]). \tag{8}$$

As can be easily seen from this definition, the CM is an antisymmetric $2n \times 2n$ real matrix, which can be transformed by a real special orthogonal matrix, O, into the normal form

$$O\gamma O^T = \oplus_{k=1}^n(-\mu_k J_2). \tag{9}$$

Note that $\mu_k = \tanh(\beta_k/2)$ [30]. This normal form is referred to as the (fermionic) Williamson normal form [39]. Note that in contrast to the case of bosons, no first moments have to be specified for fermions since due to the parity superselection rule all physical states have $\mathrm{tr}(\tilde{c}_k\rho) = 0$. Thus, GFS are completely characterized by their second moments, i.e., their CM, due to Wick's theorem [40]

$$i^p \mathrm{tr}(\rho\tilde{c}_{j_1}\dots\tilde{c}_{j_{2p}}) = \mathrm{Pf}(\gamma_{j_1,\dots,j_{2p}}), \tag{10}$$

where $1 \leqslant j_1 < \dots < j_{2p} \leqslant 2n$ and $\gamma_{j_1,\dots,j_{2p}}$ is the $2p \times 2p$ submatrix of γ with rows and columns $j_1,\dots,j_{2p}$. Here Pf denotes the Pfaffian which for a $2n \times 2n$ matrix $A = (a_{i,j})$ is defined as $\mathrm{Pf}(A) = \frac{1}{2^n n!}\sum_{\pi\in S_{2n}}\mathrm{sgn}(\pi)\prod_{i=1}^n a_{\pi(2i-1),\pi(2i)}$, where the sum is over all permutations π and $\mathrm{sgn}(\pi)$ the signature of π and satisfies $\mathrm{Pf}(A)^2 = \det(A)$.

Note that an antisymmetric real matrix γ corresponds to the CM of a GFS, in particular to a normalized positive semidefinite operator, iff $\gamma^2 \geqslant -\mathbb{1}$, i.e., iff all the eigenvalues of γ, which are all purely imaginary, have modulus smaller or equal to one. The CM corresponds to a pure state if $\gamma^2 = -\mathbb{1}$. That is, $\mu_k \in [-1,1]$ in Eq. (9) and $|\mu_k| = 1 \forall k$ in case the state is pure. For instance, the CM corresponding to the vacuum state would be $\gamma = -J_2$, whereas the one corresponding to $|1\rangle\langle 1|$ would be J_2. Hence, the CM corresponding to the completely mixed state is $\gamma = 0$.

B. Jordan-Wigner transformation

Let us recall here that there exists a one-to-one mapping between FS and qubit states, which is known as Jordan-Wigner transformation. Let us consider n modes and define the operators

$$\begin{aligned} c_{2j-1} &= Z \otimes Z \otimes \dots \otimes Z \otimes X_j \otimes \mathbb{1}\dots, \\ c_{2j} &= Z \otimes Z \otimes \dots \otimes Z \otimes Y_j \otimes \mathbb{1}\dots. \end{aligned} \tag{11}$$

These operators obey the same anticommutation relations as the Majorana operators.

Consider now a FS

$$|\Psi\rangle = \sum_{i_1\dots,i_n\in 0,1} \alpha_{i_1\dots,i_n}(b_1^\dagger)^{i_1}(b_2^\dagger)^{i_2}\dots(b_n^\dagger)^{i_n}|0\rangle, \tag{12}$$

where $|0\rangle$ denotes the vacuum state and $\alpha_{i_1\dots,i_n} \in \mathbb{C}$. Because $(b_k^\dagger)^2 = 0$, one can associate to the state given in Eq. (12) the n-qubit state

$$|\Psi\rangle = \sum_{i_1\dots,i_n} \alpha_{i_1\dots,i_n}|i_1\dots,i_n\rangle_{1,\dots,n}. \tag{13}$$

The Jordan-Wigner transformation is a unitary mapping between the antisymmetric Fock space of n modes and the Hilbert space of n qubits, relating Fermi operators $\tilde{c}_i$ with qubit operators in Eq. (11) and the states in Eq. (12) with the ones in Eq. (13).

Note, however, that the parties are ordered and one cannot simply reorder them, as the order is fixed due to the commutation relations. To give an example, the state $|00\rangle_{12} + |11\rangle_{12} = |00\rangle_{21} - |11\rangle_{21}$, where the minus sign results from permuting particles 1 and 2. To be more precise, the operation which has to be performed on the qubit state in order to swap two systems is the fermionic swap, which is the mapping $|ij\rangle \to (-1)^{ij}|ji\rangle$. In order to perform, for instance, a partial trace, also these commutation relations have to be taken into account. That is, first the party over which the trace is performed has to be swapped (with a fermionic swap) to the last position [41]. After that, the partial trace can be performed as usual. Fermionic mixed states are then convex combinations of fermionic pure states.

Note that the parity conservation implies that a FS is always a direct sum of states whose support is only in the subspace with even parity and states whose support is only in the subspace with odd parity. Here, the subspace with even (odd) parity coincides with the set of states which are a superposition of Fock states which have all an even (odd) number of 1's, respectively. Denoting by P_e (P_o) the projector onto the even (odd) subspace, we hence have that a state with density matrix ρ is fermionic iff $\rho = P_e\rho P_e + P_o\rho P_o$, i.e., iff $P_e\rho P_o = P_o\rho P_e = 0$ [42].

Especially when one is working with this representation, it is important to be able to identify which of the FS are Gaussian. Fortunately, given a FS, the following result can be used to decide whether it is Gaussian or not. Recall that any operator in the Clifford algebra generated by the Majorana operators $\tilde{c}_i$ ($i = 1,\dots,2n$) can be written as

$$x = \alpha\mathbb{1} + \sum_{p=1}^{2n}\sum_{1\leqslant a_1<a_2<\dots a_p\leqslant 2n} \alpha_{a_1,\dots,a_p}\tilde{c}_{a_1}\dots\tilde{c}_{a_p}. \tag{14}$$

An operator is called even if it involves only even powers of the generators, or stated differently and using the Jordan-Wigner transformation, if the number of X's plus the number of Y's occurring in the sum is even. As any odd operator changes the parity, it is easy to see that x is even iff $P_e x P_o = P_o x P_e = 0$. Thus, in particular, all FS have even density matrices.

It has been shown in Ref. [43] that an even operator, x, is Gaussian iff

$$[\Lambda, x^{\otimes 2}] = 0, \tag{15}$$

where

$$\Lambda = \sum_{i=1}^{2n} c_i \otimes c_i. \tag{16}$$

Thus, we have that a FS, ρ, is Gaussian iff

$$[\Lambda, \rho^{\otimes 2}] = 0. \tag{17}$$

Let us give some examples. For a single mode, a state is fermionic if its density matrix is diagonal in the computational basis. Any such state is also Gaussian. For two modes, any FS is of the form $\rho = \rho_e \oplus \rho_o$, where ρ_e (ρ_o) are density operators in the two-dimensional even (odd) parity subspace spanned by $\{|00\rangle,|11\rangle\}$ ($\{|01\rangle,|10\rangle\}$) respectively. It can be easily seen that such a state is then Gaussian, i.e., fulfills the condition given in Eq. (17) iff $|\rho_e| = |\rho_o|$, where $|\cdot|$ denotes the determinant.

An example of such a state would be $e^{i\alpha(b_1^\dagger b_2^\dagger + b_1 b_2)}$, where $\rho_e = \begin{pmatrix} \cosh(\alpha) & -i\sinh(\alpha) \\ i\sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$ and $\rho_o = \mathbb{1}$. In particular, *all* pure two-mode FS are Gaussian. However, not all mixed two-mode FS are: Examples of non-Gaussian FS are the Werner states, $\rho_W = \frac{4F-1}{3}|\psi^-\rangle\langle\psi^-| + \frac{1-F}{3}\mathbb{1}$, for $F \in (1/4,1)$. Moreover, as we will see later, any pure FS of three modes is Gaussian. However, this is not the case for four modes.

When discussing pure GFS, we either consider the Jordan-Wigner representation of the FS or the CM of the state.

C. Gaussian operations

Let us now briefly recall the definitions and properties of Gaussian unitary operations, general Gaussian operators, and Gaussian maps in the fermionic case. First note that all quantum operations (completely positive maps) that respect parity are considered as valid physical operations here and referred to as *fermionic operations*.

Gaussian operations are those that can be realized with Gaussian means: evolution under quadratic Hamiltonians, adjoining of systems in Gaussian states, discarding of subsystems, measuring Gaussian POVMs, and projecting on pure Gaussian states. A Gaussian fermionic unitary, U, acting on n modes can be written as $U = e^{-iH}$, where H is quadratic in the Majorana operators, that is,

$$H = i \sum_{kl} h_{kl} \tilde{c}_k \tilde{c}_l, \tag{18}$$

with h being a real antisymmetric $2n \times 2n$ matrix [44]. In Ref. [45], it was shown that these unitaries effect a canonical transformation of the Majorana operators

$$U^\dagger \tilde{c}_j U = \sum_{k=1}^{2n} O_{jk} \tilde{c}_k, \tag{19}$$

where $O = e^{4h} \in \mathcal{SO}(2n)$ is a real special orthogonal $2n \times 2n$ matrix. Hence, a fermionic Gaussian unitary maps the CM to $O\gamma O^T$, where $O \in \mathcal{SO}(2n)$ [46].

All Gaussian unitaries preserve the parity; i.e., they commute with the parity operator $P = (-1)^{\sum_k n_k}$. However, the parity-flipping transformation of mode k, which corresponds to an (nonspecial) orthogonal transformation $O = \oplus_{i=1}^{k-1}\mathbb{1} \oplus Z \oplus_{i=k+1}^{n} \mathbb{1}$ on the Majorana operators of the system [47] (here and in the following X,Y,Z denote the Pauli operators), also has a (local) physical realization. The transformation can be achieved for example by adjoining an ancillary mode in a Fock state and then acting on the Majorana operators of the system modes and the ancillary mode with the $\mathcal{SO}(2n+2)$ operation $O = \oplus_{i=1}^{k-1}\mathbb{1} \oplus Z \oplus_{i=k+1}^{n} \mathbb{1} \oplus Z$ [48]. This exchanges particles with holes both in mode k and in the ancilla and leaves the latter unentangled; i.e., after discarding the ancilla it realizes Z on mode k. Since for any $O \in \mathcal{O}(2n)$ there exists a $O' \in \mathcal{SO}(2n)$ such that $O = (\oplus_{i=1}^{n-1}\mathbb{1} \oplus Z)O'$, we can allow for all orthogonal operations. That adjoining local ancillas enlarge the set of implementable unitaries is in contrast to the Gaussian bosonic states and also to systems consisting of qudits. Hence, the most general operation on a single mode can be written as $\bar{O} = Z^m O$, where $m \in \{0,1\}$, i.e., an arbitrary real orthogonal matrix. Clearly these operations no longer correspond to unitaries which are generated by quadratic Hamiltonians on the system modes alone [see Eq. (19)]. However, as they can be implemented using a quadratic Hamiltonian and ancillas in a Gaussian state, we consider them as GLUs and take them into account in the following. If it is, however, the case that a particle-number superselection rule would forbid these kind of transformations, it would be straightforward to slightly modify the results derived here to exclude any operation which is not of the form given in Eq. (19).

Let us also note here that the action of any Gaussian unitary in the Jordan-Wigner representation corresponds to a product of nearest neighbor match gates [44], which are unitaries of the form $U = U_e \oplus U_o$, where both U_e and U_o are 2×2 unitary operators acting on the even and odd subspace, respectively; and moreover, $|U_e| = |U_o|$.

A general Gaussian operator is any operator of the form $x = e^{i\sum_{i,j}\chi_{ij}\tilde{c}_i\tilde{c}_j}$ for a complex antisymmetric matrix χ.

In Ref. [43], the most general Gaussian maps have been characterized via the Choi-Jamiolkowski (CJ) isomorphism. Recall that a completely positive (CP) map is called Gaussian if it maps Gaussian states to Gaussian states. We reconsider in Appendix A the CJ isomorphism for GFS. It follows that a map $\mathcal{E}$ mapping n to m modes is Gaussian iff the corresponding CJ state is Gaussian (see also Ref. [43]), i.e., if it is given by the CM $E_\mathcal{E} = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}$, with A,B,D $2m \times 2m, 2m \times 2n$, and $2n \times 2n$ matrices, respectively (for more details, see Appendix A and also Ref. [43]).

Note that the condition for Gaussian maps to map *every* Gaussian state to a Gaussian is very stringent. Consider, for instance, the situation where one wants to transform the state $|00\rangle + |11\rangle$ into a state $\alpha|00\rangle + \beta|11\rangle$. Note that these are two-mode GFS in the Jordan-Wigner representation and such a transformation is always possible for two-qubit states via LOCC. The local operations accomplishing this transformation, i.e., $A_1 = \mathrm{diag}(\alpha,\beta)$, $A_2 = \mathrm{diag}(\beta,\alpha)$, are Gaussian; however, the map

$$\mathcal{E}(\rho) = A_1 \otimes \mathbb{1}(\rho) A_1^\dagger \otimes \mathbb{1} + X A_2 \otimes X(\rho) A_2^\dagger X \otimes X \tag{20}$$

is non-Gaussian even though both terms in the sum are. A simple example of a GFS that is not mapped to a GFS by $\mathcal{E}$ is the two-mode GFS $\rho = \rho_e \oplus \rho_o$, with $\rho_e = \begin{pmatrix} z_e + 1/4 & 0 \\ 0 & 1/4 - z_e \end{pmatrix}$, $\rho_o = \begin{pmatrix} z_o + 1/4 & x_o \\ x_o & 1/4 - z_o \end{pmatrix}$ for $z_e \leqslant 1/4$, $\sqrt{x_o^2 + z_o^2} \leqslant 1/4$ and $z_e^2 = x_o^2 + z_o^2, x_o \neq 0$. Note that $\mathcal{E}$ is FLOCC [i.e., it is a local map which maps FS to FS (as it preserves parity)] and would accomplish the desired transformation. Because of that, we consider in Sec. IV not only GLOCC but also the richer class of FLOCC.

III. SEPARABILITY OF GAUSSIAN FERMIONIC STATES AND OPERATIONS

Here we specialize the three definitions of separability of general FS presented in Ref. [16] to the case of GFS. We show that they do not all coincide even for Gaussian states and that one of them is not stable when considering multiple copies of a state. We show that one of the two remaining definitions of separability is also consistent with the desired property that any separable state can be generated by a local operation. Furthermore, we derive a standard form for mixed n-mode n-partite states into which any GFS can be transformed via GLU.

A. Mixed Gaussian fermionic separable states

The notion of entanglement is complicated for fermions (compared to bosons or qubits) due to superselection rules and anticommutation relations. The former enforces that all physical states have to commute with the parity operator but makes it impossible to characterize all states uniquely by local measurements of "physical" observables, i.e., those commuting with the parity operator. The latter implies that observables acting on different sites (disjoint sets of modes) do not, in general, commute.

In Ref. [16], several notions of product state and separable state were discussed for arbitrary FS, i.e., not necessarily Gaussian states. There, the set of physical states was defined as $\Pi := \{\rho : [\rho, P] = 0\}$, with P the parity operator. This gave rise to two notions of "product states": The set of physical states for which the expectation values of all products of physical observables factorize, i.e., $\rho(A_\pi B_\pi) = \rho(A_\pi)\rho(B_\pi)$, was denoted by $\mathcal{P}1_\pi$. $\mathcal{P}2_\pi$ ($\mathcal{P}2$) is the set of all states of the form $\rho = \rho_A \tilde{\otimes} \rho_B$ with (without) the parity restriction, respectively.

Then the three separable sets $\mathcal{S}1_\pi, \mathcal{S}2_\pi, \mathcal{S}2_{\pi'}$ can be defined via the convex hull of the different product sets together with the requirement that the final state commutes with the global parity. Specifically, $\mathcal{S}1_\pi = co(\mathcal{P}1_\pi)$, $\mathcal{S}2_\pi = co(\mathcal{P}2_\pi)$ and $\mathcal{S}2_{\pi'} = co(\mathcal{P}2) \cap \Pi$.

Let us now investigate these definitions further by considering GFS. In order to identify the set of separable GFS, one might want to define the separable states as those that are not useful for any quantum information task even if arbitrarily many copies of the state are given. Another reasonable choice would be to define the set of separable states to be those that can (at least asymptotically) be prepared by LOCC. The single-copy case [16] shows that these two notions do not coincide for fermions: $\mathcal{P}2$ contains states that cannot be prepared locally but the set $\mathcal{P}1_\pi$ of states that are not useful (considering only a single copy) is strictly larger. Before we focus on the first choice, i.e., on $\mathcal{P}2$, and show that the definition using $\mathcal{P}1_\pi$ can be ruled out, let us present some observations about these sets.

Observation 1. A GFS is in the set $\mathcal{S}2_\pi$ iff its covariance matrix takes direct-sum form.

This can be easily seen by noting that all states in $\mathcal{S}2_\pi$ are convex combinations of products of states that each commute with the local parity; i.e., all terms in the mixture have a CM that is block diagonal (and all first moments vanish), and hence, the CM of the mixture is also block diagonal. In contrast, even the states in $\mathcal{P}1_\pi$ can have nonzero correlations between A and B as stated in the next observation.

Observation 2. A state in $\mathcal{P}1_\pi$ can have nonzero correlations between A and B. However, in that case the block of the CM containing the correlations between A and B has at most *one* nonvanishing singular value.

For a proof of Observation 2, see Appendix B. An example of a Gaussian state, which is separable according to definition $\mathcal{S}1_\pi$ but not according to $\mathcal{S}2_\pi$ is the two-mode Gaussian state with CM

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It describes a state in which one (nonlocal and paired) mode is prepared in a pure Fock state and the other in the maximally mixed one. In general, we could consider the first mode to be in a (finite temperature) thermal state (e.g., being occupied with probability p); then

$$\gamma_p = (1 - 2p)\gamma_0.$$

The two modes are defined by the nonlocal $\mathcal{SO}(4)$ matrix

$$O = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which maps $O\gamma_0 O^T = [-J_2 \oplus 0_2]$. The mode operators of the transformed state are $(\tilde{c}_3, \tilde{c}_1)$ and $(\tilde{c}_2, \tilde{c}_4)$; i.e., the new annihilation operators are given in terms of the old ones as $b_1' = \frac{1}{2}[b_2 + b_2^\dagger + i(b_1 + b_1^\dagger)]$ and $b_2' = \frac{1}{2}[b_2 - b_2^\dagger - i(b_1 - b_1^\dagger)]$). It is readily checked that the vacuum for these two modes in the original basis is $|0_{b_1'} 0_{b_2'}\rangle = (|0_{b_1} 0_{b_2}\rangle + i|1_{b_1} 1_{b_2}\rangle)/\sqrt{2}$. Therefore, the mixed Gaussian state with CM γ_p is given by the mixture of $|0_{b_1'} 0_{b_2'}\rangle$ and $|0_{b_1'} 1_{b_2'}\rangle = (|0_{b_1} 1_{b_2}\rangle + i|1_{b_1} 0_{b_2}\rangle)/\sqrt{2}$ (each with probability $(1 - p)/2$) and $|1_{b_1'} 0_{b_2'}\rangle, |1_{b_1'} 1_{b_2'}\rangle$ (each with probability $p/2$).

Since the Fock states in the b_1', b_2' basis correspond up to a local phase gate to Bell states in the local basis, the state can be seen as being GLU equivalent to a Bell-diagonal state with entries $((1 - p)/2, (1 - p)/2, p/2, p/2)$ in the $(\Phi_+, \Psi_+, \Phi_-, \Psi_-)$ basis. For qubits, we would argue that for all p the density matrix is separable (the maximal overlap with a maximally entangled state is $\leqslant 1/2$). Formally, a Jordan-Wigner transformation maps the fermionic two-mode state ρ_{γ_p} to the separable (up to LU) Bell-diagonal two-qubit state described above. Is the GFS ρ_{γ_p} separable or entangled? As we show below, it does not behave as a separable state, when many copies are available and allows (at least for $p = 0$) even to distill pure singlets. Consequently, separability should be defined in a way that does not include these states. Note that this has already been shown for FS in Ref. [16]. The following theorem proves that the statement also holds for the restricted set of GFS.

Theorem 3. The set of Gaussian states in $\mathcal{S}1_\pi$ is not stable. That is, there exists a GFS, ρ such that $\rho \in \mathcal{S}1_\pi$ (even in $\mathcal{P}1_\pi$); however, $\rho \tilde{\otimes} \rho \notin \mathcal{S}1_\pi$.

Proof. Given two copies of a Gaussian state, ρ, with CM $\Gamma_\rho = \begin{pmatrix} \Gamma_A & C \\ -C^T & \Gamma_B \end{pmatrix}$ and rank $C = 1$, then the full state now has a rank-2 matrix C and therefore is no longer in $\mathcal{P}1_\pi$, since we can find a pair of local observables (commuting with local parity) for which the expectation value does not factorize. That is, assuming $(\Gamma_\rho)_{kl} \propto \rho(c_k c_l) \neq 0$ and using Wick's theorem implies $\rho^{\tilde{\otimes} 2}(c_k c_k' c_l c_l') = -\rho(c_k c_l)\rho(c_k' c_l') \neq 0$, where the primed operators refer to the second copy. Hence, $\rho \tilde{\otimes} \rho \notin \mathcal{S}1_\pi$. ■

This shows that any Gaussian state ρ for which $\rho^{\tilde{\otimes} n}(A_n B_n) = \rho^{\tilde{\otimes} n}(A_n)\rho^{\tilde{\otimes} n}(B_n) \forall A_n, B_n, n$ must have a CM $\Gamma_\rho = \Gamma_A \oplus \Gamma_B$. We are going to show next that $\rho^{\tilde{\otimes} 2}$ is not only no longer in the set $\mathcal{S}1_\pi$ but that it can also be useful for quantum information theoretical tasks.

Observation 4. Some states in $\mathcal{S}1_\pi$ can be useful for quantum information processing.

Given two copies of a Gaussian state with CM γ_0, we can use local Gaussian unitaries to transform it to the form (now written in 2×2 block form)

$$\begin{pmatrix} 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which now contains a pure, maximally entangled two-mode state in the first modes of A and B [32]. These states can be used for teleportation (though only probabilistically). This shows that $\mathcal{S}1_\pi$ is not a viable definition of separability.

Because of these observations, it is clear that one relevant set of separable states is defined via $\mathcal{S}2_\pi$. Hence, we use this definition in the following. In that case, the CM of any n-partite mixture of product states has direct-sum form, i.e., $\gamma = \oplus_i \gamma_i$. That is, a GFS is separable iff its CM is of that form. Moreover, this definition of separability is meaningful in the context of the generation of separable states, as all these states can be prepared locally. To be more precise, let us note that as separability does not have such a clear meaning for FS, as it has, e.g., in the bosonic or finite-dimensional case, and it is *a priori* not clear how separable maps ought to be defined. This is especially due to the fact that the set of separable maps (SEP) does not have a clear physical meaning. In contrast to that, LOCC transformations, even if restricted to certain local operations, such as (Gaussian) fermionic operations, are operationally defined. It is the set of transformations which can be implemented by local [(Gaussian) fermionic] operations assisted by classical communication. LOCC is strictly contained in SEP and is mathematically usually much harder to characterize. However, in a situation as here, where the definition of the larger set is not clear, one is forced to deal with LOCC. Hence, we consider in Appendix A FLOCC transformations and show that this leads to a natural choice of the definition of FSEP. Moreover, we show that any separable state (according to $\mathcal{S}2_\pi$) can be generated via a FLOCC transformation. Hence, the definition of separable states being those which are elements of $\mathcal{S}2_\pi$ meets all the necessary requirements. Note that it is, however, not clear if for every Gaussian state in $\mathcal{S}2_{\pi'}$ there exists a decomposition into physical product states, i.e., it is not clear whether for Gaussians the sets $\mathcal{S}2_\pi$ and $\mathcal{S}2_{\pi'}$ coincide or not (in general, they do not [16]). However, as mentioned above and as shown in Appendix A, all states which can be reasonably prepared locally must belong to $\mathcal{S}2_\pi$.

B. Gaussian fermionic separable operations

As mentioned in Sec. II C, the CJ isomorphism provides a one-to-one mapping between quantum states and quantum operations. Moreover, it has been shown in the finite-dimensional case that a map is separable; i.e., it can be written as a convex combination of local operators iff the corresponding CJ state is separable [49]. In Appendix A, we argue that the CJ state of a Gaussian separable map has a CM of the form

$$\Gamma_{AB} = \Gamma_A \oplus \Gamma_B, \tag{21}$$

with a natural generalization to more systems. As a consequence of the previous section, this state is a separable GFS according to $\mathcal{S}2_\pi$. Thus, this definition of separability agrees with the operational viewpoint that all separable states can be generated locally (an agreement which is not maintained for all definitions in the presence of superselection rules; see, e.g., Ref. [50]). Moreover, this definition can be naturally generalized to Gaussian separable maps (GSEP) (see Appendix A 2 for more details). As stated in the following lemma, fermionic completely positive maps (FCPM), i.e., CP maps that map FSs onto FSs, can be written in Kraus decomposition with special Kraus operators (see also Ref. [22]).

Lemma 5. All fermionic completely positive maps can be written using only Kraus operators with definite parity (i.e., that are either sums of only even monomials in the Majorana operators $\tilde{c}_i$ or sums of only odd monomials).

Proof. Let $\mathcal{E}$ denote a FCPM with Kraus operators $\{A_k\}$, i.e., $\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$ for all ρ. In general, the A_k are sums of even and odd terms, that is, $A_k = A_k^{(e)} + A_k^{(o)}$. FCPMs map FSs to (unnormalized) FSs; i.e., both ρ and $\mathcal{E}(\rho)$ are even (sums of even monomials in the Majorana operators $\tilde{c}_i$). Using the Kraus representation, this implies that $\sum_k A_k^{(e)} \rho (A_k^{(o)})^\dagger + A_k^{(o)} \rho (A_k^{(e)})^\dagger = 0$ for all ρ. Consequently, the FCPM $\tilde{\mathcal{E}}$ with Kraus operators $\{A_k^{(e)}, A_k^{(o)}\}$, which we denote by $\tilde{A}_k$ in the following, represents the same channel as $\mathcal{E}(\rho) = \tilde{\mathcal{E}}(\rho)$ for all FSs ρ. To show that $\sum_k \tilde{A}_k^\dagger \tilde{A}_k = \mathbb{1}$ on the whole state space, note that $\mathrm{tr}(Y\rho) = 1\ \forall \rho = \rho_e \oplus \rho_o$ iff $Y = \begin{pmatrix} \mathbb{1} & Y_{eo} \\ Y_{oe} & \mathbb{1} \end{pmatrix}$, where $Y_{eo} = P_e Y P_o$ ($Y_{oe} = P_o Y P_e$) and both the even and odd parts of Y have to be equal to the identity. Moreover, for $Y = \sum_k \tilde{A}_k^\dagger \tilde{A}_k = A_e \oplus A_o$ it follows immediately that $Y_{eo} = Y_{oe} = 0$. Thus, the Kraus operators of the FCPM $\tilde{\mathcal{E}}$ also satisfy $\sum_k \tilde{A}_k^\dagger \tilde{A}_k = \mathbb{1}$. ■

C. Standard form and GLU equivalence for *n*-mode *n*-partite states

Here, we consider n-mode n-partite fermionic systems. That is, each mode is spatially separated from the others. We derive a standard form $S(\gamma)$ into which the CM γ can be transformed via GLU. As the standard form is unique, we have that two GFS are GLU equivalent iff their CMs in standard form coincide.

Let us start by recalling that the most general GLU operation corresponds to an arbitrary real orthogonal matrix on each mode. Hence, the CM γ is transformed to

$$S(\gamma) = (\oplus_i Z^{m_i} O_i)\gamma\big(\oplus_i O_i^T Z^{m_i}\big), \tag{22}$$

via GLU with $O_i \in \mathcal{SO}(2,\mathbb{R})$ and $m_i \in \{0,1\}$. We denote in the following by γ_{jk} the 2×2 matrix describing the covariances between modes j and k. Because $\gamma = -\gamma^T$ we have for $i \leqslant n$

$$\gamma_{ii} = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} = \lambda_i J_2, \tag{23}$$

where $\lambda_i \in [-1,1]$. As $AJ_2A^T = |A|J_2$, for any 2×2 matrix A, γ_{ii} transforms to $Z^{m_i}\gamma_{ii}Z^{m_i} = (-1)^{m_i}\gamma_{ii}$. If $\lambda_i \neq 0$, we chose m_i such that $Z^{m_i}\gamma_{ii}Z^{m_i} = \lambda_i J_2$, where $\lambda_i > 0$. In case $\lambda_i = 0$, i.e., mode i is completely mixed, we show below how the bit value m_i can be uniquely defined. In order to uniquely define $O_i = e^{i\alpha_i Y}$ we proceed as follows. Consider the first index j with $i < j$ such that the off–diagonal matrix γ_{ij} is not vanishing. If the singular values of γ_{ij} are nondegenerate ($d_{ij} \neq |d'_{ij}|$), we define $O_i, O_j \in \mathcal{SO}(2,\mathbb{R})$ by $O_i \gamma_{ij} O_j^T = D_{ij} = \mathrm{diag}(d_{ij}, d'_{ij}), d_{ij} \geqslant |d'_{ij}|$ [51]. If the singular values of γ_{ij} are degenerate, γ_{ij} is itself proportional to an orthogonal matrix. In case $|\gamma_{ij}| > 0$, γ_{ij} is proportional to a special orthogonal matrix, $e^{i\alpha_{ij}Y}$. Then, we define $O_j \propto O_i \gamma_{ij}$; that is, we set $\alpha_j = \alpha_{ij} + \alpha_i$. In case $|\gamma_{ij}| < 0$, γ_{ij} is proportional to a matrix $Ze^{i\alpha_{ij}Y}$. Then, we define $O_j \propto Z O_i \gamma_{ij}$; that is, we set $\alpha_j = \alpha_{ij} - \alpha_i$. In all cases, $S(\gamma)_{ij}$ is diagonal. We proceed in the same way for γ_{ij+1} (and then any subsequent γ_{ik}). If α_j has already been determined in a previous step, α_k is determined by diagonalizing $\gamma_{jk}^T \gamma_{jk}$. More precisely, α_k is chosen such that $O_j \gamma_{jk} O_k^T = \tilde{O}_{jk} D_{jk}$ with $D_{jk} = \mathrm{diag}(d_{jk}, d'_{jk}), d_{jk} > |d'_{jk}|$ [52], $\tilde{O}_{jk} \in \mathcal{SO}(2,\mathbb{R})$, and $(\tilde{O}_{jk})_{11} \geqslant 0$ [for $(\tilde{O}_{jk})_{11} = 0$ choose α_k such that $(\tilde{O}_{jk})_{12} \geqslant 0$] [53]. If α_j has been expressed as a function of some other $\alpha_l, l < j$, which cannot be determined by the procedure explained before, then we fix α_l by diagonalizing $\gamma_{jk}\gamma_{jk}^T$ and imposing that the singular values are ordered nonincreasingly [54]. Note that if not both O_j and O_k depend on α_l, we can choose either $(O_j \gamma_{jk} O_k^T)_{11} > 0$ or if $(O_j \gamma_{jk} O_k^T)_{11} = 0$ we impose that $(O_j \gamma_{jk} O_k^T)_{12} \geqslant 0$. In case γ_{jk} is proportional to an orthogonal matrix, then either one relates O_k and α_l using the scheme explained before or O_k has already been related to α_l in a previous step. In the second case, either $O_j \gamma_{jk} O_k^T$ is independent of α_l or one chooses $O_j \gamma_{jk} O_k^T = \mathrm{diag}(|d_{jk}|, d_{jk})$.

It is easy to see that in this way any α_j is uniquely determined unless the CM is invariant under the conjugation with O_j, that is, the mode j is decoupled from all other modes, in which case we set $\alpha_j = 0$. At this point, all the operators which are no symmetry of the CM are determined. Those which leave the CM invariant can be chosen to be equal to the identity, e.g., if for 3-modes $\gamma_{12} = O_{12}$, $\gamma_{13} = O_{13}$, and $\gamma_{23} = O_{23}$ with $O_{ij} \in \mathcal{SO}(2,\mathbb{R})$, i.e., all of them are special orthogonal matrices and invariant under O_1, we choose $O_1 = \mathbb{1}$.

It remains to consider the case where $\lambda_i = 0$. If there is no index j such that $\gamma_{ij} \neq 0$, then the mode i factorizes and we set $m_i = 0$. Hence, let us assume that $\gamma_{ij} \neq 0$ for some j. We determine $m_i + m_j$ by requiring that $Z^{m_i} O_i \gamma_{ij} O_j Z^{m_j} = D_{ij}$ such that $\mathrm{tr}(D_{ij}) > \mathrm{tr}(Z D_{ij})$. In case m_j is determined by the condition on the transformed γ_{jj}, this determines m_i. Otherwise, there exists either a k such that either $\gamma_{ik} \neq 0$ or $\gamma_{jk} \neq 0$ or, the modes i and j factorize. In this case, the CM is invariant under the transformation $Z^{m_i} \oplus Z^{m_j}$ and we set $m_i = m_j = 0$. Note that if selection rules forbid the application of the operations Z to the individual modes, we simply set $m_i = 0\ \forall i$ in the derivation above.

In summary, we have shown that any GFS can be easily transformed into its standard form by applying GLU. As the standard form is unique, we have the following theorem.

Theorem 6. Any CM γ can be transformed into its standard form, $S(\gamma)$, by Gaussian local unitaries (GLUs). Two CMs γ and Γ are GLU equivalent if and only if $S(\gamma) = S(\Gamma)$.

As the CM determines uniquely the corresponding GFS, Theorem 6 presents a criterion for GLU equivalence of GFS.

Let us consider now some examples, where we explicitly compute the standard form for the CM. As mentioned above, we consider here n-mode n-partite systems, i.e., the $1 \times 1 \times \cdots \times 1$ case. Here, we compute the standard form of 2- and 3-mode states.

1. 1 × 1

Using the definition of the standard form introduced above, it is straightforward to see that any two-mode state CM can be written (up to GLU) as

$$S(\gamma) = \begin{pmatrix} 0 & \lambda_1 & d_{12} & 0 \\ -\lambda_1 & 0 & 0 & d'_{12} \\ -d_{12} & 0 & 0 & \lambda_2 \\ 0 & -d'_{12} & -\lambda_2 & 0 \end{pmatrix}, \tag{24}$$

with $\lambda_i > 0$ for $i \in \{1,2\}$ and $d_{12} \geqslant |d'_{12}|$ or $\lambda_i = 0$ and $\lambda_j \geqslant 0$ for $\{i,j\} = \{1,2\}$ and $d_{12} \geqslant d'_{12} \geqslant 0$. Imposing that the state is pure, i.e., that $\gamma\gamma^T = \mathbb{1}$ we obtain $\lambda_1 = \lambda_2 > 0$, $d_{12} = -d'_{12}$ and $d_{12}^2 + \lambda_1^2 = 1$ or $\lambda_1 = \lambda_2 = 0$ and $d_{12} = d'_{12} = 1$ (the maximally entangled state).

2. 1 × 1 × 1

Similar to above, one can identify the standard form for mixed states of three modes to be

$$S(\gamma) = \begin{pmatrix} 0 & \lambda_1 & d_{12} & 0 & l_1 d_{13} & l_2 d'_{13} \\ -\lambda_1 & 0 & 0 & d'_{12} & -l_2 d_{13} & l_1 d'_{13} \\ -d_{12} & 0 & 0 & \lambda_2 & m_1 & m_{12} \\ 0 & -d'_{12} & -\lambda_2 & 0 & m_{21} & m_2 \\ -l_1 d_{13} & l_2 d_{13} & -m_1 & -m_{21} & 0 & \lambda_3 \\ -l_2 d'_{13} & -l_1 d'_{13} & -m_{12} & -m_2 & -\lambda_3 & 0 \end{pmatrix},$$

where $\lambda_i, d_{ij}, d'_{ij}, l_i$, and m_i, m_{ij} are real parameters and $l_1^2 + l_2^2 = 1$. Thus, there are 12 free parameters characterizing the mixed GFS, which have to obey certain conditions, given in Appendix C.

Imposing the condition that the state is pure is more involved than in the case of two modes. Even though it is straightforward to derive this decomposition for the CM, we use the Jordan-Wigner representation of the states instead. In Sec. IV B, we show that any pure GFS is either of the from $|\Phi\rangle = a_1|000\rangle + a_2|011\rangle + a_3|101\rangle + a_4|110\rangle, a_i \in \mathbb{R}_{\geqslant 0} \forall i$, $\sum_{i=1}^4 a_i^2 = 1$ or of the form $X^{\otimes 3}|\Phi\rangle$. Note that without loss of generality $(a_3^2 + a_4^2 - a_1^2 - a_2^2) \geqslant 0$, $(a_2^2 + a_4^2 - a_1^2 - a_3^2) \geqslant 0$, $(a_2^2 - a_4^2 - a_1^2 + a_3^2) \geqslant 0$ (equivalent to non-negative λ_i). For *strict* inequalities and for $a_i \neq 0\ \forall i$ (i.e., the case of a generic CM without degeneracies), the standard form of the

CM is given by

$$S(\gamma)=2\begin{pmatrix} 0 & \lambda_1 & a_1a_4+a_2a_3 & 0 & 0 & -a_1a_3+a_2a_4\\ -\lambda_1 & 0 & 0 & -a_1a_4+a_2a_3 & -(a_1a_3+a_2a_4) & 0\\ -(a_1a_4+a_2a_3) & 0 & 0 & \lambda_2 & a_3a_4-a_1a_2 & 0\\ 0 & a_1a_4-a_2a_3 & -\lambda_2 & 0 & 0 & (a_3a_4+a_1a_2)\\ 0 & a_1a_3+a_2a_4 & -a_3a_4+a_1a_2 & 0 & 0 & \lambda_3\\ a_1a_3-a_2a_4 & 0 & 0 & -(a_3a_4+a_1a_2) & -\lambda_3 & 0 \end{pmatrix}, \tag{25}$$

with $\lambda_1=1/2(a_3^2+a_4^2-a_1^2-a_2^2)$, $\lambda_2=1/2(a_2^2+a_4^2-a_1^2-a_3^2)$, $\lambda_3=1/2(a_2^2-a_4^2-a_1^2+a_3^2)$. If the above stated conditions do not hold, a similar standard form can be derived. More precisely, if one of the a_i's is equal to zero, at least one of the off-diagonal blocks is degenerate and therefore, as explained above, the standard form looks slightly different. Note that as in the bosonic Gaussian case [55] and in contrast to the qubit case [56] it can be easily seen that the purities of the reduced states; that is, the λ_i's, uniquely define the state. Let us remark here that there exists only one GFS (up to GLU) with $\rho_i\propto\mathbb{1}$ for each subsystem i, namely $|000\rangle+|011\rangle+|101\rangle+|110\rangle$. Note that although it is known that any three-*qubit* state whose single-qubit reduced density operators are completely mixed is LU—equivalent to the GHZ state—this does not immediately imply the same for GFS due to the restriction to GLU.

IV. PURE GAUSSIAN FERMIONIC STATES AND LOCAL TRANSFORMATIONS FOR *n*-MODE *n*-PARTITE STATES

Let us now investigate in more detail the entanglement contained in *pure* GFS. For this purpose, we consider the class of Gaussian separable operations (GSEP). In general, SEP contains LOCC but is a strictly larger class [12,57–59]. We show, however, that for Gaussian operations on n-mode n-partite systems any transformation among pure fully entangled states via Gaussian SEP (GSEP) can be performed via GLU. Hence, in particular, only trivial Gaussian LOCC (GLOCC) transformations exist for single modes. Note that here and in the following we consider only fully entangled states, i.e., states where no subset of modes factorizes from the remainder. Because of the triviality of GLOCC, we study then Gaussian stochastic LOCC (GSLOCC) and certain fermionic LOCC (FLOCC, see Sec. IV C), which map FSs to FSs. We characterize the various GSLOCC classes, which are, in contrast to the bosonic case, indeed equivalence classes [60]. We then show that there exist nontrivial FLOCC transformations that map a pure GFS to some other pure GFS and demonstrate how to identify all possible transformations of that kind. Interestingly, many of the pure GFS belong to the maximally entangled set (MES) [10]. That is, they cannot be obtained from any other state via local deterministic transformations. For other states, we derive a very simple local protocol which can be used to reach the state from a state in the MES.

Let us first of all show that Condition (17), which is a necessary and sufficient condition for a FS to be also Gaussian, simplifies for pure FS (see also Ref. [61]).

Lemma 7. Let $|\Psi\rangle$ be a FS. Then $|\Psi\rangle$ is a GFS iff

$$\Lambda(|\Psi\rangle\otimes|\Psi\rangle)=0. \tag{26}$$

Proof. As mentioned before, an even operator, X is Gaussian iff $[\Lambda,X\otimes X]=0$ [see Eq. (15)]. As the projector onto a FS, $|\Psi\rangle$, is even and as a Hermitian rank-1 operator commutes with another Hermitian operator, such as Λ, iff the state in the range of the projector is an eigenstate of Λ we have that $|\Psi\rangle$ is GFS iff $\Lambda(|\Psi\rangle\otimes|\Psi\rangle)=a|\Psi\rangle\otimes|\Psi\rangle$ for some $a\in\mathbb{R}$. As $|\Psi\rangle$ has well-defined parity, we have that $\langle\Psi|c_i|\Psi\rangle=0$ for any operator c_i. Hence, $(\langle\Psi|\Lambda|\Psi\rangle)|\Psi\rangle=0=a|\Psi\rangle$. ■

A. Gaussian separable operations and Gaussian LOCC

Let us start with the investigation of GSEP transformations. As argued in Appendix A 2, GSEP is defined as the class of operations for which the CJ state is Gaussian and has a CM of the form $\Gamma=\oplus_{i=1}^n\Gamma_i$. We show here that any GSEP acting on n separated modes, which maps at least one pure (fully entangled) state into a different pure (fully entangled) state is a GLU transformation. Hence, no nontrivial state transformation is possible. The following lemma allows us to show in the end that GLOCC on pure states are trivial, as GSEP strictly includes GLOCC (see Appendix A 2).

Lemma 8. Let $\mathcal{E}_{\rm sep}$ denote a Gaussian trace-preserving separable map which transforms at least one pure n-partite n-mode FS, $|\Psi\rangle$, into another pure n-partite n-mode fully entangled FS, $|\Phi\rangle$. Then, it holds that $\mathcal{E}_{\rm sep}(\rho)=(U_1\tilde{\otimes}U_2\ldots\tilde{\otimes}U_n)\rho(U_1^\dagger\tilde{\otimes}U_2^\dagger\ldots\tilde{\otimes}U_n^\dagger)$ for all ρ.

Proof. Every separable Gaussian CP trace-preserving map (GCPTM) $\mathcal{E}_{\rm sep}$ has a separable Gaussian CJ state $E_{\mathcal{E}_{\rm sep}}$; i.e., $E_{\mathcal{E}_{\rm sep}}$ is of the form $\rho_1\tilde{\otimes}\rho_2\ldots\tilde{\otimes}\rho_N$, and, consequently, $\mathcal{E}_{\rm sep}=\mathcal{E}_1\tilde{\otimes}\mathcal{E}_2\ldots\tilde{\otimes}\mathcal{E}_N$ is a product operation with GCPTMs $\mathcal{E}_i$ (see Appendix A 2). Let us denote $\mathbb{1}\tilde{\otimes}_{k\neq1}\mathcal{E}_k(|\Psi\rangle\langle\Psi|)$ by ρ and write it in its spectral decomposition $\rho=\sum_i p_i|\Psi_i\rangle\langle\Psi_i|$. It follows from $\mathcal{E}_{\rm sep}(|\Psi\rangle\langle\Psi|)=|\Phi\rangle\langle\Phi|$ that $\sum_i p_i\mathcal{E}_1\tilde{\otimes}\mathbb{1}^{\tilde{\otimes}n-1}(|\Psi_i\rangle\langle\Psi_i|)=|\Phi\rangle\langle\Phi|$. Hence, it has to hold that for $p_i\neq0$ $\mathcal{E}_1\tilde{\otimes}\mathbb{1}^{\tilde{\otimes}n-1}(|\Psi_i\rangle\langle\Psi_i|)=|\Phi\rangle\langle\Phi|$ and therefore there exists at least one pure state $|\Psi_i\rangle$ for which

$$|\Phi\rangle\propto A_k\tilde{\otimes}\mathbb{1}^{\tilde{\otimes}n-1}|\Psi_i\rangle\propto A_l\tilde{\otimes}\mathbb{1}^{\tilde{\otimes}n-1}|\Psi_i\rangle, \tag{27}$$

where by A_j we denote the Kraus operators of $\mathcal{E}_1$. Note that $|\Psi_i\rangle$ has to be entangled in the splitting mode 1 versus the remaining modes as $|\Phi\rangle$ is entangled in this splitting and $\mathcal{E}_1$ cannot generate entanglement. Hence, considering $|\Psi_i\rangle$ in its Jordan-Wigner representation its Schmidt decomposition can be written as $|\Psi_i\rangle=\sum_{j=0}^1\lambda_j^i|j\rangle_1|\psi_j^i\rangle$ with $\lambda_0^i,\lambda_1^i\neq0$. Using this in Eq. (27) as well as that due to Lemma 5, the Kraus operators of $\mathcal{E}_1$ can be chosen such that each of them commutes with $|\psi_j^i\rangle\langle\psi_j^i|$ (which is a sum of only even monomials in the Majorana operators acting on the modes $2,\ldots,n$) it is easy to see that $A_k|j\rangle|\psi_j^i\rangle=cA_l|j\rangle|\psi_j^i\rangle$ for $j\in\{0,1\}$ and $c\in\mathbb{C}$ [62]. As the action of the different Kraus operators on a basis leads to the same states (up to a constant proportionality factor), we have that $A_k \propto A_l$. Moreover, as this holds true for all possible pairs of Kraus operators, one obtains from $\sum_i A_i^\dagger A_i = \mathbb{1}$ that $A_i^\dagger A_i \propto \mathbb{1}$ and hence the map $\mathcal{E}_1$ corresponds to the application of a GLU on mode 1. Rearranging of the modes such that mode j corresponds to the first mode and using the same argumentation as before shows that $\mathcal{E}_j$ is a GLU transformation on mode j for all j. Note that here we make use of the fact that the maps $\mathcal{E}_j$ commute with each other [63]. Hence, we can apply the $\mathcal{E}_j$ sequentially in any order. This implies that under rearranging the modes the product structure of the map $\mathcal{E}_{\rm sep}$ and the Kraus operators of the local maps $\mathcal{E}_j$ are preserved [64]. Hence, we have that $\mathcal{E}_{\rm sep}$ is a GLU transformation. ■

As mentioned above, Lemma 8 allows us to directly obtain the following corollary.

Corollary 9. There exists no nontrivial GLOCC operation mapping a pure n-mode n-partite FS $|\psi\rangle$ into another pure n-mode n-partite FS $|\phi\rangle$.

Let us note here that a very similar result has recently been proven for finite-dimensional Hilbert spaces [13,14]. There, it has been shown that generically, i.e., for a full-measure set of states, there exists no LOCC (even SEP) transformation, which transforms one pure (fully entangled) state into another, which is not LU equivalent. In strong contrast to the scenario considered here, the reason for that is, however, not that all separable maps are particularly restricted but that generically a state has no local symmetry. The relevance of local symmetries for local state transformation is recalled in Sec. IV C 1. Note, however, that in the qudit case, the result only holds generically and that there exists a zero-measure set of states which can be transformed via LOCC, whereas for FS the result holds for any state.

B. Gaussian stochastic LOCC

In the previous subsection, we have shown that GLOCC transformations among pure GFS are trivial. Thus, to quantify and qualify entanglement properties of pure GFS, we have to turn to a larger class of local operations. To that end, we now consider Gaussian stochastic LOCC (GSLOCC) [65].

As mentioned before, the most general Gaussian operation consists of attaching an auxiliary system by applying a Gaussian unitary to it and the system mode and measuring the auxiliary system in the Fock basis. Hence, the most general operations (in the $1 \times 1 \times \cdots \times 1$ case) are in the Jordan-Wigner representation of the form

$$D_1 X^{k_1} \otimes D_2 X^{k_2} \otimes \cdots \otimes D_n X^{k_n}, \tag{28}$$

where D_i are diagonal (with complex coefficients as $e^{i\alpha Z}$ is a GLU) and $k_i \in \{0,1\}$. Note that as before the X operators are possible because the parity of the system mode can be changed with the auxiliary system (for total parity-preserving operations, we have $k_i = 1$ for an even number of k_i's). Note, furthermore, that for a single mode the Gaussian operations coincide with the fermionic operations (see Sec. IV C). Given the fact that these are the most general Gaussian local operations, we have that two states can be transformed into each other via GSLOCC if there exists an invertible operator of the form given in Eq. (28) which transforms one state into the other (in the Jordan-Wigner representation). In particular, we have that GSLOCC is indeed an equivalence relation.

Before studying now the possible GSLOCC classes, let us introduce a standard form for FS. We consider a FS in Jordan-Wigner representation. Note again that as shown in Lemma 7 a pure FS is Gaussian iff $\Lambda(|\Psi\rangle \otimes |\Psi\rangle) = 0$. Using the standard form of FS explained below together with this condition, one obtains a characterization of the GSLOCC classes. We then present the different GSLOCC classes for up to four-mode GFS.

The following lemma states that by consecutive application of diagonal matrices any FS can be transformed into a normal form, which can, however, also vanish. For this, we need the notion of a *critical* state, i.e., a state whose single system reduced states are all proportional to the identity.

Lemma 10. Let $|\Psi\rangle$ be a fully entangled FS. Then $|\Psi\rangle$ can constructively (by applying invertible diagonal matrices) be transformed into a unique (up to LUs) critical state, $|\Psi_s\rangle$ (up to a proportionality factor $\lambda \in \mathbb{C}$ which can tend to 0).

Proof. The lemma follows from the normal form of multipartite states describing finite-dimensional systems presented in Ref. [66]. There, it has been shown that any state can be transformed via (a sequence of) local operations into a state whose single-system reduced state is completely mixed. In the algorithm presented in Ref. [66], which achieves this transformation, the local determinant 1 operations are $X_i^{(k)} = |\rho_i^{(k)}|^{1/(2d_i)}(\sqrt{\rho_i^{(k)}})^{-1}$, where d_i denotes the local dimension of system i and $\rho_i^{(k)}$ the reduced state of party i in the kth step of the algorithm. In order to apply this result to FS, note that the reduced state of a FS has to be fermionic and hence diagonal. Moreover, as local diagonal operators are fermionic operations (even Gaussian), each state during the algorithm is a FS. Hence, in each step k and for each party i, the operators $X_i^{(k)}$ are diagonal, which proves the statement. ■

The normal form of $|\Psi\rangle$ is given by $\lambda|\Psi_s\rangle$ (where λ can tend to 0).

Depending on the normal form, one can group states in the following three (disjoint) classes of states. (i) Stable states: These are states belonging to a SLOCC class which contains a critical state, which then is their normal form. Due to the Kempf-Ness theorem [67], there exists only one critical state in a SLOCC class (up to LUs). In the following, we will call this state seed state and denote it by $|\Psi_s\rangle$. That the normal form of any stable state is the corresponding seed state follows also from the Kempf-Ness theorem. The GHZ state, $\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, is an example of a critical and therefore a stable state. (ii) Semistable states: These are states that belong to a SLOCC class without critical state; The normal form of these states tends to a nonzero normal form. More precisely, it tends to a seed state of a different SLOCC class [66]. The four-qubit state $|\psi\rangle = a(|0000\rangle + |1111\rangle) + |0110\rangle + |0101\rangle$ is an example of a semistable state, whose normal form tends to the four-qubit GHZ state (see Ref. [66]). (iii) States in the null cone: The normal form of these states vanishes. An example of such a state is the W state.

In the Hilbert space $\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$, the union of stable states is of full measure and dense [8]. Hence, for almost all states the normal form is not vanishing. Whether the same holds true for FS is currently not clear. Despite this, we will focus now on stable FS. However, in the more detailed investigations of few-mode states, we will also consider semistable states and states in the null cone.

It follows straightforwardly from the lemma above that stable FSs can be written as $X^{m_1}D_1 \otimes X^{m_2}D_2 \otimes \cdots \otimes X^{m_n}D_n|\Psi_s\rangle$ with $|\Psi_s\rangle$ being critical. Note, however, that any GFS can be written as $X^{m_1}D_1 \otimes X^{m_2}D_2 \otimes \cdots \otimes X^{m_n}D_n|\Psi_f\rangle$, where $|\Psi_f\rangle$ is some representative (not necessarily critical) of the GSLOCC class and $m_i \in \{0,1\}$. This follows from the fact that the most general Gaussian operations are of the form $X^{m_1}D_1 \otimes X^{m_2}D_2 \otimes \cdots \otimes X^{m_n}D_n$. The subsequent corollary allows us to characterize the GSLOCC classes of stable GFS.

Corollary 11. Let $|\Psi\rangle$ be a stable FS and $|\Psi\rangle = D_1 \otimes D_2 \ldots \otimes D_n|\Psi_s\rangle$. Then, $|\Psi\rangle$ is GFS iff $|\Psi_s\rangle$ is GFS.

Proof. The "if" part follows from the fact that local diagonal matrices are Gaussian operations. The "only if" part can be seen as follows. Because of Lemma 7, we have that $|\Psi\rangle$ is GFS iff $\Lambda(|\Psi\rangle \otimes |\Psi\rangle) = 0$, which is equivalent to $\Lambda(|\Psi_s\rangle \otimes |\Psi_s\rangle) = 0$. Hence, $|\Psi\rangle$ is a GFS iff $|\Psi_s\rangle$ is. ■

An interesting example of a critical GFS state is the n-mode state $|\Psi\rangle = H^{\otimes n}|\mathrm{GHZ}\rangle$ [with $|\mathrm{GHZ}\rangle = 1/\sqrt{2}(|00\ldots0\rangle + |11\ldots1\rangle)$]. To see that $|\Psi\rangle$ is a GFS, note that $|\Psi\rangle \propto \sum_{\mathbf{k}\in\{0,1\}^n}[1+(-1)^{h(\mathbf{k})}]|\mathbf{k}\rangle$ with $h(\mathbf{k})$ being the Hamming weight of the bitstring $\mathbf{k}$. Therefore, $|\Psi\rangle$ is a FS. That $\Lambda(|\Psi\rangle \otimes |\Psi\rangle) = 0$ can be easily verified by direct computation. The fact that the state is critical follows from the criticality of the GHZ state. Note that the GHZ state itself is only a FS for even n. Moreover, the fermionic swap applied to any two modes of $|\Psi\rangle$ (or of any critical state) is also critical. As there exists only one critical state in a SLOCC class, this state is either LU equivalent to $|\Psi\rangle$ or in a different SLOCC class [68].

Let us now explicitly compute the GSLOCC classes of up to four-mode GFS.

1. 1 × 1 *case*

We start with the simplest case of pure two-mode, two-partite systems. First note that the spin representation of any FS of two modes is either of the form $|\Psi_1\rangle = \alpha|00\rangle + \beta|11\rangle$ or of the form $(\mathbb{1} \otimes X)|\Psi_1\rangle = \alpha|01\rangle + \beta|10\rangle$. As $|\Psi_1\rangle \propto D \otimes \mathbb{1}|\Phi_+\rangle$, where $|\Phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle$ denotes the critical seed state of two qubits and $D = \mathrm{diag}(\alpha,\beta)$, there is only one entangled GSLOCC class. It is easy to see that these states are all Gaussian, as $\Lambda(|\Phi^+\rangle \otimes |\Phi^+\rangle) = 0$ (see Lemma 7 and Corollary 11).

2. 1 × 1 × 1 *case*

For three-mode GFS, we denote by $|\mathrm{GHZ}\rangle_3$ the Gaussian fermionic GHZ state, i.e., $|\mathrm{GHZ}\rangle_3 = H^{\otimes 3}[1/\sqrt{2}(|000\rangle + |111\rangle)] = 1/2(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$. Note that we consider from now on only even-parity FS, as the odd ones are simply given by applying $X^{\otimes 3}$. We write an arbitrary pure three-mode (not normalized) FS as $|\Psi(a_1,a_2,a_3,a_4)\rangle = a_1|000\rangle + a_2|011\rangle + a_3|101\rangle + a_4|110\rangle$, $a_i \in \mathbb{C}\ \forall i$. Applying GLUs $(e^{\alpha_i Z})$ and choosing the global phase appropriately allows us to chose all the parameters a_i to be real and nonnegative. Using Lemma 7, it can be easily seen that they are all Gaussian. Then, the following lemma characterizes all three-mode GSLOCC classes.

Lemma 12. There are two three-mode entangled GSLOCC classes, the GHZ and the W class. The state $|\Psi(a_1,a_2,a_3,a_4)\rangle$ belongs to the GHZ class iff $a_i \neq 0\,\forall i$. It belongs to the W class iff there exists exactly one i such that $a_i = 0$. Moreover, the state is biseparable iff exactly two $a_i = 0$ (else it is separable).

Proof. First consider the case where $a_i \neq 0\ \forall i$. It can be easily seen that the state can be written as $D_1 \otimes D_2 \otimes D_3|\mathrm{GHZ}\rangle_3$ with D_i invertible and hence it belongs to the GHZ class. Let us denote by $|W\rangle_3 = 1/\sqrt{3}(|011\rangle + |101\rangle + |110\rangle)$ the W state. Then it is easy to see that any state $|\Psi(a_1,a_2,a_3,a_4)\rangle$ with exactly one i such that $a_i = 0$ can be written as $X^{k_1}D_1 \otimes X^{k_2}D_2 \otimes X^{k_3}|W\rangle_3$, where $k_1 + k_2 + k_3 = 0 \mod 2$ and D_i diagonal and invertible. If two coefficients vanish, the state can be written as $X^{k_1} \otimes X^{k_2}D \otimes X^{k_3}|0\rangle|\Phi^+\rangle$ (up to particle permutation), where $k_1 + k_2 + k_3 = 0 \mod 2$ and D is invertible, which proves the statement. ■

Note that this implies that a tripartite entangled three-mode GFS is of the form $D_1 \otimes D_2 \otimes D_3|\Psi_f\rangle_3$ (up to GLUs), where $|\Psi_f\rangle_3$ is either the GHZ or the W state and all D_i's are invertible. Hence, there exist, as in the qubit case, two fully entangled GSLOCC classes. The standard forms of the corresponding CM are given in Sec. III C. To give an example for the GHZ state with $a_i = 1/2\ \forall i$, the standard form is given in Eq. (25). A similar standard form for the W state $(a_1 = 0, a_2 = a_3 = a_4 = 1/\sqrt{3})$ can be determined. However, it is slightly different, as in this case $\gamma_{12},\gamma_{13},\gamma_{23} \in \mathcal{SO}(2,\mathbb{R})$ in Eq. (25).

3. 1 × 1 × 1 × 1 *case*

For four modes, it is no longer true that any pure FS is a GFS. In fact, from Lemma 7 one easily derives the following observation.

Observation 13. A pure four-mode FS, $|\Psi\rangle$ (in Jordan-Wigner representation) is Gaussian iff

$$\langle\Psi^*|(X \otimes Y \otimes X \otimes Y)|\Psi\rangle = 0, \tag{29}$$

where X,Y denote the Pauli operators.

This condition, which resembles the SL-invariant polynomials [66] defined for qubit states, is in fact equivalent to the condition that all reduced three-mode states of $|\Psi\rangle$ (taking the partial trace of one party) are Gaussian. An arbitrary four-mode (even parity) FS is given by $|\Psi\rangle = a_1|0000\rangle + a_2|0011\rangle + a_3|0110\rangle + a_4|1100\rangle + a_5|1010\rangle + a_6|0101\rangle + a_7|1001\rangle + a_8|1111\rangle$. It can be easily seen (analogously to the three-mode case) that any such state can be written as in the following lemma [69].

Lemma 14. A pure four-mode FS, $|\Psi\rangle$ can be written as

$$|\Psi\rangle = X^{k_1}D_1 \otimes X^{k_2}D_2 \otimes X^{k_3}D_3 \otimes X^{k_4}D_4|\Psi_f\rangle, \tag{30}$$

with $|\Psi_f\rangle$ an appropriate representative of each SLOCC class, $k_i \in \{0,1\}$ and $k_1 + k_2 + k_3 + k_4 = 0 \mod 2$. Moreover, the state is GFS iff the FS $|\Psi_f\rangle$ is.

The last conclusion follows directly from Corollary 11 as in the proof it has not been used that $|\Psi_s\rangle$ is critical and the local $X^{k_i}D_i$ are Gaussian operations. Note that as in the three-mode case, some GSLOCC classes contain a critical state, whereas others do not. Moreover, in the four-mode case, there also exist semistable states, i.e., states that tend to a nonvanishing normal form, even though they are not stable. Let us state the different GSLOCC classes now in more detail based on the results on four-qubit SLOCC classes in [7]:

(1) GSLOCC classes containing a critical state.
These are states from the SLOCC classes G_{abcd} ([7]), with representatives

$$|\Psi_f\rangle = a|\Phi^+\rangle^{\otimes 2}+b|\Phi^-\rangle^{\otimes 2}+c|\Psi^+\rangle^{\otimes 2}+d|\Psi^-\rangle^{\otimes 2}. \quad (31)$$

Note that the states $|\Psi_f\rangle$ are critical. Because of Observation 13, we can easily see that the FS in Eq. (31) are Gaussian iff $ab + cd = 0$. Hence, either two or three of the parameters of $|\Psi_f\rangle$ can vanish, according to this necessary and sufficient condition. Whereas the states where two of the four parameters are equal to zero are still four-partite entangled, states with three parameters being equal to zero are biseparable states.

(2) GSLOCC classes containing semistable states.
As mentioned above, there exist classes that contain semistable states (see Ref. [70] for results on semistable four-qubit states). The SLOCC classes containing four-mode entangled GFS are L_{abc_2} and $L_{a_2b_2}$ (see Ref. [7]) with representatives

$$\begin{aligned}|\Psi_f(abc_2)\rangle &= \frac{a+b}{2}(|0000\rangle+|1111\rangle)+\frac{a-b}{2}(|0011\rangle \\ &\quad +|1100\rangle)+c(|0101\rangle+|1010\rangle)+|0110\rangle,\\ |\Psi_f(a_2b_2)\rangle &= a(|0000\rangle+|1111\rangle)+b(|0101\rangle+|1010\rangle)\\ &\quad +|0110\rangle+|0011\rangle. \end{aligned} \quad (32)$$

Note that neither $|\Psi_f(abc_2)\rangle$ nor $|\Psi_f(a_2b_2)\rangle$ are critical. Using Lemma 14 and Observation 13, we find that the FS are also Gaussian iff either $ab = -c^2$ for states in L_{abc_2} or $a^2 + b^2 = 0$ for states in $L_{a_2b_2}$. Note that if all of the parameters of a state in L_{abc_2} ($L_{a_2b_2}$) are equal to zero, the state is a product state (biseparable state) respectively.

(3) GSLOCC classes containing states in the null cone.
The states in the null cone are the ones for which the normal form vanishes. For four-mode GFS, there exists, as in the three-mode case, exactly one GSLOCC class containing these states, which is the class L_{ab_3} of Ref. [7] with $a = b = 0$. The representative is of the form

$$|\Psi_f\rangle = |1100\rangle + |1111\rangle + |1010\rangle + |0110\rangle. \quad (33)$$

This state is Gaussian and GLU equivalent to the four-qubit W state.

Hence, for four-mode GFS, there exist infinitely many entangled GSLOCC classes. More precisely, there are infinitely many GSLOCC classes that contain a critical state; that is, the states in these classes can be transformed into the normal form. Furthermore, there exist infinitely many GSLOCC classes of semistable states, which tend to a nonzero normal form without being stable. There exists also a single GSLOCC class containing states in the null cone for which the normal form vanishes.

There are less GSLOCC classes for four-mode GFS than there are for FS, which is not surprising as not all FS are GFS, due to the condition given in Eq. (26) on $|\Psi_f\rangle$. This also implies that there exist less GSLOCC classes than SLOCC classes in the qubit case (see Ref. [7]). However, as mentioned above, there are still infinitely many such classes. Examples of SLOCC classes that contain FS but no GFS are those denoted by $L_{a_2 0_{3\oplus 1}}$ in Ref. [7] for $a \neq 0$ [71].

C. Fermionic LOCC operations

As for transformations of pure n-mode n-partite GFS, there exist no nontrivial GLOCC transformations, we consider here a larger class of deterministic transformations and study fermionic LOCC (FLOCC) transformations. For such transformations, the local maps that are applied have to be fermionic and the measurement operators that are implemented in each round have to be parity-respecting and local; i.e., they have to be of the form $X^k D$ (in Jordan-Wigner representation), where $k \in \{0,1\}$ and D denotes here and in the following a diagonal matrix [72]. More precisely, in each round of an FLOCC transformation, one party implements locally a fermionic POVM measurement with measurement operators that are of the form $X^k D$, possibly discards some classical information about the outcome, and then communicates the relevant information to the other parties. These apply depending on the measurement outcome an arbitrary local completely positive trace-preserving (CPT) fermionic map. Note that the Kraus operators of such maps can be chosen to be of the form $X^k D$ (cf. Lemma 5). Note further that the operations that are implemented in a subsequent round might depend on the information about the prior outcomes.

For a concatenation of finitely many of such rounds, the Kraus operators of the map that is implemented in each branch of the protocol, i.e., for a specific sequence of outcomes (taking into account that some information might have been discarded), are of the form $X^{k_1} D_1 \otimes X^{k_2} D_2 \otimes \cdots \otimes X^{k_n} D_n$. This can be easily seen as a finite product of operators of this form results in an operator of the same form.

In order to provide a rigorous definition of FLOCC protocols which can also involve infinitely many rounds (in analogy to the one given in Ref. [59] for LOCC protocols), let us use the description of a protocol in terms of a quantum instrument, i.e., by the family of CP maps $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$. Here, $\mathcal{E}_i$ is the CP map that is implemented in a specific branch of the protocol denoted by i and it holds that $\sum_{i=1}^m \mathcal{E}_i$ is a trace-preserving map. Moreover, a quantum instrument $\mathcal{P}$ will be called FLOCC linked to an instrument $\tilde{\mathcal{P}}$ if $\mathcal{P}$ can be implemented by first implementing $\tilde{\mathcal{P}}$ followed by exactly one more round of an FLOCC protocol as defined before (where again the operations that are implemented in each branch i can depend on all previous outcomes) and then possibly by some discarding of classical information. With all that, $\mathcal{F}$ is defined as the instrument of a FLOCC transformation if there exists a sequence of instruments of finite-round FLOCC protocols where each element of the sequence is FLOCC linked to its preceding element. Furthermore, for each element there exists a way to discard information in the final round such that the resulting sequence of instruments converges to $\mathcal{F}$. In the following, we consider also infinite-round FLOCC, however, only those for which all Kraus operators are of the form $X^{k_i} D_i$. In order to highlight that there might be a difference to FLOCC as defined above, we denote this set of operations by FLOCC′. Note that, of course, any finitely-many-rounds FLOCC is contained in FLOCC′. We are interested in FLOCC′ transformations among pure GFS and, in particular, in the maximally entangled set for this scenario. We first review the concept of the maximally entangled set and then explain how it can be determined for GFS when one considers FLOCC′ transformations.

1. The maximally entangled set

In Ref. [10], some of us introduced the maximally entangled set (MES) as the minimal set of n-partite entangled states that has the property that any pure n-partite entangled state can be obtained via LOCC from a state within this set. That is, the states in the MES are those which cannot be reached via LOCC from some state that is not LU equivalent. In Ref. [55], GLOCC transformations among Gaussian states of two or three bosonic modes have been considered. There, it has been shown that not all pure bosonic three-mode Gaussian states can be obtained via GLOCC from a symmetric Gaussian state; i.e., the MES of bosonic three-mode Gaussian states under GLOCC cannot consist only of symmetric Gaussian states. In the following, we are interested in the MES of GFS under FLOCC′. It is defined analogously to before as the minimal set of n-partite n-mode GFS for which it holds that any pure n-partite n-mode entangled GFS can be obtained via FLOCC′ from a state within this set.

As we explain in the next section using the Jordan-Wigner representation, FLOCC′ reachability of GFS can be studied in a way analogous to qubit systems. There, we used the necessary and sufficient conditions of convertibility via separable maps (SEP) of Ref. [73] to identify the states that cannot be reached via SEP from a state that is not LU equivalent. As separable maps (strictly) include LOCC transformations it follows that these states are not reachable via LOCC. We outline here the basic idea of the proof of the necessary and sufficient condition derived in Ref. [73] for qudits in order to explain how this result can also be applied to study FLOCC′ transformations of GFS.

The initial state of the transformation is denoted by $g|\Psi_s\rangle$ and the final state by $h|\Psi_s\rangle$, where g,h are invertible local operators [74]. In order to perform this transformation, it has to hold for all the Kraus operators of the separable map, $A_i = A_i^{(1)} \otimes A_i^{(2)} \otimes \cdots \otimes A_i^{(n)}$, that $A_i g|\Psi_s\rangle \propto h|\Psi_s\rangle$ and therefore $(h^{-1}A_i g)|\Psi_s\rangle \propto |\Psi_s\rangle$. Using the definition for the local symmetries of a state $S_{|\Psi\rangle} = \{S : S|\Psi\rangle = |\Psi\rangle, S = S^{(1)} \otimes S^{(2)} \otimes \cdots \otimes S^{(n)}, S^{(j)} \in GL(d_j,\mathbb{C})\}$, where d_j denotes the local dimension of system j, we have that $h^{-1}A_i g \propto S_i$ where $S_i \in S_{|\Psi_s\rangle}$. That is, the measurement operators A_i are proportional to $hS_i g^{-1}$. Taking into account the proper proportionality factors and using that the separable map has to be trace-preserving, one obtains the following necessary condition for transforming $g|\Psi_s\rangle$ into $h|\Psi_s\rangle$ via SEP. There has to exist a probability distribution $\{p_i\}_{i=1}^m$ and local symmetries $S_i \in S_{\Psi_s}$ such that [73]

$$\sum_{i=1}^{m} p_i S_i^\dagger H S_i = rG, \tag{34}$$

where $H = h^\dagger h$, $G = g^\dagger g$, and $r = \frac{\langle\Psi_s|H|\Psi_s\rangle}{\langle\Psi_s|G|\Psi_s\rangle}$. Moreover, it is straightforward to see that this condition is also sufficient [73]. Using this criterion, one can determine the states that are not reachable via a SEP transformation and hence not via LOCC.

In the subsequent subsection, we discuss how one can in an analogous way obtain necessary and sufficient conditions for transformations among pure GFS via CPT maps with local fermionic Kraus operators.

2. The maximally entangled set of GFS under FLOCC′

As mentioned before, the MES of GFS under FLOCC′ corresponds to the minimal set of n-partite n-mode GFS with the property that any pure n-partite n-mode entangled GFS can be obtained via FLOCC′ from a state within this set. Hence, this set corresponds to the optimal resource under the restriction to pure GFS and FLOCC′ transformations. As we will see, it can be determined using a similar method as has been employed to characterize the MES for three- and four-qubit states. In particular, using the Jordan-Wigner representation, one can find analogously to the qudit case [73], which we reviewed in the previous subsection, the necessary and sufficient condition for transformations among GFS via separable maps whose Kraus operators are of the form $X^{m_1}D_1 \otimes X^{m_2}D_2 \otimes \cdots \otimes X^{m_n}D_n$, with $m_i \in \{0,1\}$ and where D_i is diagonal. Note that this class of separable maps includes all FLOCC′ transformations, as all local fermionic operators can be written like that (in Jordan-Wigner representation).

Before proceeding to study the separable maps, let us briefly recall the relation between the operator $X^{m_i}D_i$ in Jordan-Wigner representation and the Majorana operators. $X^{m_i}D_i$ corresponds to a sum of monomials of even ($m_i = 0$) or odd ($m_i = 1$) powers in the Majorana operators and hence it either commutes or anticommutes with the application of $X^{m_j}D_j$ for $j \neq i$. Note that as X_i (in Jordan-Wigner representation) corresponds in the Majorana operators to $(-i\tilde{c}_1\tilde{c}_2)(-i\tilde{c}_3\tilde{c}_4)\ldots(-i\tilde{c}_{2i-3}\tilde{c}_{2i-2})\tilde{c}_{2i-1}$, it follows that despite the fact that this operator is acting locally on the modes it is not only acting on mode i. Its implementation requires also other parties to apply a local unitary. Any diagonal matrix D_i can be written in the Majorana operators (up to a proportionality factor) as $e^{-i\alpha\tilde{c}_{2i-1}\tilde{c}_{2i}}$ for some $\alpha \in \mathbb{C}$ and therefore only acts on mode i.

In the previous subsection, we have seen that all Kraus operators A_i of a separable map transforming $g|\Psi_s\rangle$ to $h|\Psi_s\rangle$ have to be proportional to $hS_i g^{-1}$. Recall that S_i denotes a local symmetry of $|\Psi_s\rangle$. As for the transformations we are interested in, the operators h,g and the Kraus operators A_i are local fermionic operators, and this implies that also any symmetry $S_i \propto h^{-1}A_i g$ that contributes to the transformation is of the form $X^{m_1}D_1 \otimes X^{m_2}D_2 \otimes \ldots \otimes X^{m_n}D_n$. Hence, only symmetries of this form appear in the necessary and sufficient condition given by Eq. (34) [75] if one considers transformations among GFS via the considered class of separable maps.

Thus, the local symmetries that can contribute to such transformations are a subset of the local symmetries that are available for transformations among qubit states. It follows straightforwardly that if the qubit state corresponding to the GFS (in Jordan-Wigner representation) is not reachable via a nontrivial SEP transformation, then the GFS is not reachable via a separable map with the specific form of Kraus operators that we impose. Moreover, as exactly the same methods can be applied that we used to determine the MES for three and four qubits, one can infer from these results the MES for three- and four-mode GFS under FLOCC′ [76].

In Refs. [77,78], finite round LOCC transformations among pure n-qudit states have been investigated. Restricting the measurement operators, local unitaries, and SLOCC operators to local fermionic operators, one can use an analogous argumentation to obtain the corresponding results for finite-round FLOCC transformations among GFS. In the following subsections, we discuss explicitly the MES for three- and four-mode GFS under FLOCC′.

3. **1 × 1 × 1** *case*

As shown in Ref. [10], the MES of three-qubit states is given (up to LUs) by

$$\{D_1 \otimes D_2 \otimes D_3 |\mathrm{GHZ}\rangle_3, |\mathrm{GHZ}\rangle_3, D_1 \otimes D_2 \otimes \mathbb{1} |W\rangle\}, \quad (35)$$

where for the GHZ class none of the D_i's is proportional to the identity and all of them are real and invertible. Note that all these states are Gaussian and it follows directly that these states also have to be in the MES of three-mode GFS. As any GFS in the W class can be written (up to GLUs) as given in Eq. (35), we have that any tripartite entangled three-mode GFS is either in the MES or it is of the form $D_1 \otimes D_2 \otimes \mathbb{1}|\mathrm{GHZ}\rangle_3$, where at least one D_i is not proportional to the identity (up to GLUs and particle permutations). In the first case, the state cannot be reached from any other state (even if one would allow the most general LOCC transformation). In the second case, it can be easily reached from the GHZ state with the following FLOCC′ protocol. Party 1 applies the measurement consisting of the measurement operators $D_1, D_1 X$ and party 2 applies a measurement consisting of the measurement operators $D_2, D_2 X$[79]. Hence, the resulting state is $D_1 X^{k_1} \otimes D_2 X^{k_2} \otimes \mathbb{1}|\mathrm{GHZ}\rangle_3$. Using that $X^{k_1} \otimes X^{k_2} \otimes X^{k_1+k_2}|\mathrm{GHZ}\rangle_3 = |\mathrm{GHZ}\rangle_3$, we have that if party 3 applies the GLU $X^{k_1+k_2}$ the resulting state is for any outcome the desired state and hence the transformation is deterministic.

4. **1 × 1 × 1 × 1** *case*

The four-mode case is very similar to the previously discussed three-mode case. In order to illustrate this, let us consider a few examples of possible transformations among four-mode GFS via FLOCC′. Note that we consider here only GSLOCC classes with nondegenerate and noncyclic seed states as in Eq. (31) [80]. Because of Lemma 14, any four-mode GFS with a seed state of the above form is either a state in the MES (see Ref. [10]) or of the form (up to permutations) $|\Psi\rangle = D_1 \otimes \mathbb{1}^{\otimes 3}|\Psi_s\rangle$. If the state is in the MES, it cannot be reached by any other state (even if LOCC would be allowed). Moreover, apart from the seed states, all other states in the MES are isolated; i.e., they cannot be transformed into any other state via FLOCC′. Note that this is in contrast to the qubit case, where the states in Eq. (30) are states in the MES that are nonisolated; i.e., they can be transformed into a state with exactly one local nondiagonal operator (see Ref. [10]) via LOCC. These states are, however, no GFS. In case the four-mode GFS is not in the MES, it can be easily reached from the GFS seed state via the following FLOCC′ protocol (for more sophisticated protocols see below). Party 1 applies the measurement consisting of the measurement operators $D_1, D_1 X$. In the case of the first outcome, the other parties do not need to apply any transformation. In the case of the second outcome, all three apply X to their systems. Because the seed state is invariant under $X^{\otimes 4}$, it can be easily seen that the transformation can be achieved deterministically.

Note that for certain GSLOCC classes more transformations are possible (see Ref. [11]). For instance, if the seed parameters fulfill $a = b$, $c = d$, and $c = ia$, that is, they do not fulfill the above stated conditions, the seed state has more symmetries. As can be easily seen, this implies that the seed state can be, for example, transformed into states of the form $\mathbb{1} \otimes D_2 \otimes D_3 \otimes \mathbb{1}|\Psi_s\rangle$. The corresponding FLOCC′ protocol is given by party 2 applying the measurement operators $D_2, D_2 X$ and party 3 applying the operators $D_3, D_3 X$. Using that the seed state is invariant under $Y \otimes \mathbb{1} \otimes X \otimes Z$ and $Z \otimes X \otimes \mathbb{1} \otimes Y$, it is easy to see that the protocol can be implemented deterministically.

V. PURE GAUSSIAN FERMIONIC STATES AND LOCAL TRANSFORMATIONS FOR MULTIMODE STATES

In this section, we consider pure N-partite GFS where each party i holds m_i modes. We first investigate transformations among fully entangled multimode GFS (for the definition see below) via Gaussian trace-preserving separable transformations (GSEP), i.e., Gaussian transformations for which the CM of the CJ state is of direct sum form. We show that also in this more general setting such transformations are only possible if the map is a GLU transformation. As GSEP includes GLOCC transformations (see Appendix A 2), this implies that any GLOCC transformation that is possible among pure fully entangled GFS can be implemented via GLUs. Hence, as before we consider larger classes of operations, namely probabilistic transformations and FLOCC′ transformations. More precisely, we briefly explain how the GSLOCC classes can be characterized in the multimode case for classes which contain a critical state. We conclude this section by briefly discussing nontrivial FLOCC′ transformations among pure multimode GFS.

A. Gaussian separable transformations

We investigate Gaussian separable transformations (GSEP) among pure fully entangled multimode states, i.e., multimode FS with the property that the Schmidt decomposition (of the state in its Jordan-Wigner representation) with respect to the splitting of one party versus the rest has no zero Schmidt coefficients. As stated in the following lemma, we show that such transformations are only possible if the map corresponds to applying GLUs.

Lemma 15. Let $\mathcal{E}_{\mathrm{sep}}$ denote a Gaussian trace-preserving separable map which transforms at least one pure fully entangled $m_1 \times m_2 \times \cdots \times m_N$-mode FS, $|\Psi\rangle$, into another pure fully entangled $m_1 \times m_2 \times \cdots \times m_N$-mode FS, $|\Phi\rangle$. Then, it holds that $\mathcal{E}_{\mathrm{sep}}(\rho) = (U_1 \tilde{\otimes} U_2 \otimes \cdots \tilde{\otimes} U_N)\rho(U_1^\dagger \tilde{\otimes} U_2^\dagger \otimes \cdots \tilde{\otimes} U_N^\dagger)$ for all ρ.

Note that this lemma holds as in the n-partite n-mode case for all FS (not only GFS).

Proof. This lemma can be shown using an analogous argumentation as in the proof of Lemma 8. We recall here the main steps of the proof and comment on its generalization to the multimode case. As argued in Appendix A 2, Gaussian separable maps correspond to product operations; i.e., they are of the form $\mathcal{E}_{\text{sep}} = \mathcal{E}_1 \tilde{\otimes} \mathcal{E}_2 \tilde{\otimes} \cdots \tilde{\otimes} \mathcal{E}_N$ with GCPTMs $\mathcal{E}_i$ which act now on m_i modes. Analogously to the case of a single mode per site, we consider $\rho = \mathbb{1} \tilde{\otimes}_{k \neq 1} \mathcal{E}_k(|\Psi\rangle\langle\Psi|)$ with spectral decomposition $\sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. As before, it follows straightforwardly that for $p_i \neq 0$

$$|\Phi\rangle \propto A_k \tilde{\otimes}_j \mathbb{1}_{m_j} |\Psi_i\rangle \propto A_l \tilde{\otimes}_j \mathbb{1}_{m_j} |\Psi_i\rangle, \tag{36}$$

where the operators A_t are the Kraus operators of $\mathcal{E}_1$ and $\mathbb{1}_{m_j}$ denotes here the identity on m_j modes. We show next that there exists a Schmidt decomposition of the Jordan-Wigner representation of $|\Psi_i\rangle$ in the splitting of the first m_1 modes versus the remaining modes such that all involved local (with respect to that splitting) states are fermionic. In order to do so, note that the reduced state of the first m_1 modes has to be fermionic and therefore the range of the reduced state is spanned by FSs. Hence, any purification of this state (in particular $|\Psi_i\rangle$) is given by $\sum_{j=1}^{2^{\min(m_1,n_1)}} \lambda_j |\eta_j\rangle|\nu_j\rangle$, where $|\eta_j\rangle$ are orthogonal FSs of m_1 modes. That the $n_1 \equiv \sum_{j=2}^{N} m_j$-mode states $|\nu_j\rangle$ are also fermionic follows from the facts that the projector onto the states $|\eta_j\rangle$ are fermionic operators (as they are sums of only even monomials in the Majorana operators) and that $|\Psi_i\rangle$ is a FS. Moreover, as the final state $|\Phi\rangle$ is fully entangled, all Schmidt coefficients of $|\Psi_i\rangle$ have to be unequal to zero [see Eq. (36)]; i.e., $\lambda_j \neq 0\ \forall j \in \{1,\ldots,2^{\min(m_1,n_1)}\}$.

Analogous to the case of a single mode per site, one can now apply $|\nu_j\rangle\langle\nu_j|$ on both sides of Eq. (36) in order to see that the action of A_k on a basis is the same (up to a proportionality factor) for all Kraus operators A_k and hence $\mathcal{E}_1$ is a Gaussian unitary operation. Rearranging the modes [81] and applying the same argumentation for the various parties proves the lemma.

As GSEP is defined such that it includes all GLOCC transformations (see Appendix A 2), this lemma implies that nontrivial GLOCC transformations among pure fully entangled GFS are not possible even if one considers the case of an arbitrary (finite) number of modes per site. Hence, in the following section we will consider probabilistic local transformations and comment on the characterization of the GSLOCC classes for multimode states.

B. Gaussian stochastic LOCC

As deterministic transformations are not possible among pure fully entangled GFS, we will consider next stochastic GLOCC operations. We distinguish between bipartite and multipartite GFS, as in Ref. [39] a decomposition for bipartite states was introduced. For multipartite states, we show similar to the single-mode per site case that stable states can be brought into a normal form.

1. Bipartite case

For bipartite pure multimode states, i.e., party A (B) holds d_1 (d_2) modes respectively, it was shown in Ref. [39] that one can consider without loss of generality two subsystems of d modes each, where $d = \min(d_1,d_2)$, that is the two parties hold the same number of modes. Thus, we only consider $d \times d$ states here. A direct consequence of the results obtained in Ref. [39] is the following observation for bipartite multimode GSLOCC classes.

Observation 16. For $d \times d$ modes (GFS), there exist d different GSLOCC classes.

Proof. This can be easily shown by using that any such state is up to GLU equivalent to $\otimes_{i=1}^{d} |\Psi_i\rangle_{AB}$, with $|\Psi_i\rangle_{AB} = \cos\theta_i |00\rangle_{AB} + \sin\theta_i |11\rangle_{AB}$ [39]. Thus, A and B share d two-mode states $|\Psi_i\rangle_{AB}$, which are entangled for $\theta_i \neq 0,\pi/2$. Moreover, each GSLOCC class is characterized by the local rank of the states (the rank of the reduced states ρ_A,ρ_B does not increase under GSLOCC) [82] and, hence, we immediately arrive at the above stated result. ■

Thus, there exist as many GSLOCC classes for bipartite GFS as SLOCC classes for bipartite qudit states.

2. Multipartite case

Analogously to the case of a single mode per site, one can transform any multimode FS into a normal form by consecutively applying fermionic local invertible operators. Note again that this normal form vanishes for states in the null cone. Moreover, there exist semistable states that tend to a nonzero normal form but their SLOCC class does not contain a critical state [8].

Lemma 17. Let $|\Psi\rangle$ be an entangled $m_1 \times m_2 \times \cdots \times m_N$-mode FS. Then $|\Psi\rangle$ can be constructively transformed (by applying invertible fermionic operators) into a unique (up to LUs) critical FS, $|\Psi_s\rangle$ (up to a proportionality factor which can tend to 0).

The lemma can be proven by using the same argumentation as in the case of a single mode per site (see Lemma 10). Note that the only difference is that the local invertible operators, i.e., the reduced states, are no longer diagonal and thus not automatically also Gaussian. However, they are general fermionic operators. Note, furthermore, that any GSLOCC class containing a critical state can be easily characterized via this state. That is, if $|\Psi_s\rangle$ is a critical GFS then any other state $|\Psi\rangle$ in the same GSLOCC class is given by $M_1 \otimes M_2 \otimes \cdots \otimes M_n |\Psi_s\rangle = |\Psi\rangle$. Here, the operators M_i are Gaussian invertible operators.

C. Fermionic LOCC

Transformations among fully entangled multimode GFS via FLOCC′ [83] can be characterized analogously to the n-mode n-partite case. Note, however, that in this setting there is an additional freedom when one considers transformations to not fully entangled states. Similar to the finite-dimensional qudit case and contrary to the single-mode case, it is possible to reduce the local rank of the parties via FLOCC′, leaving still all parties entangled with each other.

VI. CONCLUSION

We investigated the entanglement of GFS. For this purpose, we first derived a standard form of the CM for mixed n-mode n-partite GFS. Any CM can be brought into this standard form via GLU. As the standard form is unique, any two GFS are GLU equivalent iff their CMs in standard form coincide. Furthermore, we showed that only two of the definitions of separable FS from Ref. [16] are reasonable for GFS. This is because any separable state should have the property that also two copies of this state are again separable. For our derivations, we used the definition of separability which declares a state separable if it is given by a convex combination of product states which commute with the local parity operator. According to this physically meaningful definition, any separable state can be prepared locally. Using this definition, we showed that for pure fully entangled n-mode n-partite as well as multimode GFS any GSEP is equivalent to a GLU. Thus, there exist no nontrivial GLOCC transformations among pure fully entangled GFS. Because of this, we consider then the larger class of GSLOCC. With the help of a result on normal forms of states from Ref. [66], we also characterized the GSLOCC classes in the Jordan-Wigner representation and furthermore explicitly derive them for few-mode systems. Then, we investigated the more general FLOCC′, which contains in particular finitely many rounds FLOCC (see Sec. IV C) to obtain insights into the various entanglement properties of GFS and we show how to identify the MES of pure n-mode n-partite GFS under FLOCC′.

Let us finally compare the fermionic case investigated here with the bosonic and the finite-dimensional scenarios. In all three cases, a computable condition for two (n-partite n-modes or n-qubit) states to be (G)LU equivalent has been presented [55,84]. Regarding the bosonic Gaussian case, we have that GSLOCC coincide with GLOCC transformations. This follows from the fact that any GSLOCC operation can be completed to a deterministic transformation. Moreover, there exist GLOCC transformation among pure bosonic Gaussian states which are not just GLU transformations (see, e.g., Ref. [55]). The MES for bosonic Gaussian states is not known; however, in Ref. [55] a class of three-mode states has been identified which can reach states which cannot be reached from any symmetric three-mode state (including the GHZ and W states). Regarding the finite-dimensional case, there exist (not surprisingly) more SLOCC classes than for GFS. Moreover, for Hilbert spaces composed of local Hilbert spaces of equal dimensions, it has been shown that almost all states are isolated; i.e., the state can neither be reached nor transformed into any other (not LU-equivalent) state via LOCC [13,14]. This resembles the fermionic case. However, as mentioned before, the reason for this to be true stems from the fact that almost no state possesses a local symmetry.

It would be interesting to investigate another physically relevant scenario by imposing a (global) particle-number selection rule (as it is observed by elementary fermions in nature) on the states considered and studying state transformations via number-preserving local operations. Moreover, as in the qudit case, the transformations from a multipartite state, where each party holds more than a single mode (a single qubit) to a state whose local rank is smaller might well allow (more) nontrivial transformations, respectively. Physically motivated, restricted set of states, such as FS or GFS, are ideally suited for this investigation, as it will be more trackable than the general qudit case. Moreover, this class of states is rich enough so that the results derived for them have the potential to lead also to additional insight into state transformations among qudit states.

ACKNOWLEDGMENTS

The research of K.S. and B.K. was funded by the Austrian Science Fund (FWF) Grant No. Y535-N16. G.G. acknowledges support by the Spanish Ministerio de Economía y Competitividad through the Project No. FIS2014-55987-P. C.S. acknowledges support by the Austrian Science Fund (FWF) Grant No. Y535-N16, the DFG, and the ERC (Consolidator Grant 683107/TempoQ).

APPENDIX A

In this Appendix. we study first the Choi-Jamiolkowski (CJ) isomorphism [49,85,86] among Gaussian states and Gaussian CP maps. Note that similar aspects of Gaussian CP maps have already been studied in Ref. [43]. However, there the author was using a different definition of the "tensor product" ($\otimes_f$) in the calculation. We summarize here the results using our notation. Then, we consider Gaussian LOCC (GLOCC) transformations and show that any GLOCC corresponds via the CJ isomorphism to a separable state. These investigations lead to the natural definition of fermionic separable maps (FSEP). Considering then the possible states which can be generated via GLOCC enables us to rule out the definition $\mathcal{S}2_{\pi'}$ for separable states. That is, if $\mathcal{S}2_{\pi'}$ does not coincide with $\mathcal{S}2_{\pi}$ for GFS, there exist states in $\mathcal{S}2_{\pi'}$ which can neither be prepared locally by Gaussian operations nor belong to the limit of such a preparation scheme.

1. Choi-Jamiolkowski isomorphism in the Gaussian case

The CJ isomorphism is a one-to-one mapping between CP maps and positive semidefinite operators. Denoting by $\mathcal{E}$ the CP map that is acting on n modes and by $\rho_{\mathcal{E}}$ the corresponding operator, we have

$$\begin{aligned}\rho_{\mathcal{E}} &= \mathcal{E}\tilde{\otimes}\mathbb{1}(|\Phi_{2n}^{+}\rangle\langle\Phi_{2n}^{+}|),\\ \mathcal{E}(\rho) &= \mathrm{tr}_{23}\big(\rho_{\mathcal{E}}^{12}\rho^{3}|\Phi_{2n}^{+}\rangle^{23}\langle\Phi_{2n}^{+}|\big),\end{aligned}\tag{A1}$$

where $|\Phi_{2n}^{+}\rangle \propto \prod_{a=1}^{2n}(\mathbb{1}+i\tilde{c}_a\tilde{c}_{2n+a})$. In Ref. [49], it has been shown that separable maps correspond to separable operations and that several other properties of the operators can be inferred from the maps and vice versa. The aim of this section is to show that the same isomorphism holds for Gaussian states. In the subsequent subsection, we will then investigate the relation between separable operators and the corresponding maps. Note that we write Gaussian states and operators in this section in the Grassmann representation; see Ref. [43] for more details. Note further that $\rho_{\mathcal{E}}$ is a GFS iff $\mathcal{E}$ is a Gaussian map. It is obvious that $\rho_{\mathcal{E}}$ is a Gaussian state if $\mathcal{E}$ is Gaussian as $|\Phi_{2n}^{+}\rangle$ is a GFS. Moreover, due to $\mathcal{E}(\rho)=\mathrm{tr}_{23}(\rho_{\mathcal{E}}^{12}\rho^{3}|\Phi_{2n}^{+}\rangle^{23}\langle\Phi_{2n}^{+}|)$ one obtains that if $\rho_{\mathcal{E}}$ is a GFS then also $\mathcal{E}(\rho)$ is Gaussian for all GFS ρ and therefore $\mathcal{E}$ corresponds to a Gaussian map.

In Ref. [43], it was shown that a linear CP map on n fermionic modes is Gaussian iff it has a (Grassmann) integral representation

$$\mathcal{E}(X)(\theta) = C\int D\eta D\mu\, \exp[S(\theta,\eta)+i\eta^{T}\mu]X(\mu),\tag{A2}$$

where

$$S(\theta,\eta)=\frac{i}{2}\begin{pmatrix}\theta\\ \eta\end{pmatrix}^T\begin{pmatrix}A & B\\ -B^T & D\end{pmatrix}\begin{pmatrix}\theta\\ \eta\end{pmatrix}\equiv\vec{\theta}^{\,T}M_{\mathcal{E}}\vec{\theta},$$

with $C\geqslant 0$, real $2n\times 2n$ matrices A,B,D, and $M_{\mathcal{E}}^T M_{\mathcal{E}}\leqslant \mathbb{1}$. The identity map on n modes is given by $A=D=0$ and $B=\mathbb{1}$. Thus, for a map $\mathcal{E}'$ on $n+m$ modes that acts nontrivially only on the first n modes, we take $A'=A\oplus 0, D'=D\oplus 0, B'=B\oplus\mathbb{1}$. Applying this map (for $m=n$) to the maximally entangled state of $2n$ modes, we get as the CM of the output state [with $\vec{\theta}=(\theta,\theta')$ (and same for $\vec{\eta},\vec{\mu}$) and $\vec{x}_{12}=(\vec{\theta},\vec{\eta}),\vec{x}_{23}=(\vec{\eta},\vec{\mu})$]

$$\int D\eta D\eta' D\mu D\mu' e^{\frac{i}{2}\vec{x}_{12}^T\begin{pmatrix}A' & B'\\ -B'^T & D'\end{pmatrix}\vec{x}_{12}+i\vec{\eta}^T\vec{\mu}}e^{\frac{i}{2}\vec{\mu}^T\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}\vec{\mu}}$$
$$=e^{\frac{i}{2}\vec{\theta}^T(A\oplus 0)\vec{\theta}}\int Dx_{23}e^{y^T\vec{x}_{23}+\frac{i}{2}\vec{x}_{23}^T\tilde{M}\vec{x}_{23}}$$
$$\propto e^{\frac{i}{2}\vec{\theta}^T(A\oplus 0)\vec{\theta}}e^{-\frac{i}{2}y^T\tilde{M}^{-1}y}.$$

In the last step, we used the Gaussian integration rule (see Eq. (13) of Ref. [43]), $y=(iB'^T\vec{\theta},0)$, and $\tilde{M}=\begin{pmatrix}D & 0 & \mathbb{1} & 0\\ 0 & 0 & 0 & \mathbb{1}\\ -\mathbb{1} & 0 & 0 & \mathbb{1}\\ 0 & -\mathbb{1} & -\mathbb{1} & 0\end{pmatrix}$. Since y is nonzero only in the first two components, we only need the upper diagonal block of the 2×2 block matrix $\tilde{M}^{-1}$, which is given by the Schur complement as $\begin{pmatrix}D & -\mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}^{-1}=\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & D\end{pmatrix}$. Thus, we end up with a Gaussian Grassmann representation with CM

$$\begin{pmatrix}A & B\\ -B^T & D\end{pmatrix}. \tag{A3}$$

Hence, the GFS with this CM is the CJ state $\rho_{\mathcal{E}}$ of the map $\mathcal{E}$. Note that by using the above-mentioned definition of a tensor product $\otimes_f$ (see Definition 5 in Ref. [43]) for the computation of the CJ state, we obtain a CM $\begin{pmatrix}A & -B\\ B^T & D\end{pmatrix}$. The corresponding state is obtained by applying the local operator $\prod_{i=1}^{2n}\tilde{c}_i$ to $\rho_{\mathcal{E}}$.

In order to confirm that the state $\rho_{\mathcal{E}}$ with CM given in Eq. (A3) allows for the physical interpretation, which is characteristic for the CJ state, and that it can be used to realize the map $\mathcal{E}$ via teleportation, we compute

$$\mathrm{tr}_{23}\big(\rho_{\mathcal{E}}^{12}\rho_\Gamma^3|\Phi_{2n}^+\rangle^{23}\langle\Phi_{2n}^+|\big).$$

Here, the superscripts indicate on which of the three different blocks of modes the state is nontrivial. Using the formula for the trace of two operators X,Y in Grassmann variables [87] (see also Eq. (15) in Ref. [43]) and with $X=\rho_{\mathcal{E}}^{12}\rho_\Gamma^3$, $Y=|\Phi_{2n}^+\rangle^{23}\langle\Phi_{2n}^+|$) the trace is given by

$$\mathrm{tr}_{23}(XY)$$
$$\propto\int D\vec{\eta}D\vec{\mu}e^{(iB^T\theta)^T\eta+\frac{i}{2}\left(\theta^TA\theta+\eta^TD\eta+\eta'^T\Gamma\eta'+\vec{\mu}^T\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}\vec{\mu}\right)}e^{\vec{\eta}^T\vec{\mu}}$$
$$=e^{\frac{i}{2}\theta^TA\theta}\int D\vec{x}_{23}e^{\xi^T\vec{x}_{23}+\frac{i}{2}\vec{x}_{23}^TM'\vec{x}_{23}}.$$

Here, again $\vec{x}_{23}=(\vec{\eta},\vec{\mu})$ and

$$\xi^T=((iB^T\theta)^T,0,0,0),$$
$$M'=\begin{pmatrix}D & 0 & -i\mathbb{1} & 0\\ 0 & \Gamma & 0 & -i\mathbb{1}\\ i\mathbb{1} & 0 & 0 & \mathbb{1}\\ 0 & i\mathbb{1} & -\mathbb{1} & 0\end{pmatrix}.$$

Using again the Gaussian integration rule (Eq. (13) in Ref. [43]), we obtain as a result a Gaussian state with CM

$$\Gamma_{\text{out}}=A-(iB)\left(\left[\begin{pmatrix}D & \\ & \Gamma\end{pmatrix}-\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}^{-1}\right]^{-1}\right)_{11}(iB^T)$$
$$=A+B\left(\begin{pmatrix}D & \mathbb{1}\\ -\mathbb{1} & \Gamma\end{pmatrix}^{-1}\right)_{11}B^T$$
$$=A+B(D+\Gamma^{-1})^{-1}B^T, \tag{A4}$$

which is just $\mathcal{E}(\rho_\Gamma)$.

Summarizing, we have shown that the state $\rho_{\mathcal{E}}=(\mathcal{E}\tilde{\otimes}\mathbb{1})(|\Phi_{2n}^+\rangle\langle\Phi_{2n}^+|)=\rho_M$, where the GFS ρ_M with CM $M=\begin{pmatrix}A & B\\ -B^T & D\end{pmatrix}$ is the CJ state of the Gaussian map $\mathcal{E}$ which is given in Eq. (A2) or equivalently which maps the CM Γ to Γ_{out} as given in Eq. (A4).

2. Gaussian LOCC (GLOCC)

Let us now investigate the relation of the entanglement properties of CJ state and the entanglement properties of the corresponding CP map. We will consider here only bipartite systems; however, all arguments hold also for the multipartite setting. In the case of finite-dimensional systems a CPTM, $\mathcal{E}$, is called separable if it can be written as

$$\mathcal{E}(\rho)=\sum_k A_k\otimes B_k\rho A_k^\dagger\otimes B_k^\dagger. \tag{A5}$$

As the set of separable maps (SEP) strictly contains the set of LOCC, i.e., the set of maps which can be realized via local operations and classical communication, SEP lacks a clear physical meaning. Hence, when considering restricted sets of maps, such as here fermionic or Gaussian maps, there is no clear way of specializing the notion of SEP to these sets. This is why we consider here the physically meaningful, however, mathematically generically much less tractable set of LOCC, for which this specialization is obvious. We will then show that this consideration suggests the natural definition of fermionic SEP (FSEP).

Let us first consider the CJ state of a local map, $\mathcal{E}=\mathcal{E}_1\tilde{\otimes}\mathcal{E}_2$, i.e., a composition of two maps, $\mathcal{E}_1$, and $\mathcal{E}_2$, which act on the first and second systems nontrivially, respectively. In this case, the CM of the CJ state splits in the form $A=A_1\oplus A_2, B=B_1\oplus B_2, D=D_1\oplus D_2$. One can easily check that $\mathcal{E}(\Phi_+)$ is separable with respect to the splitting 13|24 according to our definition (see Sec. III A). Hence, using our definition of separability ($\mathcal{S}2_\pi$), the CJ isomorphism maps local maps to separable states.

Let us next show that the CJ state of any Gaussian LOCC is separable according to the definition $\mathcal{S}2_\pi$. That is, we show that any map which describes a GLOCC corresponds to a Gaussian CJ state whose CM is given by $\Gamma_1\oplus\Gamma_2$ [88], i.e., the corresponding state factorizes. Using this result and the remark above, it is then easy to see that any GLOCC can be written as $\mathcal{E}_1\tilde{\otimes}\mathcal{E}_2$.

Operationally, a finitely-many-rounds FLOCC protocol is a protocol which can be realized by local fermionic operations and a finite number of rounds of classical communication. In order to include also FLOCC protocols which require infinitely many rounds of communication, we define FLOCC as the set of finitely-many-rounds FLOCC protocols together with those which are the limit of a sequence of such protocols. A Gaussian FLOCC is a FLOCC that can be implemented with Gaussian means and that maps Gaussian states to Gaussian states. Stated differently any map, $\mathcal{E}$, corresponding to a finitely-many-rounds FLOCC (GLOCC) can be written as in Eq. (A5), where all operators, A_k, B_k are fermionic (Gaussian) operators, respectively. Any map within FLOCC (GLOCC) can be written as the limit of a sequence of such maps where each element of the sequence is obtained by applying one more round of a FLOCC protocol to the preceding element.

Let us now show that the CJ state of a Gaussian FLOCC factorizes. We consider first finitely many round protocols and extend the result then to the limit of sequences of such protocols. The CJ state is given by $\rho_{\mathcal{E}} = (\mathcal{E}^{ab}_{\mathrm{FLOCC}}\tilde{\otimes}\mathbb{1}^{a'b'})(P^{aa'}_{\Phi^+}\tilde{\otimes}P^{bb'}_{\Phi^+})$. Using that $\mathcal{E}_{\mathrm{FLOCC}}$ is of the form given in Eq. (A5), where A_k, B_k are fermionic operators and computing the expectation value of $\tilde{c}_a\tilde{c}_b$, where $\tilde{c}_a$ ($\tilde{c}_b$) denotes any Majorana operator acting on modes in a (b), respectively, we obtain

$$\begin{aligned}&\mathrm{tr}\big[\mathcal{E}^{ab}_{\mathrm{FLOCC}}\tilde{\otimes}\mathbb{1}^{a'b'}\big(P^{aa'}_{\Phi^+}\tilde{\otimes}P^{bb'}_{\Phi^+}\big)\tilde{c}_a\tilde{c}_b\big]\\ &\quad=\sum_k(-1)^{f(A_kB_k)}\mathrm{tr}\big[A_k^\dagger\tilde{c}_aA_k\tilde{\otimes}B_k^\dagger\tilde{c}_bB_k\big(P^{aa'}_{\Phi^+}\tilde{\otimes}P^{bb'}_{\Phi^+}\big)\big], \quad (\mathrm{A6})\end{aligned}$$

where $f(A_kB_k)=0$ for even operators A_kB_k and $f(A_kB_k)=1$ for odd operators. As any operator A_k is parity respecting, i.e., is fermionic, $A_k^\dagger\tilde{c}_aA_k$ is an odd operator. Because the projector onto Φ^+ is even and that $\mathrm{tr}[A\tilde{\otimes}B(P^{aa'}_{\Phi^+}\tilde{\otimes}P^{bb'}_{\Phi^+})] = \mathrm{tr}[AP^{aa'}_{\Phi^+}]\mathrm{tr}[BP^{bb'}_{\Phi^+}]$, the trace vanishes. Hence, the off-diagonal terms in the CM of the CJ state vanish and $\Gamma = \Gamma^{aa'}\oplus\Gamma^{bb'}$. In case the CJ state is Gaussian, in particular, if $\mathcal{E}^{ab}_{\mathrm{FLOCC}}$ is a Gaussian map [89], we hence have that the CJ state factorizes. The last assertion follows from the fact that for GFS, Wick's theorem holds and thus, all higher order correlations factorize if the CM is block diagonal. In case $\mathcal{E}^{ab}_{\mathrm{FLOCC}}$ is the limit of a sequence of finitely-many-rounds protocols, the statement also holds due to continuity arguments [90].

It is evident from the discussion above that (i) any FLOCC applied to a product state is separable and that (ii) any separable GFS (according to $\mathcal{S}2_\pi$) can be generated via FLOCC from a product state. This fact, being obvious from a physical point of view, shows that the definition we choose for separability meets the necessary requirements. Moreover, this also shows that states which are convex combination of nonfermionic states (or the limit thereof) and for which no decomposition into FSs exist cannot be generated locally. Hence, in case the set $\mathcal{S}2_{\pi'}$ contains such a state, then calling states in $\mathcal{S}2_{\pi'}$ separable does not conform to the usual operational definition.

Note that in the argument above the restriction to locally realizable maps has never been used. Hence, a natural definition of Gaussian separable maps (GSEP) is the set of CPTMs whose CJ state is a separable Gaussian state, i.e., $\rho_{\mathcal{E}_{\mathrm{GSEP}}} = \rho_A\tilde{\otimes}\rho_B$ (which for GFS is equivalent to $\Gamma_{\mathcal{E}_{\mathrm{GSEP}}} = \Gamma_A\oplus\Gamma_B$). Note that this implies that $\mathcal{E}_{\mathrm{GSEP}} = \mathcal{E}_A\tilde{\otimes}\mathcal{E}_B$. FSEP is then defined as the set of CPT maps that can be written as $\mathcal{E}(\rho) = \sum_k(A_k\tilde{\otimes}B_k)\rho(A_k\tilde{\otimes}B_k)^\dagger$ where all the A_k, B_k are parity-respecting operators.

APPENDIX B: PROOF OF OBSERVATION 2

Here, we prove the observation that a product state according to definition $\mathcal{P}1_\pi$, i.e., the set of states for which the expectation values of all products of physical observables factorize, can have nonzero correlation between A and B.

Proof. Let us denote by $\mathcal{P}1_\pi$ the set of states for which all products of locally measurable observables factorize, by $\mathcal{S}1_\pi$ its convex hull, and by $\mathcal{S}_G$ the set of Gaussian states. We show that $\rho\in\mathcal{S}1_\pi\cap\mathcal{S}_G$ implies $\Gamma_\rho = \begin{pmatrix}\Gamma_A & C\\ -C^T & \Gamma_B\end{pmatrix}$ with rank $C\leqslant 1$ and that there are such states with rank $C=1$.

We consider observables of the form $\Pi_{i=1}^{2n_a}\tilde{c}^a_{k_i}$ and $\Pi_{j=1}^{2m_b}\tilde{c}^b_{l_j}$, where $\tilde{c}^{a(b)}$ refer to Majorana operators on Alice's (Bob's) modes. We exploit the fact that we can compute their expectation values in two ways: either by using the Wick formula for the $n+m$-mode Gaussian state or by using the separability condition and using the Wick formula twice for the n and m local modes separately. We show that these only coincide for all observables if the rank of the off-diagonal block C of the full CM is not larger than 1.

Considering the observable $\tilde{c}^a_{k_1}\tilde{c}^a_{k_2}\tilde{c}^b_{l_1}\tilde{c}^b_{l_2}$, we find that $C_{k_1l_2}C_{k_2l_1} = C_{k_1l_1}C_{k_2l_2}$ where $C=(C_{ij})$. Without loss of generality, we can choose to work in the basis in which C takes diagonal form [i.e., apply local basis changes O_a, O_b such that $O_aCO_b^T$ is diagonal (singular value decomposition)]. Then, considering $k_1=l_1, k_2=l_2$ one obtains that the rank of C can be at most one since two nonzero singular values would lead to a contradiction. This single nonzero entry, however, cannot lead to any difference between the two ways of computing expectation values of products of even observables and thus there can be (and are) Gaussian states in $\mathcal{S}_{S1}$ with $C\neq 0$: For example, consider any Gaussian state with CM such that $C_{k_1k_1}\neq 0$ is the only nonzero entry of C and consider any pair of even observables $A=\Pi_i\tilde{c}^a_{k_i}, B=\Pi_j\tilde{c}^b_{l_j}$, then $\rho_\Gamma(AB) = \rho_{\Gamma_A\oplus\Gamma_B}(AB) = \rho_{\Gamma_A}(A)\rho_{\Gamma_B}(B) = \rho_\Gamma(A)\rho_\Gamma(B)$, since, using Wick's formula any term that contains a pairing (k_1,k_1) must necessarily contain another AB-correlating pair (k_2,l_2) with $k_1\neq k_2, l_1\neq l_2$ since no index appears twice in the same subsystem. However, since $C_{k_1k_1}$ is the only nonvanishing entry of C the corresponding term is zero and only the local blocks Γ_A,Γ_B contribute to $\rho(AB)$. ■

APPENDIX C: STANDARD FORM OF THE CM OF 1 × 1 × 1 STATES

Here, we state the conditions on the parameters of the standard form for mixed three modes GFS, i.e.,

$$S(\gamma) = \begin{pmatrix} 0 & \lambda_1 & d_{12} & 0 & l_1d_{13} & l_2d'_{13}\\ -\lambda_1 & 0 & 0 & d'_{12} & -l_2d_{13} & l_1d'_{13}\\ -d_{12} & 0 & 0 & \lambda_2 & m_1 & m_{12}\\ 0 & -d'_{12} & -\lambda_2 & 0 & m_{21} & m_2\\ -l_1d_{13} & l_2d_{13} & -m_1 & -m_{21} & 0 & \lambda_3\\ -l_2d'_{13} & -l_1d'_{13} & -m_{12} & -m_2 & -\lambda_3 & 0 \end{pmatrix},$$

in more detail. If no mode factorizes we have for $\lambda_i > 0$ for $i \in \{1,2,3\}$ the following cases:

(1) $d_{12} > |d'_{12}|$ and
  (a) $d_{13} > |d'_{13}|$ and $l_1^2 + l_2^2 = 1$ with either $l_1 > 0$ or $l_1 = 0$ and $l_2 > 0$ or
  (b) $d_{13} = |d'_{13}| \neq 0, l_1 = 1$ and $l_2 = 0$ or
  (c) $l_1 = l_2 = 0, m_1 = l'_1 d_{23}, m_2 = l'_1 d'_{23}, m_{12} = l'_2 d'_{23}$, and $m_{21} = -l'_2 d_{23}$ with $l_1'^2 + l_2'^2 = 1, d_{23} > |d'_{23}|$ and either $l'_1 > 0$ or $l'_1 = 0$ and $l'_2 > 0$ or
  (d) $l_1 = l_2 = 0, m_1 = |m_2| \neq 0, m_{12} = 0$ and $m_{21} = 0$.

(2) $d_{12} = |d'_{12}| \neq 0$ and
  (a) $d_{13} > |d'_{13}|, l_1 = 1$, and $l_2 = 0$ or
  (b) $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = l'_1 d_{23}$, $m_2 = l'_1 d'_{23}, m_{12} = l'_2 d_{23}$, and $m_{21} = -l'_2 d'_{23}$ with $d_{23} > |d'_{23}|$ and $l_1'^2 + l_2'^2 = 1$ or
  (c) $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, \gamma_{23} \propto O(2,\mathbb{R})$, and $d'_{12} d'_{13} |\gamma_{23}| > 0$ or
  (d) $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = |m_2|$, $m_{12} = m_{21} = 0$, and $d'_{12} d'_{13} m_2 < 0$ or
  (e) $l_1 = l_2 = 0, m_1 > |m_2|, m_{12} = 0$, and $m_{21} = 0$ or
  (f) $l_1 = l_2 = 0, m_1 = |m_2| \neq 0, m_{12} = 0$, and $m_{21} = 0$ or
  (g) $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0$, and $m_1 = m_2 = m_{12} = m_{21} = 0$.

(3) $d_{12} = |d'_{12}| = 0$ and
  (a) $d_{13} > |d'_{13}|, l_1 = 1, l_2 = 0, m_1 = l'_1 d_{23}, m_2 = l'_1 d'_{23}$, $m_{12} = l'_2 d_{23}$, and $m_{21} = -l'_2 d'_{23}$ with $l_1'^2 + l_2'^2 = 1$, $d_{23} > |d'_{23}|$ and either $l'_1 > 0$ or $l'_1 = 0$ and $l'_2 > 0$ or
  (b) $d_{13} > |d'_{13}|, l_1 = 1, l_2 = 0, m_1 = |m_2| \neq 0, m_{12} = 0$, and $m_{21} = 0$ or
  (c) $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 > |m_2|, m_{12} = 0$, and $m_{21} = 0$ or
  (d) $d_{13} = |d'_{13}| \neq 0, l_1 = 1, l_2 = 0, m_1 = |m_2| \neq 0$, $m_{12} = 0$, and $m_{21} = 0$.

In case $\lambda_i = 0$ for some $i \in \{1,2,3\}$, the standard form can be obtained analogously. However, in this case m_i is not determined by γ_{ii}.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[2] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, Proc. R. Soc. London, Ser. A **465**, 2537 (2009).
[5] A. Acín, I. Bloch, H. Buhrman, T. Calarco, C. Eichler, J. Eisert, D. Esteve, N. Gisin, S. J. Glaser, F. Jelezko *et al.*, arXiv:1712.03773 (unpublished).
[6] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[7] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).
[8] G. Gour and N. R. Wallach, Phys. Rev. Lett. **111**, 060502 (2013).
[9] C. Eltschka, T. Bastin, A. Osterloh, and J. Siewert, Phys. Rev. A **85**, 022301 (2012).
[10] J. I. de Vicente, C. Spee, and B. Kraus, Phys. Rev. Lett. **111**, 110502 (2013).
[11] C. Spee, J. de Vicente, and B. Kraus, J. Math. Phys. **57**, 052201 (2016).
[12] M. Hebenstreit, C. Spee, and B. Kraus, Phys. Rev. A **93**, 012339 (2016).
[13] G. Gour, B. Kraus, and N. R. Wallach, J. Math. Phys. **58**, 092204 (2017).
[14] D. Sauerwein, N. R. Wallach, G. Gour, and B. Kraus, arXiv:1711.11056 (unpublished).
[15] K. Eckert, J. Schliemann, D. Bruss, and M. Lewenstein, Ann. Phys. **299**, 88 (2002).
[16] M.-C. Bañuls, J. I. Cirac, and M. M. Wolf, Phys. Rev. A **76**, 022311 (2007).
[17] D.-M. Schlingemann, M. Cozzini, M. Keyl, and L. C. Venuti, Phys. Rev. A **78**, 032301 (2008).
[18] A. R. Plastino, D. Manzano, and J. S. Dehesa, Europhys. Lett. **86**, 20005 (2009).
[19] N. Friis, A. R. Lee, and D. E. Bruschi, Phys. Rev. A **87**, 022338 (2013).
[20] F. Benatti, R. Floreanini, and K. Titimbo, Open Syst. Inf. Dyn. **21**, 1440003 (2014).
[21] G. Sárosi and P. Lévay, J. Phys. A: Math. Theor. **47**, 115304 (2014).
[22] G. M. D'Ariano, F. Manessi, P. Perinotti, and A. Tosini, Int. J. Mod. Phys. A **29**, 1430025 (2014).
[23] V. Eisler and Z. Zimboras, New J. Phys. **17**, 053048 (2015).
[24] R. Lo Franco and G. Compagno, Sci. Rep. **6**, 20603 (2016).
[25] C. Kloeffel and D. Loss, Annu. Rev. Condens. Matter Phys. **4**, 51 (2013).
[26] S. Hermelin, S. Takada, M. Yamamoto, S. Tarucha, A. D. Wieck, L. Saminadayar, C. Bäuerle, and T. Meunier, Nature (London) **477**, 435 (2011).
[27] R. P. G. McNeil, M. Kataoka, C. J. B. Ford, C. H. W. Barnes, D. Anderson, G. A. C. Jones, I. Farrer, and D. A. Ritchie, Nature (London) **477**, 439 (2011).
[28] E. Bocquillon, F. D. Parmentier, C. Grenier, J.-M. Berroir, P. Degiovanni, D. C. Glattli, B. Plaçais, A. Cavanna, Y. Jin, and G. Fève, Phys. Rev. Lett. **108**, 196803 (2012).
[29] S. Das Sarma, M. Freedman, and C. Nayak, NPJ Quant. Inf. **1**, 15001 (2015).
[30] C. V. Kraus, M. M. Wolf, J. I. Cirac, and G. Giedke, Phys. Rev. A **79**, 012306 (2009).
[31] T. B. Wahl, H.-H. Tu, N. Schuch, and J. I. Cirac, Phys. Rev. Lett. **111**, 236805 (2013).
[32] O. Morgenshtern, B. Reznik, and I. Zalzberg, arXiv:0807.0850 (unpublished).
[33] Z. Kadar, M. Keyl, and D. Schlingemann, Quantum Inf. Comput. **12**, 74 (2012).
[34] B. Yurke, Phys. Rev. Lett. **56**, 1515 (1986).
[35] F. Benatti, R. Floreanini, and U. Marzolino, Phys. Rev. A **89**, 032326 (2014).
[36] D. DiVincenzo and B. Terhal, Found. Phys. **35**, 1967 (2005).
[37] K. Baumann and G. C. Hegerfeldt, Pub. Res. Inst. Math. Sci. **21**, 191 (1985).
[38] D. W. Robinson, Commun. Math. Phys. **1**, 89 (1965).
[39] A. Botero and B. Reznik, Phys. Lett. A **331**, 39 (2004).

[40] V. Bach, E. H. Lieb, and J. P. Solovej, J. Stat. Phys. **76**, 3 (1994).

[41] Note that the party over which one performs the trace has to be mapped to the last position. This ensures that the expectation value of all operators A acting only on the first part of a bipartition $A|B$ fulfills $\langle A\rangle = \mathrm{tr}(A\rho) = \mathrm{tr}(A\rho_A)$, with $\rho_A = \mathrm{tr}_B(\rho)$; see Refs. [19,91].

[42] This can be easily seen by writing the parity operator, $P = i^n\prod_k c_k$, as $P = P_e - P_o$ and thus $[\rho,P] = 0$ iff $\rho = P_e\rho P_e + P_o\rho P_o$.

[43] S. Bravyi, Quantum Inf. Comput. **5**, 216 (2005).

[44] see R. Jozsa and A. Miyake, Proc. R. Soc. London, Ser. A **464**, 3089 (2008), and references therein.

[45] B. M. Terhal and D. P. DiVincenzo, Phys. Rev. A **65**, 032325 (2002).

[46] Note that the operator O can be written as $O = O_1 O_p O_2$, with $O_i \in \mathcal{SO}(2n,\mathbb{R})\bigcap SP(2n,\mathbb{R})$, for $i = 1,2$ and $O_p \in \mathcal{SO}(2n,\mathbb{R})$, the so-called pairing operator. The real orthogonal matrices O_i are passive transformations; i.e., they commute with the number operator, whereas the pairing operator stems from, e.g., $b_1^\dagger b_2^\dagger$. Note that for a single mode, i.e., $n = 1$, any real symplectic matrix is a special orthogonal matrix.

[47] The order of the Majorana operators is here $\tilde{c}_1\tilde{c}_2\tilde{c}_3\tilde{c}_4\ldots\tilde{c}_{2n-1}\tilde{c}_{2n}$.

[48] The corresponding unitary is proportional to $\tilde{c}_{2k}\tilde{c}_{2n+2}$. Note that $\tilde{c}_{2n+2}$ is a Majorana operator of ancillary mode which has to be ranked last. Note that the operation $\tilde{c}_{2k}\tilde{c}_{2k+1}\tilde{c}_{2k+2}\ldots\tilde{c}_{2n-1}\tilde{c}_{2n}\tilde{c}_{2n+2}$, which is GLU to $\tilde{c}_{2k}\tilde{c}_{2n+2}$, corresponds to a X on particle k in the JW-representation (see Sec. II B).

[49] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).

[50] N. Schuch, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. **92**, 087904 (2004).

[51] O_i diagonalizes $\gamma_{ij}\gamma_{ij}^T$ and O_j diagonalizes $\gamma_{ij}^T\gamma_{ij}$. Note that due to the restriction O_i, $O_j \in \mathcal{SO}(2,\mathbb{R})D_{ij}$ cannot be chosen to be non-negative.

[52] If $d_{jk} = |d'_{jk}|$ then γ_{jk} is proportional to an orthogonal matrix and α_k is determined as explained before.

[53] Analogously, one determines α_j by diagonalizing $\gamma_{jk}\gamma_{jk}^T$ (and imposing the same conditions on the singular values and the orthogonal matrix) if only α_k has been already determined.

[54] One applies an analogous procedure if α_k is expressed as a function of α_l.

[55] G. Giedke and B. Kraus, Phys. Rev. A **89**, 012335 (2014).

[56] A. Sawicki, M. Walter, and M. Kuś, J. Phys. A: Math. Theor. **46**, 055304 (2013).

[57] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **59**, 1070 (1999).

[58] M. Kleinmann, H. Kampermann, and D. Bruß, Phys. Rev. A **84**, 042326 (2011).

[59] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter, Commun. Math. Phys. **328**, 303 (2014).

[60] In analogy to the finite-dimensional case and in contrast to the bosonic case, all operators are bounded and hence invertible.

[61] F. de Melo, P. Ćwikliński, and B. M. Terhal, New J. Phys. **15**, 013015 (2013).

[62] It straightforwardly follows from this equation that the action of A_k is the same (up to a proportionality factor) as the action of A_l on the whole Fock basis. In order to see this, apply for $j \in \{0,1\}$ the projector $|\vec{v}_j\rangle\langle\vec{v}_j|$ where $|\vec{v}_j\rangle$ is a Fock state of modes $2,\ldots,n$ for which $|\vec{v}_j\rangle\langle\vec{v}_j||\Psi_j^i\rangle \neq 0$ to both sides of the equation. Note that $|\vec{v}_j\rangle\langle\vec{v}_j|$ commutes with the Kraus operators as they have definite parity. Then apply the operators $(\tilde{c}_3)^{m_2}(\tilde{c}_5)^{m_3}\ldots(\tilde{c}_{2n-1})^{m_n}$ for $m_i \in \{0,1\}$ to the resulting equation. It is important to note here that if the Kraus operators are chosen to have definite parity in order for $A_k|j\rangle|\Psi_j^i\rangle = cA_l|j\rangle|\Psi_j^i\rangle$ to possibly hold true either both, A_k and A_l, are sums of even monomials in the Majorana operators or both are sums of odd monomials. Hence, whenever A_k commutes (anticommutes) with $(\tilde{c}_3)^{m_2}(\tilde{c}_5)^{m_3}\ldots(\tilde{c}_{2n-1})^{m_n}$ so does A_l respectively. This procedure generates the whole Fock basis on the modes $2,\ldots,n$ and one obtains $A_k|j\rangle|\vec{w}\rangle = cA_l|j\rangle|\vec{w}\rangle$ for $j \in \{0,1\}$ and an arbitrary Fock state $|\vec{w}\rangle$ on modes $2,\ldots,n$.

[63] Due to Lemma 5, the Kraus operators can be chosen with definite parity and therefore the operations on the different modes commute. That is, $(\mathcal{E}_1\tilde{\otimes}\mathbb{1})(\mathbb{1}\tilde{\otimes}\mathcal{E}_2)(\cdot) = \sum_{k,l}(A_{1k}\tilde{\otimes}\mathbb{1})(\mathbb{1}\tilde{\otimes}A_{2l})(\cdot)(\mathbb{1}\tilde{\otimes}A_{2l}^\dagger)(A_{1k}^\dagger\otimes\mathbb{1}) = \sum_{k,l}(\mathbb{1}\tilde{\otimes}A_{2l})(A_{1k}\tilde{\otimes}\mathbb{1})(\cdot)(A_{1k}^\dagger\tilde{\otimes}\mathbb{1})(\mathbb{1}\tilde{\otimes}A_{2l}^\dagger) = (\mathbb{1}\tilde{\otimes}\mathcal{E}_2)(\mathcal{E}_1\tilde{\otimes}\mathbb{1})(\cdot)$. This follows from the fact that we get either no or two phase factors of -1 when commuting the Kraus operators.

[64] Note, moreover, that rearranging the modes cannot transform a pure n-partite n-mode entangled GFS into a state for which one mode factorizes.

[65] More precisely, we consider FLOCC operations that implement probabilistically a Gaussian operator.

[66] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A **68**, 012103 (2003).

[67] G. Kempf and L. Ness, The length of vectors in representation spaces, in *Algebraic Geometry: Summer Meeting, Copenhagen, August 7–12, 1978*, edited by K. Lønsted (Springer, Berlin, 1979), pp. 233–243.

[68] A simple example is given by the four-mode GHZ, i.e., $|\Psi\rangle = H^{\otimes 4}|\mathrm{GHZ}_4\rangle$ and the not LU-equivalent state that is obtained after applying a fermionic swap on the first two modes.

[69] Note that we consider again only even states, as the same holds for odd four-mode FS.

[70] T. Maciazek, M. Oszmaniec, and A. Sawicki, J. Math. Phys. **54**, 092201 (2013).

[71] The class with $a = 0$ does contain GFS and in this case the states are biseparable.

[72] Note that if we consider more modes per site, the operations which can be applied would be of the form $X^{k_1}\otimes X^{k_2}T$, where T has to commute with $Z\otimes Z$.

[73] G. Gour and N. R. Wallach, New J. Phys. **13**, 073013 (2011).

[74] Note that only transformations among states within the same SLOCC class are considered here. This is necessary, as we do not consider transformations reducing the local rank of the states, i.e., we consider only transformations among truly multipartite entangled states.

[75] As the SLOCC operators and local symmetries appearing in this equation are of the form $(X)^{m_1}D_1\otimes(X)^{m_2}D_2\otimes\cdots\otimes(X)^{m_n}D_n$ it can be easily seen that if one considers this equation in terms of Majorana operators it only involves even powers of the Majorana operators of a single mode. Hence, the local operators commute and partial traces can be performed without any additional reordering. It can also easily be seen that any operator X acting on system i that appears in this equation can be represented there in terms of Majorana operators by $\tilde{c}_{2i-1}$.

[76] Note that this does not imply that in order to compute the reachable GFS one simply computes the intersection of the GFS

with the reachable qubit states as the required transformations to reach this state might be nonfermionic.

[77] C. Spee, J. I. de Vicente, D. Sauerwein, and B. Kraus, Phys. Rev. Lett. **118**, 040503 (2017).

[78] J. I. de Vicente, C. Spee, D. Sauerwein, and B. Kraus, Phys. Rev. A **95**, 012323 (2017).

[79] More precisely, the measurement operators (including the proper normalization factors) for the measurement of party i are given by $\frac{1}{\sqrt{\mathrm{tr}(D_i^\dagger D_i)}} D_i$ and $\frac{1}{\sqrt{\mathrm{tr}(D_i^\dagger D_i)}} D_i X$.

[80] That is, the seed parameters fulfill $ab + cd = 0, b^2 \neq c^2 \neq d^2 \neq b^2, a^2 \neq b^2, c^2, d^2$, and $\nexists q \in \mathcal{C}/\{1\}$, such that $\{a^2,b^2,c^2,d^2\} = \{qa^2,qb^2,qc^2,qd^2\}$; see Ref. [11]. These conditions stem from a condition on the local symmetries of the states.

[81] Note that exchanging the order of the parties (but keeping the relative order among the modes belonging to one party) neither changes the product structure of the maps due to Lemma 5 nor the Kraus operators. Moreover, the Schmidt coefficients of the state in Jordan-Wigner representation are not changed by such a rearranging.

[82] H.-K. Lo and S. Popescu, Phys. Rev. A **63**, 022301 (2001).

[83] Analogous to the case of a single mode per party, we define FLOCC′ in the multimode case as the class of maps that can be implemented via FLOCC and whose Kraus operators are local fermionic operators.

[84] B. Kraus, Phys. Rev. Lett. **104**, 020504 (2010).

[85] M.-D. Choi, Can. J. Math. **24**, 520 (1972).

[86] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).

[87] The trace is given by $\mathrm{tr}(XY) = (-2)^n \int D\theta D\mu e^{\theta^T \mu} w(X,\theta) w(Y,\mu)$ with $w(\tilde{c}_p \tilde{c}_q ... \tilde{c}_r, \theta) = \theta_p \theta_q ... \theta_r$.

[88] It can be easily seen that a CM of this form can be obtained by rearranging the modes if a state has a CM $A = A_1 \oplus A_2, B = B_1 \oplus B_2, D = D_1 \oplus D_2$.

[89] Let us remark here that the CJ state of a Gaussian map is, of course, Gaussian as Φ^+ is Gaussian.

[90] In Ref. [59], instrument convergence (see also Sec. IV C) was shown by using the distance measure induced by the diamond norm of the corresponding CPTMs, i.e., $\| \mathcal{E} - \tilde{\mathcal{E}} \|_\diamond$. That is, for all $\mathcal{E}$ that are the limit of a sequence of finitely many round protocols, there exists a finitely many round protocol $\tilde{\mathcal{E}}_n$ such that $\lim_{n\to\infty} \| \mathcal{E} - \tilde{\mathcal{E}}_n \|_\diamond = 0$. This implies convergence of the corresponding CJ states $\rho_\mathcal{E}, \rho_{\tilde{\mathcal{E}}_n}$ in trace norm, i.e., $\lim_{n\to\infty} \| \rho_\mathcal{E} - \rho_{\tilde{\mathcal{E}}_n} \|_1 = 0$. This leads to the continuity of the expectation value of $\tilde{c}_a \tilde{c}_b$ in Eq. (A7), i.e., $\lim_{n\to\infty} \mathrm{tr}[(\rho_\mathcal{E} - \rho_{\tilde{\mathcal{E}}_n})\tilde{c}_a \tilde{c}_b] = 0$.

[91] S. Yang, T. B. Wahl, H.-H. Tu, N. Schuch, and J. I. Cirac, Phys. Rev. Lett. **114**, 106803 (2015).